Modern Physics: Physics 305, Section 1

NAME:

Homework 7: The Hydrogenic Atom: Homeworks are due as posted on the course web site. They are **NOT** handed in. The student reports that it is completed and receives one point for this. Solutions are already posted, but students are only permitted to look at the solutions after completion. The solutions are intended to be (but not necessarily are) super-perfect and go beyond a complete answer expected on a test.

1. In a central-force problem, the magnitude of central force depends only on:

   a) the angle of the particle.
   b) the vector $\vec{r}$ from the center to the particle.
   c) the radial distance r from the center to the particle.
   d) the magnetic quantum number of the particle.
   e) the uncertainty principle.

2. The usual approach to getting the eigenfunctions of the Hamiltonian in multi-dimensions is:

   a) non-separation of variables. b) separation of variables. c) separation of invariables.
   d) non-separation of invariables. e) non-separation of variables/invariables.

3. Say you have a differential equation of two independent variables x and y and you want to look for solutions that can be factorized thusly $f(x,y) = g(x)h(y)$. Say then it is possible to reorder equation into the form

$$\mathrm{LHS}(x) = \mathrm{RHS}(y) \ ,$$

   where LHS stands for left-hand side and RHS for right-hand side. Well LHS is explicitly independent of y and implicitly independent of x:

$$\frac{\partial \mathrm{LHS}}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial \mathrm{LHS}}{\partial x} = \frac{\partial \mathrm{RHS}}{\partial x} = 0 \ .$$

   Thus, LHS is equal to a constant C and necessarily RHS is equal to the same constant C which is called the constant of separation (e.g., Arf-383). The solutions for $g(x)$ and $h(y)$ can be found separately and are related to each other through C. The solutions for $f(x,y)$ that cannot be factorized are not obtained, of course, by the described procedured. However, if one obtains complete sets of $g(x)$ and $h(y)$ solutions for the x-y region of interest, then any solution $f(x,y)$ can be constructed at least to within some approximation (WA-510). Thus, the generalization of the described procedure is very general and powerful. It is called: then

   a) separation of the left- and right-hand sides. b) partitioning.
   c) separation of the variables. d) solution factorization. e) the King Lear method.

4. "Let's play *Jeopardy*! For \$100, the answer is: By writing the two-body Schrödinger equation in relative/center-of-mass coordinates."

   How do you ____________, Alex?

   a) reduce a **ONE-BODY** problem to a **TWO-BODY** problem
   b) reduce a **TWO-BODY** problem to a **ONE-BODY** problem
   c) solve a one-dimensional infinite square well problem
   d) solve for the simple harmonic oscillator eigenvalues
   e) reduce a **TWO-BODY** problem to a **TWO-BODY** problem

5. The formula for the reduced mass m for two-body system (with bodies labeled 1 and 2) is:

   a) $m = m_1 m_2$. b) $m = \dfrac{1}{m_1 m_2}$. c) $m = \dfrac{m_1 + m_2}{m_1 m_2}$. d) $m = \dfrac{m_1 m_2}{m_1 + m_2}$. e) $m = \dfrac{1}{m_1}$.

6. The eigensolutions of the angular part of the Hamiltonian for the central force problem are the:

   a) linear harmonics. b) spherical harmonics. c) square harmonics.
   d) Pythagorean harmonics. e) Galilean harmonics.

7. Just about the only spherical harmonic that people remember—and they really should remember it too—is Y_{00} =:

   a) $e^{im\phi}$. b) r^2. c) $\frac{1}{\sqrt{4\pi}}$. d) θ^2. e) $2a^{-3/2}e^{-r/a}$.

8. Conventionally, the spherical harmonic eigenstates for angular momentum quantum numbers

$$\ell = 0, 1, 2, 3, 4, ...$$

   are designated by:

   a) a, b, c, d, e, etc.
   b) s, p, d, f, and then alphabetically following f: i.e., g, h, etc.
   c) x, y, z, xx, yy, zz, xxx, etc.
   d) A, C, B, D, E, etc.
   e) $@%&*!!

9. The 2-body time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m_1}\nabla_1^2\psi - \frac{\hbar^2}{2m_2}\nabla_2^2\psi + V\psi = E_{\text{total}}\psi \ .$$

   If the V depends only on $\vec{r} = \vec{r}_2 - r_1$ (the relative vector), then the problem can be separate into two problems: a relative problem 1-body equivalent problem and a center-of-mass 1-body equivalent problem. The center of mass vector is

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} \ ,$$

   where $M = m_1 + m_2$.

   a) Determine the expressions for $\vec{r}_1$ and $\vec{r}_2$ in terms of $\vec{R}$ and $\vec{r}$.

   b) Determine the expressions for ∇_1^2 and ∇_2^2 in terms of ∇_{cm}^2 (the center-of-mass Laplacian operator) and ∇^2 (the relative Laplacian operator). Then re-express kinetic operator

$$-\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2$$

   in terms of ∇_{cm}^2 and ∇^2. **HINTS:** The x, y, and z direction components of vectors can all be treated separately and identically since x components of $\vec{R}$ and $\vec{r}$) (i.e., X and x) depend only on x_1 and x_2, etc. You can introduce a reduced mass to make the transformed kinetic energy operator simpler.

   c) Now separate the 2-body Schrödinger equation assuming $V = V(\vec{r})$. What are the solutions of the center-of-mass problem? How would you interpret the solutions of the relative problem? **HINT:** I'm only looking for a short answer to the interpretation question.

10. In the central force problem the separated azimuthal part of the Schrödinger equation is:

$$\frac{d^2\Phi}{d\phi^2} = -m_\ell^2\Phi \ ,$$

   where $-m_\ell^2$ is the constant of separation for the azimuthal part. The constant has been parameterized in terms of m_ℓ (which is not mass) since it turns out that for normalizable (and therefore physically allowed) solutions that m must be an integer. The m_ℓ quantity is the z-component angular momentum quantum number or magnetic quantum number (MEL-59; ER-240). The latter name arises since the z-components of the angular momentum manifest themselves most noticeably in magnetic field phenomena.

   a) Since the differential equation is second order, there should should be two independent solutions for each value of m_ℓ^2. Solve for the general solution Φ for each m_ℓ^2: i.e., the solution that is a linear combination of the two independent solutions with undetermined coefficients. Note that writing the separation constant as m_ℓ^2 is so far just a parameterization and nothing yet demands that m_ℓ^2 be greater than zero or pure real. **HINT:** Use an exponential trial function with exponent $\pm(a+ib)$ with a and b real. Also remember the special case of $m_\ell^2 = 0$.

b) Impose the single-valuedness and continuity condition for quantum mechanical solutions on

$$\Phi = Ae^{(a+ib)\phi} + Be^{-(a+ib)\phi} ,$$

and show that $a = 0$ and m_ℓ must be an integer. What happens to the special case where $m_\ell = 0$?

c) What are the eigenfunction solutions for the z-component of the angular momentum operator

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} .$$

What are the eigenvalues that satisfy single-valuedness and continuity? What is the relationship between these eigenfunction solutions and the azimuthal angle part of the hydrogenic atom wave functions?

d) Normalize the allowed eigensolutions of L_z Note these solutions are in fact conventionally left unnormalized: i.e., the coefficient of the special function that is the solution is left as just 1. Normalization is conventionally imposed on the total orbital angular momentum solutions, spherical harmonics.

11. The hydrogen atom is the simplest of all neutral atoms because:

a) it is a 2-body system. b) it is a 3-body system. c) it has no electrons.
d) it has many electrons. e) hydrogen is the most abundant element in the universe.

12. What basic requirements must the radial part of hydrogenic atom wave function meet in order to be a physical radial wave function?

a) Satisfy the radial part of the Schrödinger equation and grow exponentially as $r \to \infty$.
b) Not satisfy the radial part of the Schrödinger equation and grow exponentially as $r \to \infty$.
c) Satisfy the radial part of the Schrödinger equation and be normalizable.
d) Not satisfy the radial part of the Schrödinger equation and be normalizable.
e) None at all.

13. What special functions are factors in the radial part of the of the hydrogenic atom eigenstate wave functions?

a) The associated Laguerre polynomials. b) The unassociated Laguerre polynomials.
c) The associated Jaguar polynomials. d) The unassociated jaguar polynomials.
e) The Hermite polynomials.

14. Almost all would agree that the most important empirical means for learning about atomic energy eigenstates is:

a) spectroscopy. b) microscopy. c) telescopy. d) pathology. e) astrology.

15. In the vector model for angular momentum of a quantum system with the standard axis for the eigenstates being the z axis, the particles in the eigenstates are thought of as having definite z-components of angular momentum $m_\daleth\hbar$ and definite total angular momenta of magnitude $\sqrt{j(j+1)}\hbar$, where j can stand for orbital, spin, or total angular momentum quantum number and m_j is the z-component quantum number. Recall j can be only be integer or half-integer and there are $2j+1$ possible values of m_j given by $-j, -j+1, \ldots, j-1, j$. The x-y component of the angular momementum has magnitude $\sqrt{j(j+1) - m_j^2}\hbar$, but it has no definite direction. Rather this component can be thought of as pointing all x-y directions in simultaneous: i.e., it is in a superposition state of all direction states and is equally likely to be found in any of them. Diagramatically, the momentum vectors can be represented by

a) cones with axis aligned with the x-axis. b) cones with axis aligned with the y-axis.
c) cones with axis aligned with the z-axis. d) cones with axis aligned with the x-y-axis.
e) the cones of silence.

Equation Sheet for Modern Physics

These equation sheets are intended for students writing tests or reviewing material. Therefore they are neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

1 Constants

$$
\begin{aligned}
c &= 2.99792458 \times 10^{8}\,\mathrm{m/s} \approx 2.998 \times 10^{8}\,\mathrm{m/s} \approx 3 \times 10^{8}\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \\
e &= 1.602176487(40) \times 10^{-19}\,\mathrm{C} \\
E_{\mathrm{Rydberg}} &= 13.60569193(34)\,\mathrm{eV} \\
g_e &= 2.0023193043622 \qquad \text{(electron g-factor)} \\
h &= 6.62606896(33) \times 10^{-34}\,\mathrm{J\,s} = 4.13566733(10) \times 10^{-15}\,\mathrm{eV\,s} \\
hc &= 12398.419\,\mathrm{eV\,\AA} \approx 10^{4}\,\mathrm{eV\,\AA} \\
\hbar &= 1.054571628(53) \times 10^{-34}\,\mathrm{J\,s} = 6.58211899(16) \times 10^{-16}\,\mathrm{eV\,s} \\
k &= 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} = 0.8617343(15) \times 10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,eV/K \\
m_e &= 9.10938215(45) \times 10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV} \\
m_p &= 1.672621637(83) \times 10^{-27}\,\mathrm{kg} = 938.272013(23), \mathrm{MeV} \\
\alpha &= e^2/(4\pi\epsilon_0\hbar c) = 7.2973525376(50) \times 10^{-3} = 1/137.035999679(94) \approx 1/137 \\
\lambda_{\mathrm{C}} &= h/(m_e c) = 2.4263102175(33) \times 10^{-12}\,\mathrm{m} = 0.0024263102175(33)\,\mathrm{\AA} \\
\mu_{\mathrm{B}} &= 5.7883817555(79) \times 10^{-5}\,\mathrm{eV/T}
\end{aligned}
$$

2 Geometrical Formulae

$$
C_{\mathrm{cir}} = 2\pi r \qquad A_{\mathrm{cir}} = \pi r^2 \qquad A_{\mathrm{sph}} = 4\pi r^2 \qquad V_{\mathrm{sph}} = \frac{4}{3}\pi r^3
$$

3 Trigonometry

$$
\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta \qquad \cos^2\theta + \sin^2\theta = 1
$$

$$
\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)
$$

$$
\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)] \qquad \sin^2\theta = \frac{1}{2}[1-\cos(2\theta)] \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)
$$

$$
\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right] \qquad \sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right]
$$

$$
\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b)+\sin(a+b)\right]
$$

4 Blackbody Radiation

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{[e^{h\nu/(kT)}-1]} \qquad B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{[e^{hc/(kT\lambda)}-1]}$$

$$B_\lambda\,d\lambda = B_\nu\,d\nu \qquad \nu\lambda = c \qquad \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$$

$$E = h\nu = \frac{hc}{\lambda} \qquad p = \frac{h}{\lambda}$$

$$F = \sigma T^4 \qquad \sigma = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3} = 5.670400(40)\times 10^{-8}\,\mathrm{W/m^2/K^4}$$

$$\lambda_{\mathrm{max}}T = \mathrm{constant} = \frac{hc}{kx_{\mathrm{max}}} \approx \frac{1.4387751\times 10^{-2}}{x_{\mathrm{max}}}$$

$$B_{\lambda,\mathrm{Wien}} = \frac{2hc^2}{\lambda^5}e^{-hc/(kT\lambda)} \qquad B_{\lambda,\mathrm{Rayleigh-Jeans}} = \frac{2ckT}{\lambda^4}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\nu = \frac{\omega}{c} \qquad k_i = \frac{\pi}{L}n_i \quad \text{standing wave BCs} \qquad k_i = \frac{2\pi}{L}n_i \quad \text{periodic BCs}$$

$$n(k)\,dk = \frac{k^2}{\pi^2}\,dk = \pi\left(\frac{2}{c}\right)\nu^2\,d\nu = n(\nu)\,d\nu$$

$$\ln(z!) \approx \left(z+\frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \ldots$$

$$\ln(N!) \approx N\ln(N) - N$$

$$\rho(E)\,dE = \frac{e^{-E/(kT)}}{kT}\,dE \qquad P(n) = (1-e^{-\alpha})e^{-n\alpha} \qquad \alpha = \frac{h\nu}{kT}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad f(x-vt) \qquad f(kx-\omega t)$$

5 Photons

$$KE = h\nu - w \qquad \Delta\lambda = \lambda_{\mathrm{scat}} - \lambda_{\mathrm{inc}} = \lambda_{\mathrm{C}}(1-\cos\theta)$$

$$\ell = \frac{1}{n\sigma} \qquad \rho = \frac{e^{-s/\ell}}{\ell} \qquad \langle s^m\rangle = \ell^m m!$$

6 Matter Waves

$$\lambda = \frac{h}{p} \qquad p = \hbar k \qquad \Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\Psi(x,t) = \int_{-\infty}^{\infty} \phi(k)\Psi_k(x,t)\,dk \qquad \phi(k) = \int_{-\infty}^{\infty} \Psi(x,0)\frac{e^{-ikx}}{\sqrt{2\pi}}\,dk$$

$$v_{\mathrm{g}} = \left.\frac{d\omega}{dk}\right|_{k_0} = \frac{\hbar k_0}{m} = \frac{p_0}{m} = v_{\mathrm{clas},0}$$

7 Non-Relativistic Quantum Mechanics

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V \qquad T = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \qquad H\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$$\rho = \Psi^*\Psi \qquad \rho\,dx = \Psi^*\Psi\,dx$$

$$A\phi_i = a_i\phi_i \qquad f(x) = \sum_i c_i\phi_i \qquad \int_a^b \phi_i^*\phi_j\,dx = \delta_{ij} \qquad c_j = \int_a^b \phi_j^* f(x)\,dx$$

$$[A,B] = AB - BA$$

$$P_i = |c_i|^2 \qquad \langle A\rangle = \int_{-\infty}^{\infty} \Psi^* A\Psi\,dx = \sum_i |c_i|^2 a_i \qquad H\psi = E\psi \qquad \Psi(x,t) = \psi(x)e^{-i\omega t}$$

$$p_{\mathrm{op}}\phi = \frac{\hbar}{i}\frac{\partial\phi}{\partial x} = p\phi \qquad \phi = \frac{e^{ikx}}{\sqrt{2\pi}} \qquad \frac{\partial^2\psi}{\partial x^2} = \frac{2m}{\hbar^2}(V-E)\psi$$

$$|\Psi\rangle \qquad \langle\Psi| \qquad \langle x|\Psi\rangle = \Psi(x) \qquad \langle\vec{r}|\Psi\rangle = \Psi(\vec{r}) \qquad \langle k|\Psi\rangle = \Psi(k) \qquad \langle\Psi_i|\Psi_j\rangle = \langle\Psi_j|\Psi_i\rangle^*$$

$$\langle\phi_i|\Psi\rangle = c_i \qquad 1_{\mathrm{op}} = \sum_i |\phi_i\rangle\langle\phi_i| \qquad |\Psi\rangle = \sum_i |\phi_i\rangle\langle\phi_i|\Psi\rangle = \sum_i c_i|\phi_i\rangle$$

$$1_{\mathrm{op}} = \int_{-\infty}^{\infty} dx\,|x\rangle\langle x| \qquad \langle\Psi_i|\Psi_j\rangle = \int_{-\infty}^{\infty} dx\,\langle\Psi_i|x\rangle\langle x|\Psi_j\rangle \qquad A_{ij} = \langle\phi_i|A|\phi_j\rangle$$

$$Pf(x) = f(-x) \qquad P\frac{df(x)}{dx} = \frac{df(-x)}{d(-x)} = -\frac{df(-x)}{dx} \qquad Pf_{\mathrm{e/o}}(x) = \pm f_{\mathrm{e/o}}(x)$$

$$P\frac{df_{\mathrm{e/o}}(x)}{dx} = \mp\frac{df_{\mathrm{e/o}}(x)}{dx}$$

8 Spherical Harmonics

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}} \qquad Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos(\theta) \qquad Y_{1,\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin(\theta) e^{\pm i\phi}$$

$$L^2 Y_{\ell m} = \ell(\ell+1)\hbar^2 Y_{\ell m} \qquad L_z Y_{\ell m} = m\hbar Y_{\ell m} \qquad |m| \leq \ell \qquad m = -\ell, -\ell+1, \ldots, \ell-1, \ell$$

0	1	2	3	4	5	6	...
s	p	d	f	g	h	i	...

9 Hydrogenic Atom

$$\psi_{n\ell m} = R_{n\ell}(r) Y_{\ell m}(\theta, \phi) \qquad \ell \leq n-1 \qquad \ell = 0, 1, 2, \ldots, n-1$$

$$a_z = \frac{a_0}{Z}\left(\frac{m_e}{m_{\text{reduced}}}\right) \qquad a_0 = \frac{\hbar}{m_e c\alpha} = \frac{\lambda_{\text{C}}}{2\pi\alpha} \qquad m_{\text{reduced}} = \frac{m_1 m_2}{m_1 + m_2}$$

$$R_{10} = 2a_Z^{-3/2} e^{-r/a_Z} \qquad R_{20} = \frac{1}{\sqrt{2}} a_Z^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a_Z}\right) e^{-r/(2a_Z)}$$

$$R_{21} = \frac{1}{\sqrt{24}} a_Z^{-3/2} \frac{r}{a_Z} e^{-r/(2a_Z)}$$

$$R_{n\ell} = -\left\{\left(\frac{2}{na_Z}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}\right\}^{1/2} e^{-\rho/2} \rho^\ell L_{n+\ell}^{2\ell+1}(\rho) \qquad \rho = \frac{2r}{nr_Z}$$

$$L_q(x) = e^x \left(\frac{d}{dx}\right)^q \left(e^{-x} x^q\right) \quad \text{Rodrigues's formula for the Laguerre polynomials}$$

$$L_q^j(x) = \left(\frac{d}{dx}\right)^j L_q(x) \qquad \text{Associated Laguerre polynomials}$$

$$\langle r \rangle_{n\ell m} = \frac{a_Z}{2}\left[3n^2 - \ell(\ell+1)\right]$$

$$\text{Nodes} = (n-1) - \ell \qquad \text{not counting zero or infinity}$$

$$E_n = -\frac{1}{2} m_e c^2 \alpha^2 \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} = -E_{\text{Ryd}} \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} \approx -13.606 \times \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} \text{ eV}$$

10 Special Relativity

$$c = 2.99792458 \times 10^8 \text{ m/s} \approx 2.998 \times 10^8 \text{ m/s} \approx 3 \times 10^8 \text{ m/s} \approx 1 \text{ lyr/yr} \approx 1 \text{ ft/ns}$$

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \gamma(\beta << 1) = 1 + \frac{1}{2}\beta^2 \qquad \tau = ct$$

Galilean Transformations	Lorentz Transformations
$x' = x - \beta\tau$	$x' = \gamma(x - \beta\tau)$
$y' = y$	$y' = y$
$z' = z$	$z' = z$
$\tau' = \tau$	$\tau' = \gamma(\tau - \beta x)$
$\beta'_{\text{obj}} = \beta_{\text{obj}} - \beta$	$\beta'_{\text{obj}} = \dfrac{\beta_{\text{obj}} - \beta}{1 - \beta\beta_{\text{obj}}}$

$$\ell = \ell_{\text{proper}}\sqrt{1-\beta^2} \qquad \Delta\tau_{\text{proper}} = \Delta\tau\sqrt{1-\beta^2}$$

$$m = \gamma m_0 \qquad p = mv = \gamma m_0 c\beta \qquad E_0 = m_0 c^2 \qquad E = \gamma E_0 = \gamma m_0 c^2 = mc^2$$

$$E = mc^2 \qquad E = \sqrt{(pc)^2 + (m_0c^2)^2}$$

$$KE = E - E_0 = \sqrt{(pc)^2 + (m_0c^2)^2} - m_0c^2 = (\gamma - 1)m_0c^2$$

$$f = f_{\text{proper}}\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{for source and detector separating}$$

$$f(\beta << 1) = f_{\text{proper}}\left(1 - \beta + \frac{1}{2}\beta^2\right)$$

$$f_{\text{trans}} = f_{\text{proper}}\sqrt{1-\beta^2} \qquad f_{\text{trans}}(\beta << 1) = f_{\text{proper}}\left(1 - \frac{1}{2}\beta^2\right)$$

$$\tau = \beta x + \gamma^{-1}\tau' \quad \text{for lines of constant } \tau'$$

$$\tau = \frac{x - \gamma^{-1}x'}{\beta} \quad \text{for lines of constant } x'$$

$$x' = \frac{x_{\text{intersection}}}{\gamma} = x'_{x\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}} \qquad \tau' = \frac{\tau_{\text{intersection}}}{\gamma} = \tau'_{\tau\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}}$$

$$\theta_{\text{Mink}} = \tan^{-1}(\beta)$$