Quantum Mechanics **NAME:**

Homework 4: Quantum Mechanics in Three Dimensions: Homeworks are not handed in or marked. But you get a mark for reporting that you have done them. Once you've reported completion, you may look at the already posted supposedly super-perfect solutions.

Answer Table

	a	b	c	d	e
1.	O	O	O	O	O
2.	O	O	O	O	O
3.	O	O	O	O	O
4.	O	O	O	O	O
5.	O	O	O	O	O
6.	O	O	O	O	O
7.	O	O	O	O	O
8.	O	O	O	O	O
9.	O	O	O	O	O
10.	O	O	O	O	O
11.	O	O	O	O	O
12.	O	O	O	O	O
13.	O	O	O	O	O
14.	O	O	O	O	O
15.	O	O	O	O	O
16.	O	O	O	O	O
17.	O	O	O	O	O
18.	O	O	O	O	O
19.	O	O	O	O	O
20.	O	O	O	O	O
21.	O	O	O	O	O
22.	O	O	O	O	O
23.	O	O	O	O	O
24.	O	O	O	O	O
25.	O	O	O	O	O
26.	O	O	O	O	O
27.	O	O	O	O	O
28.	O	O	O	O	O
29.	O	O	O	O	O
30.	O	O	O	O	O

Name:

	a	b	c	d	e
31.	O	O	O	O	O
32.	O	O	O	O	O
33.	O	O	O	O	O
34.	O	O	O	O	O
35.	O	O	O	O	O
36.	O	O	O	O	O
37.	O	O	O	O	O
38.	O	O	O	O	O
39.	O	O	O	O	O
40.	O	O	O	O	O
41.	O	O	O	O	O
42.	O	O	O	O	O
43.	O	O	O	O	O
44.	O	O	O	O	O
45.	O	O	O	O	O
46.	O	O	O	O	O
47.	O	O	O	O	O
48.	O	O	O	O	O
49.	O	O	O	O	O
50.	O	O	O	O	O
51.	O	O	O	O	O
52.	O	O	O	O	O
53.	O	O	O	O	O
54.	O	O	O	O	O
55.	O	O	O	O	O
56.	O	O	O	O	O
57.	O	O	O	O	O
58.	O	O	O	O	O
59.	O	O	O	O	O
60.	O	O	O	O	O

1. In a central-force problem, the magnitude of central force depends only on:

   a) the angle of the particle.
   b) the vector $\vec{r}$ from the center to the particle.
   c) the radial distance r from the center to the particle.
   d) the magnetic quantum number of the particle.
   e) the uncertainty principle.

2. The usual approach to getting the eigenfunctions of the Hamiltonian in multi-dimensions is:

   a) non-separation of variables. b) separation of variables. c) separation of invariables.
   d) non-separation of invariables. e) non-separation of variables/invariables.

3. Say you have a differential equation of two independent variables x and y and you want to look for solutions that can be factorized thusly $f(x,y) = g(x)h(y)$. Say then it is possible to reorder equation into the form

$$\mathrm{LHS}(x) = \mathrm{RHS}(y) \ ,$$

   where LHS stands for left-hand side and RHS for right-hand side. Well LHS is explicitly independent of y and implicitly independent of x:

$$\frac{\partial \mathrm{LHS}}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial \mathrm{LHS}}{\partial x} = \frac{\partial \mathrm{RHS}}{\partial x} = 0 \ .$$

   Thus, LHS is equal to a constant C and necessarily RHS is equal to the same constant C which is called the constant of separation (e.g., Arf-383). The solutions for $g(x)$ and $h(y)$ can be found separately and are related to each other through C. The solutions for $f(x,y)$ that cannot be factorized are not obtained, of course, by the described procedured. However, if one obtains complete sets of $g(x)$ and $h(y)$ solutions for the x-y region of interest, then any solution $f(x,y)$ can be constructed at least to within some approximation (Arf-443). Thus, the generalization of the described procedure is very general and powerful. It is called:

   a) separation of the left- and right-hand sides. b) partitioning.
   c) separation of the variables. d) solution factorization. e) the King Lear method.

4. The eigensolutions of the angular part of the Hamiltonian for the central force problem are the:

   a) linear harmonics. b) spherical harmonics. c) square harmonics.
   d) Pythagorean harmonics. e) Galilean harmonics.

5. Just about the only spherical harmonic that people remember—and they really should remember it too—is $Y_{00} =$:

   a) $e^{im\phi}$. b) r^2. c) $\dfrac{1}{\sqrt{4\pi}}$. d) θ^2. e) $2a^{-3/2}e^{-r/a}$.

6. In the central force problem, the separated azimuthal part of the Schrödinger equation is:

$$\frac{d^2\Phi}{d\phi^2} = -m_\ell^2\Phi \ ,$$

   where $-m_\ell^2$ is the constant of separation for the azimuthal part. The constant has been parameterized in terms of m_ℓ (which is not mass) since it turns out that for normalizable (and therefore physically allowed) solutions that m must be an integer. The m_ℓ quantity is the z-component angular momentum quantum number or magnetic quantum number (MEL-59; ER-240). The latter name arises since the z-components of the angular momentum manifest themselves most noticeably in magnetic field phenomena.

   a) Since the differential equation is second order, there should should be two independent solutions for each value of m_ℓ^2. Solve for the general solution Φ for each m_ℓ^2: i.e., the solution that is a linear combination of the two independent solutions with undetermined coefficients. Note that writing the separation constant as m_ℓ^2 is so far just a parameterization and nothing yet demands that m_ℓ^2 be greater than zero or pure real. **HINT:** Use an exponential trial function with exponent $\pm(a+ib)$ with a and b real. Also remember the special case of $m_\ell^2 = 0$.

   b) The solutions are continuous and so that quantum mechanical requirement is met. But another one must be imposed for the azimuthal coordinate: the single-valuedness condition. Since we have no interpretation for multi-valuedness, we micropostulate that it doesn't happen. Impose the single-valuedness condition on the generl solution

$$\Phi = Ae^{(a+ib)\phi} + Be^{-(a+ib)\phi} ,$$

and show that $a = 0$ and m_ℓ must be an integer. Remember to consider the special case where $m_\ell = 0$?

c) What are the eigenfunction solutions for the z-component of the angular momentum operator

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} .$$

What are the eigenvalues that satisfy single-valuedness and continuity? What is the relationship between these eigenfunction solutions and the azimuthal angle part of the hydrogenic atom wave functions?

d) Normalize the allowed eigensolutions of L_z Note these solutions are, in fact, conventionally left unnormalized: i.e., the coefficient of the special function that is the solution is left as just 1. Normalization is conventionally imposed on the total orbital angular momentum solutions, spherical harmonics.

7. The hydrogen atom is the simplest of all neutral atoms because:

   a) it is a 2-body system. b) it is a 3-body system. c) it has no electrons.
   d) it has many electrons. e) hydrogen is the most abundant element in the universe.

8. What basic requirements must the radial part of hydrogenic atom wave function meet in order to be a physical radial wave function?

   a) Satisfy the radial part of the Schrödinger equation and grow exponentially as $r \to \infty$.
   b) Not satisfy the radial part of the Schrödinger equation and grow exponentially as $r \to \infty$.
   c) Satisfy the radial part of the Schrödinger equation and be normalizable.
   d) Not satisfy the radial part of the Schrödinger equation and be normalizable.
   e) None at all.

9. The hydrogenic atom eigenstate wave functions contain a factor that causes them to:

   a) increase exponentially with radius. b) decrease exponentially with radius.
   c) increase logarithmically with radius. d) increase quadratically with radius.
   e) increase linearly with wavelength.

10. What special functions are factors in the radial part of the of the hydrogenic atom eigenstate wave functions?

    a) The associated Laguerre polynomials. b) The unassociated Laguerre polynomials.
    c) The associated Jaguar polynomials. d) The unassociated jaguar polynomials.
    e) The Hermite polynomials.

11. Almost all would agree that the most important empirical means for learning about atomic energy eigenstates is:

    a) spectroscopy. b) microscopy. c) telescopy. d) pathology. e) astrology.

12. Let us consider the probability that the electron of a hydrogenic atom in the ground state will be in the nucleus. Recall the wave function for ground state is

$$\Psi_{100}(\vec{r}) = R_{10}(r)Y_{00}(\theta,\phi) = 2a^{-3/2}e^{-r/a} \times \frac{1}{\sqrt{4\pi}}$$

(Gr2005-154), where $a = a_{\text{Bohr}}[m_e/(mZ_{\text{N}})]$: $a_{\text{Bohr}} \approx 0.529$Å is the Bohr radius, Z_{N} is the nuclear charge, m_e is the electron mass, and m is the reduced mass of the actual hydrogenic atom.

a) First assume that the wave function is accurate down to $r = 0$. It actually can't be, of course. The wave function was derived assuming a point nucleus and the nucleus is, in fact, extended. However, the extension of the nucleus is of order 10^5 times smaller than the Bohr radius, and so the effect of a finite nucleus is a small perturbation. Given that the nuclear radius is b, calculate the probability of finding the electron in the nucleus. Use $\epsilon = b/(a/2) = 2b/a$ to simplify the formula. **HINT:** The formula

$$g(n,x) = \int_0^x e^{-t} t^n \, dt = n! \left(1 - e^{-x} \sum_{\ell=0}^{n} \frac{x^\ell}{\ell!} \right)$$

could be of use.

b) Expand the part (a) answer in ϵ power series and show to lowest non-zero order that

$$P(r < b, \epsilon << 1) = \frac{1}{6}\epsilon^3 = \frac{4}{3}\left(\frac{b}{a}\right)^3 .$$

c) An alternate approach to find the probability of the electron being in the nucleus is assume $\Psi(\vec{r})$ can be approximated by $\Psi(0)$ over nucleus. Thus

$$P(r < b) \approx \left(\frac{4\pi}{3}\right) b^3 |\Psi(0)|^2 .$$

Is this result consistent with the part (b) answer?

d) Assume $b \approx 10^{-15}$ m and $a = 0.5 \times 10^{-10}$ m. What is the approximate numerical value for finding the electron in the nucleus? You can't interpret this result as "the fraction of the time the electron spends in the nucleus". Nothing in quantum mechanics tells us that the electron spends time definitely anywhere. One should simply stop with what quantum mechanics gives: the result is the probability of finding the electron in nucleus.

13. The fundamental angular momentum commutation relation and a key corollary are, respectively:

a) $[J_i, J_j] = 0$ and $[J^2, J_i] = J_i$. b) $[J_i, J_j] = J_k$ and $[J^2, J_i] = 0$.
c) $[J_i, J_j] = 0$ and $[J^2, J_i] = 0$. d) $[J_i, J_j] = i\hbar\varepsilon_{ijk} J_k$ and $[J^2, J_i] = 0$.
e) $[x_i, p_j] = i\hbar\delta_{ij}$, $[x_i, x_j] = 0$, and $[p_i, p_j] = 0$.

14. In the vector model for angular momentum of a quantum system with the standard axis for the eigenstates being the z axis, the particles in the eigenstates are thought of as having definite z-components of angular momentum $m_j\hbar$ and definite total angular momenta of magnitude $\sqrt{j(j+1)}\hbar$, where j can stand for orbital, spin, or total angular momentum quantum number and m_j is the z-component quantum number. Recall j can be only be integer or half-integer and there are $2j+1$ possible values of m_j given by $-j, -j+1, \ldots, j-1, j$. The x-y component of the angular momementum has magnitude $\sqrt{j(j+1) - m_j^2}\hbar$, but it has no definite direction. Rather this component can be thought of as pointing all x-y directions in simultaneous: i.e., it is in a superposition state of all direction states. Diagramatically, the momentum vectors can be represented by

a) cones with axis aligned with the x-axis. b) cones with axis aligned with the y-axis.
c) cones with axis aligned with the z-axis. d) cones with axis aligned with the x-y-axis.
e) the cones of silence.

15. Does the fundamental commutation relation for angular momentum operators (i.e., $[J_i, J_j] = i\hbar\varepsilon_{ijk} J_k$) apply to angular momentum operators formed by summation from angular momentum operators applying to individual particles or to spatial and spin degrees of freedom? The answer is:

a) Yes. b) No. c) Maybe. d) All of the above. e) None of the above.

16. "Let's play *Jeopardy*! For \$100, the answer is: The name for the coefficients used in the expansion of a total angular momentum state for 2 angular momentum degrees of freedom in terms of products of individual angular momemtum states."

What are the ____________, Alex?

a) Racah W coefficients b) Wigner $6j$ symbols c) Buck-Rogers coefficients
d) Flash-Gordon coefficients e) Clebsch-Gordan coefficients

17. "Let's play *Jeopardy*! For \$100, the answer is: In constructing a set of $|j_1 j_2 j m\rangle$ states from a set of $|j_1 j_2 m_1 m_2\rangle$ states using Clebsch-Gordan coefficients, this is a strict constraint on the non-zero coefficients."

What is the rule ____________, Alex?

a) of complete overtures b) of incomplete overtures c) $m = m_1^2 + m_2^2$ d) $m = m_1 - m_2$
e) $m = m_1 + m_2$

18. Prove the following angular momentum operator identities. **HINT:** Recall the fundamental angular momentum commutator identity,

$$[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k \ , \qquad \text{and the definition} \qquad J_\pm \equiv J_x \pm iJ_y \ .$$

a) $[J_i, J^2] = 0$.
b) $[J^2, J_\pm] = 0$.
c) $[J_z, J_\pm] = \pm\hbar J_\pm$.
d) $J_\pm^\dagger J_\pm = J_\mp J_\pm = J^2 - J_z(J_z \pm \hbar)$.
e)

$$J_x = \frac{1}{2}(J_+ + J_-) \qquad \text{and} \qquad J_y = \frac{1}{2i}(J_+ - J_-) \ .$$

f) $[J_+, J_-] = 2\hbar J_z$.
g)

$$J^2_{\left\{\substack{x \\ y}\right\}} = \pm\frac{1}{4}\left(J_+^2 + J_-^2 \pm \{J_+, J_-\}\right) \ ,$$

where the upper case is for J_x^2 and the lower case for J_y^2 and where recall that $\{A, B\} = AB + BA$ is the anticommutator of A and B.

h)

$$J^2 = \frac{1}{2}\{J_+, J_-\} + J_z^2 \ .$$

19. The x-component angular momentum operator matrix in a three-dimensional angular momentum space expressed in terms of the z-component orthonormal basis (i.e., the standard basis with eigenvectors $|1\rangle$, $|0\rangle$, and $|-1\rangle$) is:

$$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(Co-659) and yes the $1/\sqrt{2}$ factor is correct. Is this matrix Hermitian? Diagonalize this matrix: i.e., solve for its eigenvalues and normalized eigenvectors (written in terms of the standard basis ket eigenvectors) or, if you prefer in column vector form. Note the solution is somewhat simpler if you solve the reduced eigen problem. Just divide both sides of the eigen equation by $\hbar/\sqrt{2}$ and solve for the reduced eigenvalues. The physical eigenvalues are the reduced ones times $\hbar/\sqrt{2}$. Verify that the eigenvectors are orthonormal.

NOTE: Albeit some consider it a sloppy notation since kets and bras are abstract vectors and columns vectors are from a concrete representation, its concretely useful to equate them at times. In the present case, the kets equate like so

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ , \qquad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ , \qquad |-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ ,$$

and the bras, like so

$$\langle 1| = (1, 0, 0)^* \ , \qquad \langle 0| = (0, 1, 0)^* \ , \qquad \langle -1| = (0, 0, 1)^* \ .$$

20. "Let's play *Jeopardy*! For \$100, the answer is: It is the intrinsic angular momentum of a fundamental (or fundamental-for-most-purposes) particle. It is invariant and its quantum number s is always an integer or half-integer.

What is ____________, Alex?

a) rotation b) quantum number c) magnetic moment d) orbital angular momentum
e) spin

21. "Let's play *Jeopardy*! For \$100, the answer is: Goudsmit and Ulhenbeck."

    a) Who are the original proposers of electron spin in 1925, Alex?
    b) Who performed the Stern-Gerlach experiment, Alex?
    c) Who are Wolfgang Pauli's evil triplet brothers, Alex?
    d) What are two delightful Dutch cheeses, Alex?
    e) What were Rosencrantz and Gildenstern's first names, Alex?

22. A spin s particle's angular momentum vector magnitude (in the vector model picture) is

    a) $\sqrt{s(s+1)}\hbar$. b) $s\hbar$ c) $\sqrt{s(s-1)}\hbar$ d) $-s\hbar$ e) $s(s+1)\hbar^2$

23. The eigenvalues of a **COMPONENT** of the spin of a spin 1/2 particle are always:

    a) $\pm\hbar$. b) $\pm\frac{\hbar}{3}$. c) $\pm\frac{\hbar}{4}$. d) $\pm\frac{\hbar}{5}$. e) $\pm\frac{\hbar}{2}$.

24. The quantum numbers for the component of the spin of a spin s particle are always:

    a) ± 1. b) $s, s-1, s-2, \ldots, -s+1, -s$. c) $\pm\frac{1}{2}$. d) ± 2. e) $\pm\frac{1}{4}$.

25. Is the spin (not spin component) of an electron dependent on the electron's environment?

    a) Always.
    b) No. Spin is an intrinsic, unchanging property of a particle.
    c) In atomic systems, no, but when free, yes.
    d) Both yes and no.
    e) It depends on a recount in Palm Beach.

26. "Let's play *Jeopardy*! For \$100, the answer is:

$$[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k \text{ .''}$$

    What is ____________, Alex?

    a) the spin anticommutator relation b) an implicit equation for ε_{ijk} c) an impostulate
    d) an inobservable e) the fundamental spin commutation relation

27. "Let's play *Jeopardy*! For \$100, the answer is:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ .''}$$

    What are ____________, Alex?

    a) dimensioned spin 1/2 matrices b) the Pauli spin matrices
    c) the Pauli principle matrices d) non-Hermitian matrices
    e) matrix-look-alikes, not matrices

28. The expression

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{1}$$

    is

    a) the Pauli spin matrix anticommutator relation. b) the Pauli spin matrix commutator relation.
    c) the fundamental spin commutator relation. d) the covariance of two standard deviations.
    e) an oddish relation.

29. A spatial operator and a spin operator commute:

    a) never. b) sometimes. c) always. d) always and never. e) to the office.

30. "Let's play *Jeopardy*! For \$100, the answer is:

$$\vec{\mu} = g\frac{q}{2m}\vec{J} , \qquad \vec{F} = \nabla(\vec{\mu}\cdot\vec{B}) , \qquad \vec{\tau} = \vec{\mu}\times\vec{B} , \qquad H = -\vec{\mu}\cdot\vec{B} \text{ .''}$$

a) What are Maxwell's equations, Alex?
b) What are incorrect formulae, Alex?
c) What are classical formulae sans any quantum mechanical analogs, Alex?
d) What are quantum mechanical formulae sans any classical analogs, Alex?
e) What are formulae needed to treat the interaction of angular momentum of a particle and magnetic field in classical and quantum mechanics, Alex?

31. What is

$$\mu_{\rm B} = \frac{e\hbar}{2m_e} = 9.27400915(23) \times 10^{-24}\,{\rm J/T} = 5.7883817555(79) \times 10^{-5}\,{\rm eV/T}\ ?$$

a) The nuclear magneton, the characteristic magnetic moment of nuclear systems.
b) The Bohr magneton, the characteristic magnetic moment of electronic systems.
c) The intrinsic magnetic dipole moment of an electron.
d) The coefficient of sliding friction.
e) The zero-point energy of an electron.

32. The g factor in quantum mechanics is the dimensionless factor for some system that multiplied by the appropriate magneton (e.g., Bohr magneton for electron systems) times the angular momentum of the system divided by $\hbar$ gives the magnetic moment of the system. Sometimes the sign of the magnetic moment is included in the g factor and sometimes it is just shown explicitly. The modern way seems to be to include it, but yours truly finds that awkward and so for now yours truly doesn't do it. For the electron, the intrinsic magnetic moment operator associated with intrinsic spin is given by

$$\vec{\mu}_{\rm op} = -g\mu_{\rm B}\frac{\vec{S}_{\rm op}}{\hbar}\ ,$$

where $\mu_{\rm B}$ is the Bohr magneton and $S_{\rm op}$ is the spin vector operator. What is g for the intrinsic magnetic moment operator of an electron to modern accuracy?

a) 1. b) 2. c) 2.0023193043622(15). d) 1/137. e) 137.

33. An object in a uniform magnetic field with magnetic moment due to the object's angular momentum will both classically and quantum mechanically:

a) Lancy progress. b) Lorenzo regress. c) London recess. d) Larmor precess.
e) Lamermoor transgress.

34. The Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ , \qquad \text{and} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ .$$

a) Are the Pauli matrices Hermitian?

b) What is the result when Pauli matrices act on general vector

$$\begin{pmatrix} a \\ b \end{pmatrix}\ ?$$

c) Diagonalize the Pauli matrices: i.e., solve for their eigenvalues and **NORMALIZED** eigenvectors. **NOTE:** The verb 'diagonalize' takes its name from the fact that a matrix transformed to the representation of its own eigenvectors is diagonal with the eigenvalues being the diagonal elements. One often doesn't actually write the diagonal matrix explicitly.

d) Prove that

$$\sigma_i\sigma_j = \delta_{ij}\mathbf{1} + i\varepsilon_{ijk}\sigma_k\ ,$$

where i, j, and k stand for any of x, y, and z, $\mathbf{1}$ is the unit matrix (which can often be left as understood), δ_{ij} is the Kronecker delta, ε_{ijk} is the Levi-Civita symbol, and Einstein summation is used. **HINT:** I rather think by exhaustion is the only way: i.e., extreme tiredness.

e) Prove

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k \qquad \text{and} \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\ ,$$

where $\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i$ is the anticommutator of Pauli matrices. **HINT:** You should make use of the part (d) expression.

f) Show that a general 2×2 matrix can be expanded in the Pauli spin matrices plus the unit matrix: i.e.,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha\mathbf{1} + \vec{\beta}\cdot\vec{\sigma} \ ,$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of the Pauli matrices. **HINT:** Find expressions for the expansion coefficients α, β_x, β_y, and β_z.

g) Let $\vec{A}$ and $\vec{B}$ be vectors of operators in general and let the components of $\vec{B}$ commute with the Pauli matrices. Prove

$$(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) = \vec{A}\cdot\vec{B} + i(\vec{A}\times\vec{B})\cdot\vec{\sigma} \ .$$

HINT: Make use of the part (d) expression.

35. What is the Hamiltonian fragment (piece, part) that describes the energy of an electron spin magnetic moment in a magnetic field? This fragment in a Schrödinger equation can sometimes be separated from the rest of the equation and solved as separate eigenvalue problem. Solve this separated problem. The intrinsic angular momentum operator is $\vec{S}$ and assume the magnetic field points in the z direction. **HINTS:** Think of the classical energy of a magnetic dipole in a magnetic field and use the correspondence principle. This is not a long question.

36. Let's tackle the classical Larmor precession.

a) What is Newton's 2nd law in rotational form?

b) What is the torque on a magnetic dipole moment $\vec{\mu}$ in a magnetic field $\vec{B}$? **HINT:** Any first-year text will tell you.

c) Say that the magnetic moment of a system is given by $\vec{\mu} = \gamma\vec{L}$, where γ is the gyromagnetic ratio and $\vec{L}$ is the system's angular momemtum. Say also that there is a magnetic field $\vec{B} = (0, 0, B_z)$. Solve for the time evolution of $\vec{L}$ using Newton's 2nd law in rotational form assuming the **INITIAL CONDITION** $\vec{L}(t = 0) = (L_{x,0}, 0, L_{z,0})$. **HINTS:** You should get coupled differential equations for two components of $\vec{L}$. They are not so hard to solve. For niceness you should define an appropriate Larmor frequency ω.

Appendix 2 Quantum Mechanics Equation Sheet

Note: This equation sheet is intended for students writing tests or reviewing material. Therefore it neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things.

1 Constants not to High Accuracy

Constant Name	Symbol	Derived from CODATA 1998
Bohr radius	$a_{\text{Bohr}} = \dfrac{\lambda_{\text{Compton}}}{2\pi\alpha}$	$= 0.529\,\text{Å}$
Boltzmann's constant	k	$= 0.8617 \times 10^{-6}\,\text{eV}\,\text{K}^{-1}$ $= 1.381 \times 10^{-16}\,\text{erg}\,\text{K}^{-1}$
Compton wavelength	$\lambda_{\text{Compton}} = \dfrac{h}{m_e c}$	$= 0.0246\,\text{Å}$
Electron rest energy	$m_e c^2$	$= 5.11 \times 10^5\,\text{eV}$
Elementary charge squared	e^2	$= 14.40\,\text{eV}\,\text{Å}$
Fine Structure constant	$\alpha = \dfrac{e^2}{\hbar c}$	$= 1/137.036$
Kinetic energy coefficient	$\dfrac{\hbar^2}{2m_e}$	$= 3.81\,\text{eV}\,\text{Å}^2$
	$\dfrac{\hbar^2}{m_e}$	$= 7.62\,\text{eV}\,\text{Å}^2$
Planck's constant	h	$= 4.15 \times 10^{-15}\,\text{eV}$
Planck's h-bar	$\hbar$	$= 6.58 \times 10^{-16}\,\text{eV}$
	hc	$= 12398.42\,\text{eV}\,\text{Å}$
	$\hbar c$	$= 1973.27\,\text{eV}\,\text{Å}$
Rydberg Energy	$E_{\text{Ryd}} = \dfrac{1}{2} m_e c^2 \alpha^2$	$= 13.606\,\text{eV}$

2 Some Useful Formulae

Leibniz's formula
$$\frac{d^n(fg)}{dx^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}}$$

Normalized Gaussian
$$P = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x - \langle x \rangle)^2}{2\sigma^2}\right]$$

3 Schrödinger's Equation

$$H\Psi(x,t) = \left[\frac{p^2}{2m} + V(x)\right]\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

$$H\psi(x) = \left[\frac{p^2}{2m} + V(x)\right]\psi(x) = E\psi(x)$$

$$H\Psi(\vec{r},t) = \left[\frac{p^2}{2m} + V(\vec{r})\right]\Psi(\vec{r},t) = i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} \qquad H|\Psi\rangle = i\hbar\frac{\partial}{\partial t}|\Psi\rangle$$

$$H\psi(\vec{r}) = \left[\frac{p^2}{2m} + V(\vec{r})\right]\psi(\vec{r}) = E\psi(\vec{r}) \qquad H|\psi\rangle = E|\psi\rangle$$

4 Some Operators

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x} \qquad p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$$

$$H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

$$p = \frac{\hbar}{i}\nabla \qquad p^2 = -\hbar^2 \nabla^2$$

$$H = \frac{p^2}{2m} + V(\vec{r}\,) = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}\,)$$

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\theta}\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

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5 Kronecker Delta and Levi-Civita Symbol

$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & \text{otherwise} \end{cases} \qquad \varepsilon_{ijk} = \begin{cases} 1, & ijk \text{ cyclic;} \\ -1, & ijk \text{ anticyclic;} \\ 0, & \text{if two indices the same.} \end{cases}$$

$$\varepsilon_{ijk}\varepsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell} \qquad \text{(Einstein summation on } i\text{)}$$

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6 Time Evolution Formulae

$$\text{General} \qquad \frac{d\langle A\rangle}{dt} = \left\langle \frac{\partial A}{\partial t}\right\rangle + \frac{1}{\hbar}\langle i[H(t), A]\rangle$$

$$\text{Ehrenfest's Theorem} \qquad \frac{d\langle \vec{r}\,\rangle}{dt} = \frac{1}{m}\langle \vec{p}\,\rangle \qquad \text{and} \qquad \frac{d\langle \vec{p}\,\rangle}{dt} = -\langle \nabla V(\vec{r}\,)\rangle$$

$$|\Psi(t)\rangle = \sum_j c_j(0)e^{-iE_j t/\hbar}|\phi_j\rangle$$

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7 Simple Harmonic Oscillator (SHO) Formulae

$$V(x) = \frac{1}{2}m\omega^2 x^2 \qquad \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\right)\psi = E\psi$$

$$\beta = \sqrt{\frac{m\omega}{\hbar}} \qquad \psi_n(x) = \frac{\beta^{1/2}}{\pi^{1/4}}\frac{1}{\sqrt{2^n n!}}H_n(\beta x)e^{-\beta^2 x^2/2} \qquad E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

$$H_0(\beta x) = H_0(\xi) = 1 \qquad H_1(\beta x) = H_1(\xi) = 2\xi$$

$$H_2(\beta x) = H_2(\xi) = 4\xi^2 - 2 \qquad H_3(\beta x) = H_3(\xi) = 8\xi^3 - 12\xi$$

8 Position, Momentum, and Wavenumber Representations

$$p = \hbar k \qquad E_{\text{kinetic}} = E_T = \frac{\hbar^2 k^2}{2m}$$

$$|\Psi(p,t)|^2\,dp = |\Psi(k,t)|^2\,dk \qquad \Psi(p,t) = \frac{\Psi(k,t)}{\sqrt{\hbar}}$$

$$x_{\text{op}} = x \qquad p_{\text{op}} = \frac{\hbar}{i}\frac{\partial}{\partial x} \qquad Q\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}, t\right) \qquad \text{position representation}$$

$$x_{\text{op}} = -\frac{\hbar}{i}\frac{\partial}{\partial p} \qquad p_{\text{op}} = p \qquad Q\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}, p, t\right) \qquad \text{momentum representation}$$

$$\delta(x) = \int_{-\infty}^{\infty} \frac{e^{ipx/\hbar}}{2\pi\hbar}\,dp \qquad \delta(x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{2\pi}\,dk$$

$$\Psi(x,t) = \int_{-\infty}^{\infty} \Psi(p,t)\frac{e^{ipx/\hbar}}{(2\pi\hbar)^{1/2}}\,dp \qquad \Psi(x,t) = \int_{-\infty}^{\infty} \Psi(k,t)\frac{e^{ikx}}{(2\pi)^{1/2}}\,dk$$

$$\Psi(p,t) = \int_{-\infty}^{\infty} \Psi(x,t)\frac{e^{-ipx/\hbar}}{(2\pi\hbar)^{1/2}}\,dx \qquad \Psi(k,t) = \int_{-\infty}^{\infty} \Psi(x,t)\frac{e^{-ikx}}{(2\pi)^{1/2}}\,dx$$

$$\Psi(\vec{r},t) = \int_{\text{all space}} \Psi(\vec{p},t)\frac{e^{i\vec{p}\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}}\,d^3p \qquad \Psi(\vec{r},t) = \int_{\text{all space}} \Psi(\vec{k},t)\frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}}\,d^3k$$

$$\Psi(\vec{p},t) = \int_{\text{all space}} \Psi(\vec{r},t)\frac{e^{-i\vec{p}\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}}\,d^3r \qquad \Psi(\vec{k},t) = \int_{\text{all space}} \Psi(\vec{r},t)\frac{e^{-i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}}\,d^3r$$

9 Commutator Formulae

$$[A, BC] = [A,B]C + B[A,C] \qquad \left[\sum_i a_i A_i, \sum_j b_j B_j\right] = \sum_{i,j} a_i b_j [A_i, b_j]$$

$$\text{if} \qquad [B,[A,B]] = 0 \qquad \text{then} \qquad [A, F(B)] = [A,B]F'(B)$$

$$[x,p] = i\hbar \qquad [x, f(p)] = i\hbar f'(p) \qquad [p, g(x)] = -i\hbar g'(x)$$

$$[a, a^\dagger] = 1 \qquad [N, a] = -a \qquad [N, a^\dagger] = a^\dagger$$

10 Uncertainty Relations and Inequalities

$$\sigma_x\sigma_p = \Delta x\Delta p \geq \frac{\hbar}{2} \qquad \sigma_Q\sigma_Q = \Delta Q\Delta R \geq \frac{1}{2}\left|\langle i[Q,R]\rangle\right|$$

$$\sigma_H\Delta t_{\text{scale time}} = \Delta E\Delta t_{\text{scale time}} \geq \frac{\hbar}{2}$$

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11 Probability Amplitudes and Probabilities

$$\Psi(x,t) = \langle x|\Psi(t)\rangle \qquad P(dx) = |\Psi(x,t)|^2\,dx \qquad c_i(t) = \langle\phi_i|\Psi(t)\rangle \qquad P(i) = |c_i(t)|^2$$

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12 Spherical Harmonics

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}} \qquad Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2}\cos(\theta) \qquad Y_{1,\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin(\theta)e^{\pm i\phi}$$

$$L^2Y_{\ell m} = \ell(\ell+1)\hbar^2Y_{\ell m} \qquad L_zY_{\ell m} = m\hbar Y_{\ell m} \qquad |m| \leq \ell \qquad m = -\ell, -\ell+1, \ldots, \ell-1, \ell$$

0	1	2	3	4	5	6	...
s	p	d	f	g	h	i	...

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13 Hydrogenic Atom

$$\psi_{n\ell m} = R_{n\ell}(r)Y_{\ell m}(\theta,\phi) \qquad \ell \leq n-1 \qquad \ell = 0, 1, 2, \ldots, n-1$$

$$a_z = \frac{a}{Z}\left(\frac{m_e}{m_{\text{reduced}}}\right) \qquad a_0 = \frac{\hbar}{m_e c\alpha} = \frac{\lambda_{\text{C}}}{2\pi\alpha} \qquad \alpha = \frac{e^2}{\hbar c}$$

$$R_{10} = 2a_Z^{-3/2}e^{-r/a_Z} \qquad R_{20} = \frac{1}{\sqrt{2}}a_Z^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a_Z}\right)e^{-r/(2a_Z)}$$

$$R_{21} = \frac{1}{\sqrt{24}}a_Z^{-3/2}\frac{r}{a_Z}e^{-r/(2a_Z)}$$

$$R_{n\ell} = -\left\{\left(\frac{2}{na_Z}\right)^3\frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}\right\}^{1/2}e^{-\rho/2}\rho^\ell L_{n+\ell}^{2\ell+1}(\rho) \qquad \rho = \frac{2r}{nr_Z}$$

$$L_q(x) = e^x\left(\frac{d}{dx}\right)^q\left(e^{-x}x^q\right) \quad \text{Rodrigues's formula for the Laguerre polynomials}$$

$$L_q^j(x) = \left(\frac{d}{dx}\right)^jL_q(x) \qquad \text{Associated Laguerre polynomials}$$

$$\langle r\rangle_{n\ell m} = \frac{a_Z}{2}\left[3n^2 - \ell(\ell+1)\right]$$

$$\text{Nodes} = (n-1) - \ell \qquad \text{not counting zero or infinity}$$

$$E_n = -\frac{1}{2} m_e c^2 \alpha^2 \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} = -E_{\text{Ryd}} \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} = -13.606 \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} \text{ eV}$$

14 **General Angular Momentum Formulae**

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k \qquad \text{(Einstein summation on } k\text{)} \qquad [J^2, \vec{J}] = 0$$

$$J^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle \qquad J_z |jm\rangle = m\hbar |jm\rangle$$

$$J_\pm = J_x \pm i J_y \qquad J_\pm |jm\rangle = \hbar \sqrt{j(j+1) - m(m\pm 1)} |jm \pm 1\rangle$$

$$J_{\left\{ \substack{x \\ y} \right\}} = \left\{ \begin{matrix} \frac{1}{2} \\ \frac{1}{2i} \end{matrix} \right\} (J_+ \pm J_-) \qquad J_\pm^\dagger J_\pm = J_\mp J_\pm = J^2 - J_z(J_z \pm \hbar)$$

$$[J_{fi}, J_{gj}] = \delta_{fg} i\hbar \varepsilon_{ijk} J_k \qquad \vec{J} = \vec{J}_1 + \vec{J}_2 \qquad J^2 = J_1^2 + J_2^2 + J_{1+} J_{2-} + J_{1-} J_{2+} + 2 J_{1z} J_{2z}$$

$$J_\pm = J_{1\pm} + J_{2\pm} \qquad |j_1 j_2 jm\rangle = \sum_{m_1 m_2, m = m_1 + m_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 jm\rangle j_1 j_2 jm\rangle$$

$$|j_1 - j_2| \leq j \leq j_1 + j_2 \qquad \sum_{|j_1 - j_2|}^{j_1 + j_2} (2j+1) = (2j_1 + 1)(2j_2 + 1)$$

15 **Spin 1/2 Formulae**

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\pm\rangle_x = \frac{1}{\sqrt{2}} (|+\rangle \pm |-\rangle) \qquad |\pm\rangle_y = \frac{1}{\sqrt{2}} (|+\rangle \pm i|-\rangle) \qquad |\pm\rangle_z = |\pm\rangle$$

$$|++\rangle = |1,+\rangle|2,+\rangle \qquad |+-\rangle = \frac{1}{\sqrt{2}} (|1,+\rangle|2,-\rangle \pm |1,-\rangle|2,+\rangle) \qquad |--\rangle = |1,-\rangle|2,-\rangle$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ijk} \sigma_k \qquad [\sigma_i, \sigma_j] = 2i\varepsilon_{ijk} \sigma_k \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

$$(\vec{A} \cdot \vec{\sigma})(\vec{B} \cdot \vec{\sigma}) = \vec{A} \cdot \vec{B} + i(\vec{A} \times \vec{B}) \cdot \vec{\sigma}$$

$$\frac{d(\vec{S} \cdot \hat{n})}{d\alpha} = -\frac{i}{\hbar} [\vec{S} \cdot \hat{\alpha}, \vec{S} \cdot \hat{n}] \qquad \vec{S} \cdot \hat{n} = e^{-i\vec{S} \cdot \vec{\alpha}} \vec{S} \cdot \hat{n}_0 e^{i\vec{S} \cdot \vec{\alpha}} \qquad |\hat{n}_\pm\rangle = e^{-i\vec{S} \cdot \vec{\alpha}} |\hat{z}_\pm\rangle$$

$$e^{ixA} = \mathbf{1}\cos(x) + iA\sin(x) \quad \text{if } A^2 = \mathbf{1} \qquad\qquad e^{-i\vec{\sigma}\cdot\vec{\alpha}/2} = \mathbf{1}\cos(x) - i\vec{\sigma}\cdot\hat{\alpha}\sin(x)$$

$$\sigma_i f(\sigma_j) = f(\sigma_j)\sigma_i\delta_{ij} + f(-\sigma_j)\sigma_i(1-\delta_{ij})$$

$$\mu_{\text{Bohr}} = \frac{e\hbar}{2m} = 0.927400915(23)\times 10^{-24}\,\text{J/T} = 5.7883817555(79)\times 10^{-5}\,\text{eV/T}$$

$$g = 2\left(1 + \frac{\alpha}{2\pi} + \ldots\right) = 2.0023193043622(15)$$

$$\vec{\mu}_{\text{orbital}} = -\mu_{\text{Bohr}}\frac{\vec{L}}{\hbar} \qquad \vec{\mu}_{\text{spin}} = -g\mu_{\text{Bohr}}\frac{\vec{S}}{\hbar} \qquad \vec{\mu}_{\text{total}} = \vec{\mu}_{\text{orbital}} + \vec{\mu}_{\text{spin}} = -\mu_{\text{Bohr}}\frac{(\vec{L}+g\vec{S})}{\hbar}$$

$$H_\mu = -\vec{\mu}\cdot\vec{B} \qquad H_\mu = \mu_{\text{Bohr}}B_z\frac{(L_z + gS_z)}{\hbar}$$

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16 Time-Independent Approximation Methods

$$H = H^{(0)} + \lambda H^{(1)} \qquad |\psi\rangle = N(\lambda)\sum_{k=0}^{\infty}\lambda^k|\psi_n^{(k)}\rangle$$

$$H^{(1)}|\psi_n^{(m-1)}\rangle(1-\delta_{m,0}) + H^{(0)}|\psi_n^{(m)}\rangle = \sum_{\ell=0}^{m}E^{(m-\ell)}|\psi_n^{(\ell)}\rangle \qquad |\psi_n^{(\ell>0)}\rangle = \sum_{m=0,\ m\neq n}^{\infty}a_{nm}|\psi_n^{(0)}\rangle$$

$$|\psi_n^{\text{1st}}\rangle = |\psi_n^{(0)}\rangle + \lambda\sum_{\text{all }k,\ k\neq n}\frac{\left\langle\psi_k^{(0)}|H^{(1)}|\psi_n^{(0)}\right\rangle}{E_n^{(0)} - E_k^{(0)}}|\psi_k^{(0)}\rangle$$

$$E_n^{\text{1st}} = E_n^{(0)} + \lambda\left\langle\psi_n^{(0)}|H^{(1)}|\psi_n^{(0)}\right\rangle$$

$$E_n^{\text{2nd}} = E_n^{(0)} + \lambda\left\langle\psi_n^{(0)}|H^{(1)}|\psi_n^{(0)}\right\rangle + \lambda^2\sum_{\text{all }k,\ k\neq n}\frac{\left|\left\langle\psi_k^{(0)}|H^{(1)}|\psi_n^{(0)}\right\rangle\right|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$E(\phi) = \frac{\langle\phi|H|\phi\rangle}{\langle\phi|\phi\rangle} \qquad \delta E(\phi) = 0$$

$$H_{kj} = \langle\phi_k|H|\phi_j\rangle \qquad H\vec{c} = E\vec{c}$$

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17 Time-Dependent Perturbation Theory

$$\pi = \int_{-\infty}^{\infty}\frac{\sin^2(x)}{x^2}\,dx$$

$$\Gamma_{0\to n} = \frac{2\pi}{\hbar}|\langle n|H_{\text{perturbation}}|0\rangle|^2\delta(E_n - E_0)$$

18 Interaction of Radiation and Matter

$$\vec{E}_{\text{op}} = -\frac{1}{c}\frac{\partial \vec{A}_{\text{op}}}{\partial t} \qquad \vec{B}_{\text{op}} = \nabla \times \vec{A}_{\text{op}}$$

19 Box Quantization

$$kL = 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots \qquad k = \frac{2\pi n}{L} \qquad \Delta k_{\text{cell}} = \frac{2\pi}{L} \qquad \Delta k_{\text{cell}}^3 = \frac{(2\pi)^3}{V}$$

$$dN_{\text{states}} = g\frac{k^2\,dk\,d\Omega}{(2\pi)^3/V}$$

20 Identical Particles

$$|a,b\rangle = \frac{1}{\sqrt{2}}\left(|1,a;2,b\rangle \pm |1,b;2,a\rangle\right)$$

$$\psi(\vec{r}_1,\vec{r}_2) = \frac{1}{\sqrt{2}}\left(\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)\right)$$

21 Second Quantization

$$[a_i, a_j^\dagger] = \delta_{ij} \qquad [a_i, a_j] = 0 \qquad [a_i^\dagger, a_j^\dagger] = 0 \qquad |N_1, \ldots, N_n\rangle = \frac{(a_n^\dagger)^{N_n}}{\sqrt{N_n!}} \cdots \frac{(a_1^\dagger)^{N_1}}{\sqrt{N_1!}}|0\rangle$$

$$\{a_i, a_j^\dagger\} = \delta_{ij} \qquad \{a_i, a_j\} = 0 \qquad \{a_i^\dagger, a_j^\dagger\} = 0 \qquad |N_1, \ldots, N_n\rangle = (a_n^\dagger)^{N_n} \cdots (a_1^\dagger)^{N_1}|0\rangle$$

$$\Psi_s(\vec{r}\,)^\dagger = \sum_{\vec{p}} \frac{e^{-i\vec{p}\cdot\vec{r}}}{\sqrt{V}} a_{\vec{p}s}^\dagger \qquad \Psi_s(\vec{r}\,) = \sum_{\vec{p}} \frac{e^{i\vec{p}\cdot\vec{r}}}{\sqrt{V}} a_{\vec{p}s}$$

$$[\Psi_s(\vec{r}\,), \Psi_{s'}(\vec{r}\,')]_\mp = 0 \qquad [\Psi_s(\vec{r}\,)^\dagger, \Psi_{s'}(\vec{r}\,')^\dagger]_\mp = 0 \qquad [\Psi_s(\vec{r}\,), \Psi_{s'}(\vec{r}\,')^\dagger]_\mp = \delta(\vec{r} - \vec{r}\,')\delta_{ss'}$$

$$|\vec{r}_1 s_1, \ldots, \vec{r}_n s_n\rangle = \frac{1}{\sqrt{n!}}\Psi_{s_n}(\vec{r}_n)^\dagger \ldots \Psi_{s_n}(\vec{r}_n)^\dagger|0\rangle$$

$$\Psi_s(\vec{r}\,)^\dagger|\vec{r}_1 s_1, \ldots, \vec{r}_n s_n\rangle\sqrt{n+1}|\vec{r}_1 s_1, \ldots, \vec{r}_n s_n, \vec{r} s\rangle$$

$$|\Phi\rangle = \int d\vec{r}_1 \ldots d\vec{r}_n\, \Phi(\vec{r}_1, \ldots, \vec{r}_n)|\vec{r}_1 s_1, \ldots, \vec{r}_n s_n\rangle$$

$$1_n = \sum_{s_1 \ldots s_n} \int d\vec{r}_1 \ldots d\vec{r}_n\, |\vec{r}_1 s_1, \ldots, \vec{r}_n s_n\rangle\langle\vec{r}_1 s_1, \ldots, \vec{r}_n s_n| \qquad 1 = |0\rangle\langle 0| + \sum_{n=1}^{\infty} 1_n$$

$$N = \sum_{\vec{p}s} a^\dagger_{\vec{p}s} a_{\vec{p}s} \qquad T = \sum_{\vec{p}s} \frac{p^2}{2m} a^\dagger_{\vec{p}s} a_{\vec{p}s}$$

$$\rho_s(\vec{r}\,) = \Psi_s(\vec{r}\,)^\dagger \Psi_s(\vec{r}\,) \qquad N = \sum_s \int d\vec{r}\, \rho_s(\vec{r}\,) \qquad T = \frac{1}{2m} \sum_s \int d\vec{r}\, \nabla\Psi_s(\vec{r}\,)^\dagger \cdot \nabla\Psi_s(\vec{r}\,)$$

$$\vec{j}_s(\vec{r}\,) = \frac{1}{2im} \left[\Psi_s(\vec{r}\,)^\dagger \nabla\Psi_s(\vec{r}\,) - \Psi_s(\vec{r}\,)\nabla\Psi_s(\vec{r}\,)^\dagger \right]$$

$$G_s(\vec{r} - \vec{r}\,') = \frac{3n}{2} \frac{\sin(x) - x\cos(x)}{x^3} \qquad g_{ss'}(\vec{r} - \vec{r}\,') = 1 - \delta_{ss'} \frac{G_s(\vec{r} - \vec{r}\,')^2}{(n/2)^2}$$

$$v_{\text{2nd}} = \frac{1}{2} \sum_{ss'} \int d\vec{r} d\vec{r}\,'\, v(\vec{r} - \vec{r}\,') \Psi_s(\vec{r}\,)^\dagger \Psi_{s'}(\vec{r}\,')^\dagger \Psi_{s'}(\vec{r}\,') \Psi_s(\vec{r}\,)$$

$$v_{\text{2nd}} = \frac{1}{2V} \sum_{pp'qq'} \sum_{ss'} v_{\vec{p}-\vec{p}\,'} \delta_{\vec{p}+\vec{q},\vec{p}'+\vec{q}'} a^\dagger_{\vec{p}s} a^\dagger_{\vec{q}s'} a_{\vec{q}\,'s'} a_{\vec{p}\,'s} \qquad v_{\vec{p}-\vec{p}\,'} = \int d\vec{r}\, e^{-i(\vec{p}-\vec{p}\,')\cdot\vec{r}} v(\vec{r}\,)$$

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22 Klein-Gordon Equation

$$E = \sqrt{p^2c^2 + m^2c^4} \qquad \frac{1}{c^2} \left(i\hbar \frac{\partial}{\partial t} \right)^2 \Psi(\vec{r}, t) = \left[\left(\frac{\hbar}{i} \nabla \right)^2 + m^2c^2 \right] \Psi(\vec{r}, t)$$

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar} \right)^2 \right] \Psi(\vec{r}, t) = 0$$

$$\rho = \frac{i\hbar}{2mc^2} \left(\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t} \right) \qquad \vec{j} = \frac{\hbar}{2im} \left(\Psi^* \nabla\Psi - \Psi \nabla\Psi^* \right)$$

$$\frac{1}{c^2} \left(i\hbar \frac{\partial}{\partial t} - e\Phi \right)^2 \Psi(\vec{r}, t) = \left[\left(\frac{\hbar}{i} \nabla - \frac{e}{c} \vec{A} \right)^2 + m^2c^2 \right] \Psi(\vec{r}, t)$$

$$\Psi_+(\vec{p}, E) = e^{i(\vec{p}\cdot\vec{r} - Et)/\hbar} \qquad \Psi_-(\vec{p}, E) = e^{-i(\vec{p}\cdot\vec{r} - Et)/\hbar}$$