Modern Physics: Physics 305, Section 1

NAME:

Homework 7: The Hydrogenic Atom: Homeworks are due as posted on the course web site. They are **NOT** handed in. The student reports that it is completed and receives one point for this. Solutions are already posted, but students are only permitted to look at the solutions after completion. The solutions are intended to be (but not necessarily are) super-perfect and go beyond a complete answer expected on a test.

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011 qmult 00100 1 4 3 easy deducto-memory: central-force

1. In a central-force problem, the magnitude of central force depends only on:

   a) the angle of the particle.
   b) the vector $\vec{r}$ from the center to the particle.
   c) the radial distance r from the center to the particle.
   d) the magnetic quantum number of the particle.
   e) the uncertainty principle.

   SUGGESTED ANSWER: (c)

   Wrong Answers:
   a) Nah.
   b) Exactly wrong.

   Redaction: Jeffery, 2001jan01

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011 qmult 00200 1 1 2 easy memory: separation of variables

2. The usual approach to getting the eigenfunctions of the Hamiltonian in multi-dimensions is:

   a) non-separation of variables. b) separation of variables. c) separation of invariables.
   d) non-separation of invariables. e) non-separation of variables/invariables.

   SUGGESTED ANSWER: (b) Yes separation of variables is the conventional name. See Ar-86.

   Wrong Answers:
   e) A nonsense answer

   Redaction: Jeffery, 2001jan01

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011 qmult 00210 1 1 3 easy memory: separation of variables

3. Say you have a differential equation of two independent variables x and y and you want to look for solutions that can be factorized thusly $f(x,y) = g(x)h(y)$. Say then it is possible to reorder equation into the form

   $$\mathrm{LHS}(x) = \mathrm{RHS}(y) ,$$

   where LHS stands for left-hand side and RHS for right-hand side. Well LHS is explicitly independent of y and implicitly independent of x:

   $$\frac{\partial \mathrm{LHS}}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial \mathrm{LHS}}{\partial x} = \frac{\partial \mathrm{RHS}}{\partial x} = 0 .$$

   Thus, LHS is equal to a constant C and necessarily RHS is equal to the same constant C which is called the constant of separation (e.g., Arf-383). The solutions for $g(x)$ and $h(y)$ can be found separately and are related to each other through C. The solutions for $f(x,y)$ that cannot be factorized are not obtained, of course, by the described procedured. However, if one obtains complete sets of $g(x)$ and $h(y)$ solutions for the x-y region of interest, then any solution $f(x,y)$ can be constructed at least to within some approximation (WA-510). Thus, the generalization of the described procedure is very general and powerful. It is called: then

   a) separation of the left- and right-hand sides. b) partitioning.
   c) separation of the variables. d) solution factorization. e) the King Lear method.

   SUGGESTED ANSWER: (c)

Wrong answers:

d) Seems reasonable.

e) Metaphorical names due turn up in physics like the Monte Carlo method (named after a famous casino in Monaco) and the Urca process (named after a casino in Rio de Janeiro. One sometimes gets the feeling that theoretical physicists spend a lot of time in casinos.

Redaction: Jeffery, 2008jan01

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011 qmult 00300 1 4 2 easy deducto-memory: relative/cm reduction

4. "Let's play *Jeopardy*! For \$100, the answer is: By writing the two-body Schrödinger equation in relative/center-of-mass coordinates."

How do you ____________, Alex?

a) reduce a **ONE-BODY** problem to a **TWO-BODY** problem
b) reduce a **TWO-BODY** problem to a **ONE-BODY** problem
c) solve a one-dimensional infinite square well problem
d) solve for the simple harmonic oscillator eigenvalues
e) reduce a **TWO-BODY** problem to a **TWO-BODY** problem

SUGGESTED ANSWER: (b)

Wrong answers:

e) Seems a bit pointless.

Redaction: Jeffery, 2001jan01

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011 qmult 00310 1 4 4 easy deducto-memory: reduced mass

5. The formula for the reduced mass m for two-body system (with bodies labeled 1 and 2) is:

a) $m = m_1 m_2$. b) $m = \dfrac{1}{m_1 m_2}$. c) $m = \dfrac{m_1 + m_2}{m_1 m_2}$. d) $m = \dfrac{m_1 m_2}{m_1 + m_2}$. e) $m = \dfrac{1}{m_1}$.

SUGGESTED ANSWER: (d)

Wrong Answers:

a) Dimensionally wrong.
b) Dimensionally wrong.
c) Dimensionally wrong.
e) Dimensionally wrong and it only refers to one mass.

Redaction: Jeffery, 2001jan01

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011 qmult 00400 1 4 2 easy deducto memory: spherical harmonics 1

6. The eigensolutions of the angular part of the Hamiltonian for the central force problem are the:

a) linear harmonics. b) spherical harmonics. c) square harmonics.
d) Pythagorean harmonics. e) Galilean harmonics.

SUGGESTED ANSWER: (b)

Wrong Answers:

d) Legend has it that Pythagoras discovered the harmonic properties of strings.
e) Vincenzo Galileo, father of the other Galileo, was a scientist too and studied music scientifically.

Redaction: Jeffery, 2001jan01

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011 qmult 00420 1 4 3 easy deducto memory: spherical harmonic Y_{00}

7. Just about the only spherical harmonic that people remember—and they really should remember it too—is Y_{00} =:

a) $e^{im\phi}$. b) r^2. c) $\dfrac{1}{\sqrt{4\pi}}$. d) θ^2. e) $2a^{-3/2}e^{-r/a}$.

SUGGESTED ANSWER: (c)

Wrong Answers:

a) This is the general azimuthal component of the spherical harmonics: $m = 0, \pm 1, \pm 2, \ldots$
b) This is radial and it's not normalizable.
d) Except for Y_{00} itself, the spherical harmonics are all combinations of sinusoidal functions of the θ and ϕ.
e) This is the R_{10} hydrogenic radial wave function where a is the scale radius

$$a = a_0 \frac{m_e}{m} \frac{1}{Z} ,$$

where m_e is the electron mass, m is the reduced mass, Z is the number of unit charges of the central particle, and a_0 is the Bohr radius (Gr-137). The Bohr radius is given by

$$a_0 = \frac{\hbar^2}{m_e e^2} = \frac{\lambda_{\rm C}}{2\pi} \frac{}{\alpha} \approx 0.529 \text{ Å} ,$$

where e is the elementary charge in CGS-Gaussian units, $\lambda_C = \hbar/(m_e c)$ is the Compton wavelength, and $alpha \approx /137$ is the fine structure constant.

Redaction: Jeffery, 2001jan01

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011 qmult 00500 1 4 2 easy deducto-memory: spdf designations

8. Conventionally, the spherical harmonic eigenstates for angular momentum quantum numbers

$$\ell = 0, 1, 2, 3, 4, ...$$

are designated by:

a) a, b, c, d, e, etc.
b) s, p, d, f, and then alphabetically following f: i.e., g, h, etc.
c) x, y, z, xx, yy, zz, xxx, etc.
d) A, C, B, D, E, etc.
e) $@%&*!!

SUGGESTED ANSWER: (b)

Wrong Answers:

a) This is the way it should be, not the way it is.
e) Only in Tasmanian devilish.

Redaction: Jeffery, 2001jan01

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011 qfull 00100 2 5 0 moderate thinking: 2-body reduced to 1-body problem
Extra keywords: (Gr-178:5.1)

9. The 2-body time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m_1} \nabla_1^2 \psi - \frac{\hbar^2}{2m_2} \nabla_2^2 \psi + V\psi = E_{\rm total} \psi .$$

If the V depends only on $\vec{r} = \vec{r}_2 - r_1$ (the relative vector), then the problem can be separate into two problems: a relative problem 1-body equivalent problem and a center-of-mass 1-body equivalent problem. The center of mass vector is

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M} ,$$

where $M = m_1 + m_2$.

a) Determine the expressions for $\vec{r}_1$ and $\vec{r}_2$ in terms of $\vec{R}$ and $\vec{r}$.

b) Determine the expressions for ∇_1^2 and ∇_2^2 in terms of $\nabla_{\rm cm}^2$ (the center-of-mass Laplacian operator) and ∇^2 (the relative Laplacian operator). Then re-express kinetic operator

$$-\frac{\hbar^2}{2m_1}\nabla_1^2-\frac{\hbar^2}{2m_2}\nabla_2^2$$

in terms of $\nabla_{\rm cm}^2$ and ∇^2. **HINTS:** The x, y, and z direction components of vectors can all be treated separately and identically since x components of $\vec{R}$ and $\vec{r}$) (i.e., X and x) depend only on x_1 and x_2, etc. You can introduce a reduced mass to make the transformed kinetic energy operator simpler.

c) Now separate the 2-body Schrödinger equation assuming $V=V(\vec{r})$. What are the solutions of the center-of-mass problem? How would you interpret the solutions of the relative problem? **HINT:** I'm only looking for a short answer to the interpretation question.

SUGGESTED ANSWER:

a) Well substituting for $\vec{r}_2$ using the relative expression gives

$$\vec{R}=\frac{m_1\vec{r}_1+m_2\vec{r}_2}{M}=\vec{r}_1+\frac{m_2}{M}\vec{r}\ ,$$

and so

$$\vec{r}_1=\vec{R}-\frac{m_2}{M}\vec{r}$$

and

$$\vec{r}_2=\vec{R}+\frac{m_1}{M}\vec{r}\ .$$

b) Well

$$\frac{\partial}{\partial x_{\binom{1}{2}}}=\frac{\partial X}{\partial x_{\binom{1}{2}}}\frac{\partial}{\partial X}+\frac{\partial x}{\partial x_{\binom{1}{2}}}\frac{\partial}{\partial x}=\frac{m_{\binom{1}{2}}}{M}\frac{\partial}{\partial X}\mp\frac{\partial}{\partial x}\ .$$

Thus

$$\frac{\partial^2}{\partial x^2_{\binom{1}{2}}}=\left[\frac{m_{\binom{1}{2}}}{M}\right]^2\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial x^2}\mp 2\frac{m_{\binom{1}{2}}}{M}\frac{\partial}{\partial X}\frac{\partial}{\partial x}\ .$$

The other coordinate directions are treated identically mutatis mutandis. We then find that

$$-\frac{\hbar^2}{2m_1}\nabla_1^2-\frac{\hbar^2}{2m_2}\nabla_2^2=-\frac{\hbar^2}{M}\nabla_{\rm cm}^2-\frac{\hbar^2}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\nabla^2=-\frac{\hbar^2}{2M}\nabla_{\rm cm}^2-\frac{\hbar^2}{2m}\nabla^2\ ,$$

where define the reduced mass by

$$\frac{1}{m}=\frac{1}{m_1}+\frac{1}{m_2}\qquad\text{or}\qquad m=\frac{m_1m_2}{m_1+m_2}\ .$$

The symbol μ is often used for reduced mass, but I think that is unnecessarily obscure myself. Note

$$\frac{1}{m}\geq\frac{1}{m_i}\ ,$$

where i stands for 1 or 2 and equality only olds if the dropped mass is infinite. Thus

$$m\leq m_i\qquad\text{or}\qquad m\leq\min(m_1,m_2)\ .$$

If $m_1=m_2$, then

$$m=\frac{m_1}{2}\ .$$

If $m_1/m_2<1$, then one can expand the reduced mass expression in the power series (e.g., Ar-238)

$$m=\frac{m_1}{1+m_1/m_2}=m_1\sum_k(-1)^k\left(\frac{m_1}{m_2}\right)^k\approx m_1\left(1-\frac{m_1}{m_2}\right)\ ,$$

where the last expression holds for $m_1/m_2 << 1$.

c) We make the anzatz that we can set

$$\psi_{\rm total}(\vec r_1, \vec r_2) = \psi_{\rm cm}(\vec R)\psi(\vec r) \ .$$

The Schrödinger equation can then be written at once as

$$-\frac{\hbar^2}{2M}\frac{\nabla_{\rm cm}^2\psi_{\rm cm}(\vec R)}{\psi(\vec R)} - \frac{\hbar^2}{2m}\frac{\nabla^2\psi(\vec r)}{\psi(\vec r)} + V(\vec r) = E_{\rm total} \ .$$

For the differential equation to hold for all $\vec R$ and $\vec r$, we must have

$$-\frac{\hbar^2}{2M}\frac{\nabla_{\rm cm}^2\psi_{\rm cm}(\vec R)}{\psi(\vec R)} = E_{\rm cm} \qquad \text{and} \qquad -\frac{\hbar^2}{2m}\frac{\nabla^2\psi(\vec r)}{\psi(\vec r)} + V(\vec r) = E \ ,$$

where $E_{\rm cm}$ and E are constants of separation that sum to $E_{\rm total}$. We then have two 1-body Schrödinger equation problems:

$$\frac{\hbar^2}{2M}\nabla_{\rm cm}^2\psi_{\rm cm}(\vec R) = E_{\rm cm}\psi(\vec R) \qquad \text{and} \qquad -\frac{\hbar^2}{2m}\nabla^2\psi(\vec r) + V(\vec r) = E\psi(\vec r) \ .$$

The center-of-mass problem is just the free particle Schrödinger equation. The relative problem is just the central force Schrödinger equation. The wave functions that solve the relative problem give the position of particle 2 relative to particle 1. Of course, one can get the reverse by a change of sign of the relative vector. The relative problem is not an inertial frame, but it can be treated as if it were. I always think that the reduced mass must account for the non-inertiality, but no text book I know of spits out that notion.

The classical 2-body problem with only a central force separates in analogous way to the quantum 2-body problem. For example, the identical formula of the reduced mass appears.

Redaction: Jeffery, 2001jan01

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011 qfull 00200 2 3 0 moderate math: central-force azimuthal component solution

Extra keywords: solving the azimuthal component of the central force problem

10. In the central force problem the separated azimuthal part of the Schrödinger equation is:

$$\frac{d^2\Phi}{d\phi^2} = -m_\ell^2\Phi \ ,$$

where $-m_\ell^2$ is the constant of separation for the azimuthal part. The constant has been parameterized in terms of m_ℓ (which is not mass) since it turns out that for normalizable (and therefore physically allowed) solutions that m must be an integer. The m_ℓ quantity is the z-component angular momentum quantum number or magnetic quantum number (MEL-59; ER-240). The latter name arises since the z-components of the angular momentum manifest themselves most noticeably in magnetic field phenomena.

a) Since the differential equation is second order, there should should be two independent solutions for each value of m_ℓ^2. Solve for the general solution Φ for each m_ℓ^2: i.e., the solution that is a linear combination of the two independent solutions with undetermined coefficients. Note that writing the separation constant as m_ℓ^2 is so far just a parameterization and nothing yet demands that m_ℓ^2 be greater than zero or pure real. **HINT:** Use an exponential trial function with exponent $\pm(a+ib)$ with a and b real. Also remember the special case of $m_\ell^2 = 0$.

b) Impose the single-valuedness and continuity condition for quantum mechanical solutions on

$$\Phi = Ae^{(a+ib)\phi} + Be^{-(a+ib)\phi} \ ,$$

and show that $a = 0$ and m_ℓ must be an integer. What happens to the special case where $m_\ell = 0$?

c) What are the eigenfunction solutions for the z-component of the angular momentum operator

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} .$$

What are the eigenvalues that satisfy single-valuedness and continuity? What is the relationship between these eigenfunction solutions and the azimuthal angle part of the hydrogenic atom wave functions?

d) Normalize the allowed eigensolutions of L_z Note these solutions are in fact conventionally left unnormalized: i.e., the coefficient of the special function that is the solution is left as just 1. Normalization is conventionally imposed on the total orbital angular momentum solutions, spherical harmonics.

SUGGESTED ANSWER:

a) The trial solution

$$\Phi = e^{\pm(a+ib)\phi}$$

obviously satisfies the differential equation for

$$(a+ib)^2 = -m_\ell^2 \qquad \text{or} \qquad (a+ib) = i\sqrt{m_\ell^2} .$$

The general solution for each m_ℓ is then

$$\Phi = Ae^{(a+ib)\phi} + Be^{-(a+ib)\phi} ,$$

where A and B are undetermined constants.

In the special case of $m_\ell^2 = 0$, we have

$$\Phi = A\phi + B ,$$

where A and B are undetermined constants.

b) To be single-valued and continuous, we demand that $\Phi(\phi + 2\pi) = \Phi(\phi)$ for all ϕ. Thus we must have

$$Ae^{(a+ib)(\phi+2\pi)} + Be^{-(a+ib)(\phi+2\pi)} = Ae^{(a+ib)\phi} + Be^{-(a+ib)\phi}$$

which we can rearrange to get

$$Ae^{(a+ib)(\phi+\pi)}\left[e^{(a+ib)\pi} - e^{-(a+ib)\pi}\right) = Be^{-(a+ib)(\phi+\pi)}\left(e^{(a+ib)\pi} - e^{-(a+ib)\pi}\right) .$$

If

$$e^{(a+ib)\pi} - e^{-(a+ib)\pi} \neq 0 ,$$

then

$$Ae^{2(a+ib)(\phi+\pi)} = B$$

which can only be true for general ϕ if $a + ib = 0$ which implies $e^{(a+ib)\pi} - e^{-(a+ib)\pi} = 0$ which contradicts our assumptin that $e^{(a+ib)\pi} - e^{-(a+ib)\pi} \neq 0$. (One could also satisfy the last equation with $A = B = 0$ which is not satisfactory solution for quantum mechanics.) If on the other hand

$$e^{(a+ib)\pi} - e^{-(a+ib)\pi} = 0 ,$$

then

$$e^{2a}e^{ib(2\pi)} = 1 \qquad \text{and} \qquad e^{2a}e^{-ib(2\pi)} = 1$$

which implies $e^{4a} = 1$ which implies $a = 0$ which implies that $e^{ib(2\pi)} = 1$ which implies b is an integer. Thus, for the exponential solution case, we conclude that the only allowed m_ℓ values are given by

$$m_\ell = 0, \pm1, \pm2, \pm3, \ldots$$

and the general exponential solution is

$$\Phi = Ae^{i|m_\ell|\phi} + Be^{-i|m_\ell|\phi} .$$

For the linear solution

$$\Phi = A\phi + B \ ,$$

single-valuedness and continuity require that $A = 0$. The constant solution is just the $m_\ell = 0$ solution again.

c) Say we parameterize the eigenvalues as $m_\ell\hbar$. Thus, the eigenproblem is

$$L_z\Phi = m\hbar\Phi \ .$$

The solutions that satisfy single-valuedness and continuity based on parts (a) and (b) are obviously

$$\Phi = e^{im_\ell\phi}$$

where

$$m_\ell = 0, \pm 1, \pm 2, \pm 3, \ldots$$

The azimuthal angle parts of the hydrogenic atom wave function can be constructed from the eigenstates of the L_z operator.

d) By inspection, all the allowed normalized solutions are given by

$$\Phi = \frac{1}{\sqrt{2\pi}} e^{im\phi} \ .$$

Redaction: Jeffery, 2001jan01

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012 qmult 00050 1 1 1 easy memory: hydrogen atom, 2-body

11. The hydrogen atom is the simplest of all neutral atoms because:

a) it is a 2-body system. b) it is a 3-body system. c) it has no electrons.
d) it has many electrons. e) hydrogen is the most abundant element in the universe.

SUGGESTED ANSWER: (a)

Wrong answers:

e) It is the most abundant element in the universe. But this doesn't make it the simplest element. In fact perhaps it is the other way: because it is the simplest element, it is most abundant. However, even this is not necessarily so. The abundances of the elements depend on how things were cooked up in the beginning. A different set of initial conditions would lead to different universal abundances.

Redaction: Jeffery, 2001jan01

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012 qmult 00100 1 1 3 easy memory: radial wave function requirements

12. What basic requirements must the radial part of hydrogenic atom wave function meet in order to be a physical radial wave function?

a) Satisfy the radial part of the Schrödinger equation and grow exponentially as $r \to \infty$.
b) Not satisfy the radial part of the Schrödinger equation and grow exponentially as $r \to \infty$.
c) Satisfy the radial part of the Schrödinger equation and be normalizable.
d) Not satisfy the radial part of the Schrödinger equation and be normalizable.
e) None at all.

SUGGESTED ANSWER: (c) The Schrödinger equation is our basics physics in nonrelativistic quantum mechanics. It must be satisfied. And of course a radial function must also be normalizable (i.e., be square-integrable).

Wrong Answers:

b) Everything is wrong.
e) Oh c'mon.

Redaction: Jeffery, 2001jan01

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012 qmult 00200 1 4 1 easy deducto-memory: associated Laguerre polyn.

13. What special functions are factors in the radial part of the of the hydrogenic atom eigenstate wave functions?

    a) The associated Laguerre polynomials. b) The unassociated Laguerre polynomials.
    c) The associated Jaguar polynomials. d) The unassociated jaguar polynomials.
    e) The Hermite polynomials.

    SUGGESTED ANSWER: (a)

    Wrong Answers:

    e) These are factors in the simple harmonic oscillator wave functions.

    Redaction: Jeffery, 2001jan01

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012 qmult 01000 1 4 1 easy deducto-memory: atomic spectroscopy

14. Almost all would agree that the most important empirical means for learning about atomic energy eigenstates is:

    a) spectroscopy. b) microscopy. c) telescopy. d) pathology. e) astrology.

    SUGGESTED ANSWER: (a)

    Wrong Answers:

    e) It doesn't even pretend to reveal atomic energy eigenstates.

    Redaction: Jeffery, 2001jan01

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013 qmult 00910 1 1 3 easy memory: vector model

15. In the vector model for angular momentum of a quantum system with the standard axis for the eigenstates being the z axis, the particles in the eigenstates are thought of as having definite z-components of angular momentum $m_\supset\hbar$ and definite total angular momenta of magnitude $\sqrt{j(j+1)}\hbar$, where j can stand for orbital, spin, or total angular momentum quantum number and m_j is the z-component quantum number. Recall j can be only be integer or half-integer and there are $2j+1$ possible values of m_j given by $-j, -j+1, \ldots, j-1, j$. The x-y component of the angular momementum has magnitude $\sqrt{j(j+1)-m_j^2}\hbar$, but it has no definite direction. Rather this component can be thought of as pointing all x-y directions in simultaneous: i.e., it is in a superposition state of all direction states and is equally likely to be found in any of them. Diagramatically, the momentum vectors can be represented by

    a) cones with axis aligned with the x-axis. b) cones with axis aligned with the y-axis.
    c) cones with axis aligned with the z-axis. d) cones with axis aligned with the x-y-axis.
    e) the cones of silence.

    SUGGESTED ANSWER: (c)

    Wrong answers:

    e) "I demand the cones of silence Chief."
    "But Max you know those never work."
    "I insist Chief."
    Whirr.
    "What did you say?"

    Redaction: Jeffery, 2008jan01

Equation Sheet for Modern Physics

These equation sheets are intended for students writing tests or reviewing material. Therefore they are neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

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1 Constants

$$
\begin{aligned}
c &= 2.99792458\times10^{8}\,\mathrm{m/s} \approx 2.998\times10^{8}\,\mathrm{m/s} \approx 3\times10^{8}\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns}\\
e &= 1.602176487(40)\times10^{-19}\,\mathrm{C}\\
E_{\mathrm{Rydberg}} &= 13.60569193(34)\,\mathrm{eV}\\
g_e &= 2.0023193043622 \qquad \text{(electron g-factor)}\\
h &= 6.62606896(33)\times10^{-34}\,\mathrm{J\,s} = 4.13566733(10)\times10^{-15}\,\mathrm{eV\,s}\\
hc &= 12398.419\,\mathrm{eV\,\AA} \approx 10^{4}\,\mathrm{eV\,\AA}\\
\hbar &= 1.054571628(53)\times10^{-34}\,\mathrm{J\,s} = 6.58211899(16)\times10^{-16}\,\mathrm{eV\,s}\\
k &= 1.3806504(24)\times10^{-23}\,\mathrm{J/K} = 0.8617343(15)\times10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,eV/K\\
m_e &= 9.10938215(45)\times10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV}\\
m_p &= 1.672621637(83)\times10^{-27}\,\mathrm{kg} = 938.272013(23),\mathrm{MeV}\\
\alpha &= e^2/(4\pi\epsilon_0\hbar c) = 7.2973525376(50)\times10^{-3} = 1/137.035999679(94) \approx 1/137\\
\lambda_{\mathrm{C}} &= h/(m_e c) = 2.4263102175(33)\times10^{-12}\,\mathrm{m} = 0.0024263102175(33)\,\mathrm{\AA}\\
\mu_{\mathrm{B}} &= 5.7883817555(79)\times10^{-5}\,\mathrm{eV/T}
\end{aligned}
$$

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2 Geometrical Formulae

$$C_{\mathrm{cir}} = 2\pi r \qquad A_{\mathrm{cir}} = \pi r^2 \qquad A_{\mathrm{sph}} = 4\pi r^2 \qquad V_{\mathrm{sph}} = \frac{4}{3}\pi r^3$$

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3 Trigonometry

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)] \qquad \sin^2\theta = \frac{1}{2}[1-\cos(2\theta)] \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right] \qquad \sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b)+\sin(a+b)\right]$$

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4 Blackbody Radiation

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{[e^{h\nu/(kT)}-1]} \qquad B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{[e^{hc/(kT\lambda)}-1]}$$

$$B_\lambda\,d\lambda = B_\nu\,d\nu \qquad \nu\lambda = c \qquad \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$$

$$E = h\nu = \frac{hc}{\lambda} \qquad p = \frac{h}{\lambda}$$

$$F = \sigma T^4 \qquad \sigma = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3} = 5.670400(40)\times10^{-8}\,\mathrm{W/m^2/K^4}$$

$$\lambda_{\mathrm{max}}T = \text{constant} = \frac{hc}{kx_{\mathrm{max}}} \approx \frac{1.4387751\times10^{-2}}{x_{\mathrm{max}}}$$

$$B_{\lambda,\mathrm{Wien}} = \frac{2hc^2}{\lambda^5}e^{-hc/(kT\lambda)} \qquad B_{\lambda,\mathrm{Rayleigh-Jeans}} = \frac{2ckT}{\lambda^4}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\nu = \frac{\omega}{c} \qquad k_i = \frac{\pi}{L}n_i \quad \text{standing wave BCs} \qquad k_i = \frac{2\pi}{L}n_i \quad \text{periodic BCs}$$

$$n(k)\,dk = \frac{k^2}{\pi^2}\,dk = \pi\left(\frac{2}{c}\right)\nu^2\,d\nu = n(\nu)\,d\nu$$

$$\ln(z!) \approx \left(z+\frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \ldots$$

$$\ln(N!) \approx N\ln(N) - N$$

$$\rho(E)\,dE = \frac{e^{-E/(kT)}}{kT}\,dE \qquad P(n) = (1-e^{-\alpha})e^{-n\alpha} \qquad \alpha = \frac{h\nu}{kT}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad f(x-vt) \qquad f(kx-\omega t)$$

5 Photons

$$KE = h\nu - w \qquad \Delta\lambda = \lambda_{\mathrm{scat}} - \lambda_{\mathrm{inc}} = \lambda_{\mathrm{C}}(1-\cos\theta)$$

$$\ell = \frac{1}{n\sigma} \qquad \rho = \frac{e^{-s/\ell}}{\ell} \qquad \langle s^m\rangle = \ell^m m!$$

6 Matter Waves

$$\lambda = \frac{h}{p} \qquad p = \hbar k \qquad \Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\Psi(x,t) = \int_{-\infty}^{\infty} \phi(k) \Psi_k(x,t)\, dk \qquad \phi(k) = \int_{-\infty}^{\infty} \Psi(x,0) \frac{e^{-ikx}}{\sqrt{2\pi}}\, dk$$

$$v_{\mathrm{g}} = \left.\frac{d\omega}{dk}\right|_{k_0} = \frac{\hbar k_0}{m} = \frac{p_0}{m} = v_{\mathrm{clas},0}$$

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7 Non-Relativistic Quantum Mechanics

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V \qquad T = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \qquad H\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial \Psi}{\partial t}$$

$$\rho = \Psi^* \Psi \qquad \rho\, dx = \Psi^* \Psi\, dx$$

$$A\phi_i = a_i \phi_i \qquad f(x) = \sum_i c_i \phi_i \qquad \int_a^b \phi_i^* \phi_j\, dx = \delta_{ij} \qquad c_j = \int_a^b \phi_j^* f(x)\, dx$$

$$[A, B] = AB - BA$$

$$P_i = |c_i|^2 \qquad \langle A \rangle = \int_{-\infty}^{\infty} \Psi^* A \Psi\, dx = \sum_i |c_i|^2 a_i \qquad H\psi = E\psi \qquad \Psi(x,t) = \psi(x) e^{-i\omega t}$$

$$p_{\mathrm{op}}\phi = \frac{\hbar}{i}\frac{\partial \phi}{\partial x} = p\phi \qquad \phi = \frac{e^{ikx}}{\sqrt{2\pi}} \qquad \frac{\partial^2 \psi}{\partial x^2} = \frac{2m}{\hbar^2}(V - E)\psi$$

$$|\Psi\rangle \qquad \langle\Psi| \qquad \langle x|\Psi\rangle = \Psi(x) \qquad \langle\vec{r}|\Psi\rangle = \Psi(\vec{r}) \qquad \langle k|\Psi\rangle = \Psi(k) \qquad \langle\Psi_i|\Psi_j\rangle = \langle\Psi_j|\Psi_i\rangle^*$$

$$\langle\phi_i|\Psi\rangle = c_i \qquad 1_{\mathrm{op}} = \sum_i |\phi_i\rangle\langle\phi_i| \qquad |\Psi\rangle = \sum_i |\phi_i\rangle\langle\phi_i|\Psi\rangle = \sum_i c_i|\phi_i\rangle$$

$$1_{\mathrm{op}} = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x| \qquad \langle\Psi_i|\Psi_j\rangle = \int_{-\infty}^{\infty} dx\, \langle\Psi_i|x\rangle\langle x|\Psi_j\rangle \qquad A_{ij} = \langle\phi_i|A|\phi_j\rangle$$

$$Pf(x) = f(-x) \qquad P\frac{df(x)}{dx} = \frac{df(-x)}{d(-x)} = -\frac{df(-x)}{dx} \qquad Pf_{\mathrm{e/o}}(x) = \pm f_{\mathrm{e/o}}(x)$$

$$P\frac{df_{\mathrm{e/o}}(x)}{dx} = \mp\frac{df_{\mathrm{e/o}}(x)}{dx}$$

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8 Spherical Harmonics

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}} \qquad Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2}\cos(\theta) \qquad Y_{1,\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin(\theta)e^{\pm i\phi}$$

$$L^2 Y_{\ell m} = \ell(\ell+1)\hbar^2 Y_{\ell m} \qquad L_z Y_{\ell m} = m\hbar Y_{\ell m} \qquad |m| \leq \ell \qquad m = -\ell, -\ell+1, \ldots, \ell-1, \ell$$

0	1	2	3	4	5	6	...
s	p	d	f	g	h	i	...

9 Hydrogenic Atom

$$\psi_{n\ell m} = R_{n\ell}(r)Y_{\ell m}(\theta,\phi) \qquad \ell \leq n-1 \qquad \ell = 0,1,2,\ldots,n-1$$

$$a_z = \frac{a_0}{Z}\left(\frac{m_e}{m_{\text{reduced}}}\right) \qquad a_0 = \frac{\hbar}{m_e c\alpha} = \frac{\lambda_{\text{C}}}{2\pi\alpha} \qquad m_{\text{reduced}} = \frac{m_1 m_2}{m_1+m_2}$$

$$R_{10} = 2a_Z^{-3/2}e^{-r/a_Z} \qquad R_{20} = \frac{1}{\sqrt{2}}a_Z^{-3/2}\left(1-\frac{1}{2}\frac{r}{a_Z}\right)e^{-r/(2a_Z)}$$

$$R_{21} = \frac{1}{\sqrt{24}}a_Z^{-3/2}\frac{r}{a_Z}e^{-r/(2a_Z)}$$

$$R_{n\ell} = -\left\{\left(\frac{2}{na_Z}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}\right\}^{1/2} e^{-\rho/2}\rho^\ell L_{n+\ell}^{2\ell+1}(\rho) \qquad \rho = \frac{2r}{nr_Z}$$

$$L_q(x) = e^x\left(\frac{d}{dx}\right)^q\left(e^{-x}x^q\right) \quad \text{Rodrigues's formula for the Laguerre polynomials}$$

$$L_q^j(x) = \left(\frac{d}{dx}\right)^j L_q(x) \qquad \text{Associated Laguerre polynomials}$$

$$\langle r\rangle_{n\ell m} = \frac{a_Z}{2}\left[3n^2 - \ell(\ell+1)\right]$$

$$\text{Nodes} = (n-1)-\ell \qquad \text{not counting zero or infinity}$$

$$E_n = -\frac{1}{2}m_e c^2\alpha^2\frac{Z^2}{n^2}\frac{m_{\text{reduced}}}{m_e} = -E_{\text{Ryd}}\frac{Z^2}{n^2}\frac{m_{\text{reduced}}}{m_e} \approx -13.606 \times \frac{Z^2}{n^2}\frac{m_{\text{reduced}}}{m_e}\ \text{eV}$$

10 Special Relativity

$$c = 2.99792458\times 10^8\,\text{m/s} \approx 2.998\times 10^8\,\text{m/s} \approx 3\times 10^8\,\text{m/s} \approx 1\,\text{lyr/yr} \approx 1\,\text{ft/ns}$$

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \gamma(\beta << 1) = 1 + \frac{1}{2}\beta^2 \qquad \tau = ct$$

Galilean Transformations	Lorentz Transformations
$x' = x - \beta\tau$	$x' = \gamma(x - \beta\tau)$
$y' = y$	$y' = y$
$z' = z$	$z' = z$
$\tau' = \tau$	$\tau' = \gamma(\tau - \beta x)$
$\beta'_{\mathrm{obj}} = \beta_{\mathrm{obj}} - \beta$	$\beta'_{\mathrm{obj}} = \dfrac{\beta_{\mathrm{obj}} - \beta}{1 - \beta\beta_{\mathrm{obj}}}$

$$\ell = \ell_{\mathrm{proper}}\sqrt{1-\beta^2} \qquad \Delta\tau_{\mathrm{proper}} = \Delta\tau\sqrt{1-\beta^2}$$

$$m = \gamma m_0 \qquad p = mv = \gamma m_0 c\beta \qquad E_0 = m_0c^2 \qquad E = \gamma E_0 = \gamma m_0 c^2 = mc^2$$

$$E = mc^2 \qquad E = \sqrt{(pc)^2 + (m_0c^2)^2}$$

$$KE = E - E_0 = \sqrt{(pc)^2 + (m_0c^2)^2} - m_0c^2 = (\gamma - 1)m_0c^2$$

$$f = f_{\mathrm{proper}}\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{for source and detector separating}$$

$$f(\beta << 1) = f_{\mathrm{proper}}\left(1 - \beta + \frac{1}{2}\beta^2\right)$$

$$f_{\mathrm{trans}} = f_{\mathrm{proper}}\sqrt{1-\beta^2} \qquad f_{\mathrm{trans}}(\beta << 1) = f_{\mathrm{proper}}\left(1 - \frac{1}{2}\beta^2\right)$$

$$\tau = \beta x + \gamma^{-1}\tau' \quad \text{for lines of constant } \tau'$$

$$\tau = \frac{x - \gamma^{-1}x'}{\beta} \quad \text{for lines of constant } x'$$

$$x' = \frac{x_{\mathrm{intersection}}}{\gamma} = x'_{x\ \mathrm{scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}} \qquad \tau' = \frac{\tau_{\mathrm{intersection}}}{\gamma} = \tau'_{\tau\ \mathrm{scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}}$$

$$\theta_{\mathrm{Mink}} = \tan^{-1}(\beta)$$