## Modern Physics: Physics 305, Section 1

# NAME:

**Homework 2: Blackbody Radiation** Homeworks are due as posted on the course web site. They are **NOT** handed in. The student reports that it is completed and receives one point for this. Solutions are already posted, but students are only permitted to look at the solutions after completion. The solutions are intended to be (but not necessarily are) super-perfect and go beyond a complete answer expected on a test.

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002 qmult 03010 1 4 1 easy deducto-memory: Newton bucket

**Extra keywords:** mathematical physics

1. "Let's play *Jeopardy*! For $100, the answer is: A thought experiment (which can actually be done) that has been used for arguing for absolute space as a physically active thing."

   What is ____________, Alex?

   a) Newton's bucket experiment
   b) Maxwell's demon experiment
   c) Einstein's elevator experiment
   d) Bohr's microscope experiment
   e) Schrödinger's cat experiment

   **SUGGESTED ANSWER:** (a)

   **Wrong answers:**

   d) See ER-67.
   e) Schrödinger described this as a hellish contraption (Gri-382). Still he could have picked a dog or a rat. But no-o-o-o, he had to choose a cat.

   **Redaction:** Jeffery, 2008jan01

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002 qmult 05030 1 4 1 easy deducto-memory: Ernst Mach glory

**Extra keywords:** Gre-33

2. Ersnt Mach (1838–1916), a Czech-Austrian physicist, is noted for Mach's principle (first so called by Einstein). This principle, which is really vague hypothesis, is that inertia is determined somehow by the universal distribution of mass. How it does so and what formulae apply have never adequately established. Mach is also famous:

   a) for Mach number. This is the speed of an object relative to a fluid medium in units of the sound speed in that medium.
   b) in ornithology. He is the eponym of the Machingbird.
   c) for work in pyrology. You've heard of machsticks.
   d) the invention of the mach: a kind of cooking pan.
   e) for his forceful character whence "macho", "machismo", and "Sado-Machocism".

   **SUGGESTED ANSWER:** (a) According to Wikipedia, Mach out of sheer Czechedness pronounced his name "max", but I don't buy that. Anyway there are other things. You've heard of Maching points. And then there's the major-leauge career of one Ernst "Gene" Mach. Oh well, one can talk the talk, but can one Mach the Mach?

   **Wrong answers:**

   d) Or is that wok?

   **Redaction:** Jeffery, 2008jan01

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002 qmult 06010 1 4 3 easy deducto-memory: Mach principle

**Extra keywords:** mathematical physics

3. "Let's play *Jeopardy*! For $100, the answer is: It is the vague hypothesis that the bulk distribution of matter in the universe determines the inertial mass of bodies."

   What is ____________, Alex?

   a) Zeno's paradox
   b) Fermat's last theorem
   c) Mach's principle
   d) Poincaré's conjecture
   e) the Merton thesis

**SUGGESTED ANSWER:** (c)

**Wrong answers:**
a) Zeno had a bit of a problem with the concept of limit.
b) This has actually been proven by Andrew Wiles in 1995.

**Redaction:** Jeffery, 2008jan01

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001 qmult 00410 1 1 4 easy memory: blackbody radiation temperature

4. The blackbody radiation spectrum depends only on the:

a) density of the receiver. b) density of the emitter. c) temperature of the receiver. d) temperature of the emitter. e) the color of the emitter.

**SUGGESTED ANSWER:** (c)

**Wrong answers:**
e) As Lurch would say AAAaargh.

**Redaction:** Jeffery, 2008jan01

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001 qmult 01120 2 1 2 moderate memory: density of states box quantization

5. The density of wavenumber states in wavenumber space (or $k$-space) per space space volume $V$ in the continuum limit for the box-quantization system (or particle-in-a-box system) is:

a) linear $1/V$. b) independent of $V$. c) linear in $V$. d) quadratic in $V$. e) cubic in $V$.

**SUGGESTED ANSWER:** (b) This is actually a remarkable result. Even though the states are non-local in space space (i.e., they are spread through the whole volume), the density is still volume independent. If the states were localized in space space, it would be easy to imagine that there would be a fixed density per volume $V$.

**Wrong answers:**
c) Reasonable guess.

**Redaction:** Jeffery, 2008jan01

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001 qmult 02010 1 4 3 easy deducto-memory: Planck idea
**Extra keywords:** modern physics

6. "Let's play *Jeopardy*! For \$100, the answer is: The person who first proposed that energy states of microscopic systems could form a discrete (or quantized) set instead of a continuum."

Who is ____________, Alex?

a) Thomas Young (1773–1829) b) Lord Rayleigh (1842–1919) c) Max Planck (1858–1947) d) Wilhelm Wien (1864–1928) e) James Jeans (1877–1946)

**SUGGESTED ANSWER:** (c)

**Wrong answers:**
d) Wien blew it. All he had to do was think in the box.

**Redaction:** Jeffery, 2008jan01

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015 qfull 00300 3 5 0 tough thinking: parabolic surface of rotating fluid

7. You have an incompressible fluid of density $\rho$ in a bucket. The bucket is rotating at angular speed $\omega$. Due to viscosity the fluid has come to a steady state where it is co-rotating with the bucket: i.e., each fluid element is rotating with angular speed $\omega$. In the rotating frame of the bucket the fluid is at rest. The bucket frame is, of course, non-inertial, but it can be treated as an inertial frame by introducing the centrifugal force (which is categorized as an inertial or fictitious force). Since fluid is at rest in the rotating frame, the Coriolis force (another inertial force) is zero (Fr-523). The external air pressure is assumed to be a constant $p_{\text{ext}}$.

a) Why can we assume the external air pressure is a constant?

b) In a general sense, what do you expect the shape of the surface of the water to be. **HINT:** Try swirling water in a clear glass. In a rough qualitative way it only matters that the water is rotating something like a rigid rotator: the cup doesn't have to be.

c) What is the centrifugal force per unit mass in a rotating frame (treated as an inerital frame)? Take $r$ to be the horizontal coordinate measured from axis of rotation. **HINT:** In the rotating frame, the centrifugal force is the force that cancels the centripetal force on any body at rest in the rotating frame. The centrifugal force is always present in the rotating frame no matter what other forces are acting.

d) Derive the expression for static equilibrium pressure in the vertical direction $y$ taking $y$ positive as upward. Leave the constant of integration undetermined for the moment: it will depend on $r$ and $p_{\text{ext}}$.

e) Derive the expression for static equilibrium pressure in the horizontal direction $r$. This is done treating the rotating frame as a static reference frame with a centrifugal force. Leave the constant of integration undetermined for the moment: it will depend on $y$ and $p_{\text{ext}}$. **HINT:** Consider the inner and outer horizontal pressures on a differential hollow cylinder of thickness $dr$ and mean surface area $dA$ centered on the rotation axis.

f) Set the zero level of $y$ to be the height of the fluid at the $r = 0$ point. Now combine the two pressure expressions with the boundary condition $p(r = 0, y = 0) = p_{\text{ext}}$, to find pressure as a function of $r$ and $y$. **HINT:** Pressure must have a unique value for each location $(r, y)$.

g) From the general pressure expression found in the part (f) answer, determine the formula for the height $y$ of the surface as a function of $r$.

**SUGGESTED ANSWER:**

a) Air density is about 800 times smaller than water density near the Earth's surface. Thus, air pressure varies about 800 times more slowly with height than water pressure. Since the water pressure variation in the bucket is itself only a small fraction of air pressure (recall it takes a 10-meter depth in water to double air pressure), we can assume that the relative variation of the air pressure in this problem is both much less than 1 and tiny compared to the relative variation in the water pressure. Thus, we can assume air pressure variation is negligible. This conclusion accords with our common perception that air pressure variations are negligible on the human size scale.

b) The water curves up the sides with increasing slope: i.e., it is concave relative the air above.

c) In the rotating frame (e.g., of the bucket), there is a centrifugal force that is the centripetal force is the force that cancels the centripetal force on any objects at rest in the rotating frame. This fact tells us that the centrifugal force per unit mass is

$$\vec{F}_{\text{centrifugal}} = \omega^2 r\hat{r} \ .$$

The centrifugal force is always present in the rotating frame no matter if there is a centripetal force or not.

d) Consider a differential box of mass $m$ in the fluid. For equilibrium in the $y$-direction we have

$$0 = p\,dA - (p + dp)\,dA - mg \ ,$$

where $dA$ is the horizontal area of the box, $p$ is the pressure at the bottom of the box, $p + dp$ is the pressure at the top, and $m = \rho\,dA\,dy$. Thus

$$\frac{dp}{dy} = -\rho g$$

and

$$p(r, y) = -\rho g y + C(r) \ ,$$

where the constant of integration depends on $r$ and the external pressure.

e) Consider a differential hollow cylinder in the fluid centered on the rotation axis. The cylinder has mass $m$ and thickness $dr$, and the mean surface area evaluated in the center of $dr$ is $dA$. For equilibrium in the $r$-direction we have

$$0 = p\,dA - (p + dp)\,dA + m\omega^2 r \ ,$$

where $p$ is the pressure radially outward on the cylinder, $p + dp$ is the pressure radially inward on the cylinder, and $m = \rho\,dA\,dr$. Thus

$$\frac{dp}{dr} = \rho\omega^2 r$$

and

$$p(r, y) = \frac{1}{2}\rho\omega^2 r^2 + C(y) \ ,$$

where the constant of integration depends on $y$ and the external pressure.

f) From equating the two pressure expressions and imposing the boundary condition $p(0, 0) = p_{\rm ext}$, we obtain

$$p(r, y) = \rho\left(\frac{1}{2}\omega^2 r^2 - gy\right) + p_{\rm ext} \ .$$

g) At the surface, $p(r, y) = p_{\rm ext}$, and thus

$$y = \frac{1}{2}\frac{\omega^2}{g} r^2 \ .$$

Thus the surface is parabolic, or more strictly a paraboloid of revolution. Note that the parabola gets curvier as $\omega$ increases or as $g$ decreases. We have neglected cohesion (like-molecule attraction) and adhesion (unlike molecule attraction) effects, but they would be small for anything like water in a bucket.

**Redaction:** Jeffery, 2001jan01

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001 qfull 01020 1 3 0 easy math: wave equation

8. The standard wave equation in 1 dimension is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \ ,$$

where $y$ is the oscillating quantity, $x$ is the 1 space dimension, $t$ is time, and $v$ is the constant phase speed of wave propagation (WA-710). Because this differential equation has more than one independent variable (it has $x$ and $t$ as independent variables), it is a partial differential equation.

a) Verify that $f(x - vt)$ is a general traveling wave solutions of the wave equation where $f(x)$ is any function. What is the initial condition of the solution? What is the direction of propagation of the solution? Consider the wave system as nonrelativistic.

b) In quantum mechanics, it is traditional to write the argument of a 1-dimensional wave as $kx - \omega t$ (rather than $x - vt$), where $k$ is the wavenumber and $\omega$ is the angular frequency. The $\omega$ is always taken as positive and the sign of $k$ determines the direction of a traveling wave: $k > 0$ gives travel in the positive direction and $k < 0$ gives travel in the negative direction.

Since the wave equation is a linear equation, any two solutions can be added to give another solution. You are given two traveling wave solutions $A\sin(kx - \omega t)$ and $A\sin(-kx - \omega t)$, where $A$ is a constant amplitude, $k$ is a positive wave number, $\omega$ is angular frequency, and $\omega/|k| = v$, the phase speed. What is the superposition of the waves (i.e., what is their sum) and what does this superposition amount to physically. **HINT:** The trivial answer is not an answer.

**SUGGESTED ANSWER:**

a) Let $z = x - vt$ and note that

$$\frac{\partial z}{\partial x} = 1 \qquad \text{and} \qquad \frac{\partial z}{\partial t} = -v \ .$$

Now

$$\frac{\partial^2 f(x-vt)}{\partial x^2} = \frac{\partial^2 f(z)}{\partial z^2}\left(\frac{\partial z}{\partial x}\right)^2 = \frac{\partial^2 f(z)}{\partial z^2}$$

and

$$\frac{\partial^2 f(x-vt)}{\partial t^2} = \frac{\partial^2 f(z)}{\partial z^2}\left(\frac{\partial z}{\partial t}\right)^2 = \frac{\partial^2 f(z)}{\partial z^2}v^2 \ .$$

It follows that

$$\frac{\partial^2 f(x-vt)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f(x-vt)}{\partial t^2} \ ,$$

and thus $f(x-vt)$ is a general solutions. The initial condition for the solution is clearly $f(x)$: i.e., $f(x-vt)$ with $t=0$.

The solution is a traveling wave solution. Let $x' = x - vt$ and consider $f(x')$. The function $f(x')$ is just a function shape at rest relative to the $x'$-axis which defines a primed frame. Now if an observer's position $x$ changes as $vt$ just so as to keep $x'$ constant at some value, then the observer observes the rest function shape $f(x')$. Clearly, then the solution is a traveling solution with the phase velocity being $v$. The phase velocity is the velocity at which the shape moves. If $v > 0$, the wave travels in the positive direction. If $v < 0$, the wave travels in the negative direction. If $v = 0$, the wave is static.

b) The superposition gives

$$y = A[\sin(kx-\omega t) + \sin(-kx-\omega t)] = -2A\cos(kx)\sin(\omega t) \ ,$$

where we have used the trignometric identity

$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B) + \sin(A+B)] \ .$$

The superposition is a standing wave since there is no moving waveform, but just an up and down oscillation of the medium at each point $x$ At each point $x$, $|2A\cos(kx)|$ is the amplitude for a simple harmonic motion.

**Redaction:** Jeffery, 2008jan01

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001 qfull 01310 2 3 0 moderate math: Stirling series

**Extra keywords:** This is needed to find the Boltzmann distribution

9. Prove the Stirling's approximation version

$$\ln(N!) \approx \left(N+\frac{1}{2}\right)\ln(N) - N + \frac{3}{2} - \frac{3}{2}\ln\left(\frac{3}{2}\right) + \frac{1}{8N} \ ,$$

where $N$ is an integer greater than or equal to 1. For very large $N$ (as in most of statistical mechanics), one usually uses the simpler and more memorable approximation

$$\ln(N!) \approx N\ln(N) - N \ .$$

Actually, there is a more exact Stirling's approximation. This is the real Stirling's series given by Arf-464 and WA-542. Both our Stirling approximation and the Stirling's series become more accurate as $N$ increases. **HINT:** Write $\ln(N!)$ as a sum and approximate the sum by an analytical integral. A sketch comparing the sum in histogram form and the integrand curve helps to get the best simple choices for the integration boundaries. You will also need to a Taylor's series expansion of a form $\ln(1+x)$ for small $x$.

**SUGGESTED ANSWER:** The integral of the histogram is in fact the sum. But this histogram can't be done analytically. So we make the approximation

$$\ln(N!) = \sum_{n=1}^{N}\ln(n) \approx \int_{3/2}^{N+1/2}\ln(x)\,dx \ .$$

To get the integral $x$, the region associated with each center point of the histogram columns is 1, and thus $dx/1 = dx$ integrates over a column with the right weighting. The first column is zero since $\ln(1) = 0$, and so starting the integral at the beginning of the second column at $x = 3/2$ is the best simple choice. To get the whole of the last column included, the best simple choice is to integrate to $N + 1/2$.

Integrating by parts,

$$\int \ln(x)\,dx = x\ln(x) - \int dx = x\ln(x) - x \ .$$

Thus,

$$\ln(N!) \approx \left(N+\frac{1}{2}\right)\ln\left(N+\frac{1}{2}\right) - \left(N+\frac{1}{2}\right) + \frac{3}{2} - \frac{3}{2}\ln\left(\frac{3}{2}\right) \ .$$

Now

$$\ln\left(N+\frac{1}{2}\right) = \ln(N) + \ln\left(1+\frac{1}{2N}\right) = \ln(N) + \frac{1}{2N} - \frac{1}{8N^2} \ ,$$

where we have Taylor's expanded the second logarithm expression to 2nd order in small $1/N$ and

$$\left(N+\frac{1}{2}\right)\left(\frac{1}{2N} - \frac{1}{8N^2}\right) \approx \frac{1}{2} - \frac{1}{8N} + \frac{1}{4N} = \frac{1}{2} + \frac{1}{8N} \ ,$$

where we have kept only terms up to order $1/N$. Substituting the last expression into the penultimate expression, simplifying, and keeping terms up to order $1/N$ gives required formula

$$\ln(N!) \approx \left(N+\frac{1}{2}\right)\ln(N) - N + \frac{3}{2} - \frac{3}{2}\ln\left(\frac{3}{2}\right) + \frac{1}{8N} \ .$$

This version of the Stirling approximation is pretty good for one obtained by simple means. The real Stirling's series is

$$\ln(z!) \approx \left(z+\frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \ldots \ ,$$

where $z$ is a complex number. One can see tht $1/z$ terms alternate in sign and it is true that the absolute error in the series is less than the absolute value of the first of these terms neglected (Arf-464; WA-542). Our Stirling approximation is the same as the Stirling's series up to the constant term. The difference in the constant terms is, however, quite small:

$$\frac{1}{2}\ln(2\pi) = 0.9189385\ldots \qquad \text{and} \qquad \frac{3}{2} - \frac{3}{2}\ln\left(\frac{3}{2}\right) = 0.8918023\ldots \ .$$

The first order terms in $1/z$ do not agree so well: the Stirling series term has coefficient $1/12$ and our integral approximation gives $1/8$. The $1/8$ may be an over-correction, and so it's as broad as it's long to ignore it. But, of course, our complete integral version should never be used in preference to Stirling's series.

Stirling's series to the 5th order in $1/z$ is actually very accurate. To show this, the following table gives exact results for $\ln(N!)$ and the relative deviation of Stirling's series to various orders. For the case of $N = 1$ the table gives the absolute deviation since one can't take a relative deviation from zero (which $\ln(1!)$ is). The table has been calculated with double precision. As one can see even the lowest order Stirling series (which is what is usually used in statistical mechanics) is quite accurate for $N \gtrsim 50$.

Table: $\ln(N!)$ and the Relative Deviations of Stirling's Series to Various Orders

| N | $\ln(N!)$ | lowest | next | 0th | 1st | 2nd | 3rd |
|---|---|---|---|---|---|---|---|
| 1.0 | 0.00000 | $-0.10E+01$ | $-0.10E+01$ | $-0.81E-01$ | $0.23E-02$ | $-0.51E-03$ | $0.29E-03$ |
| 2.0 | 0.69315 | $-0.19E+01$ | $-0.14E+01$ | $-0.60E-01$ | $0.47E-03$ | $-0.31E-04$ | $0.51E-05$ |
| 10.0 | 15.10441 | $-0.14E+00$ | $-0.61E-01$ | $-0.55E-03$ | $0.18E-06$ | $-0.52E-09$ | $0.39E-11$ |
| 20.0 | 42.33562 | $-0.57E-01$ | $-0.22E-01$ | $-0.98E-04$ | $0.82E-08$ | $-0.58E-11$ | $0.11E-13$ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 50.0 | 148.47777 | $-0.19E-01$ | $-0.62E-02$ | $-0.11E-04$ | $0.15E-09$ | $-0.17E-13$ | $-0.19E-15$ |

Because the series is not a straight power series, it is necessary to define the orders. From the constant term on, we define the orders by their powers of $1/z$. The first two orders are given reasonable names. The approximations to the various orders are:

$$\ln(z!) = \begin{cases} z\ln(z) - z & \text{lowest;} \\ \text{lowest} + \dfrac{1}{2}\ln(z) & \text{next;} \\ \text{next} + \dfrac{1}{2}\ln(2\pi) & \text{0th;} \\ \text{0th} + \dfrac{1}{12z} & \text{1st;} \\ \text{1st} - \dfrac{1}{360z^3} & \text{2nd;} \\ \text{2nd} + \dfrac{1}{1260z^5} & \text{3rd.} \end{cases}$$

**Redaction:** Jeffery, 2008jan01

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001 qfull 01710 2 3 0 moderate math: Planck spectrum

10. There are several different ways of presenting the Planck or blackbody spectrum. They are all equivalent in a sense, but each is most useful in some special case. The commonest one in astrophysical radiative transfer circles is probably the frequency representation of the Planck specific intensity:

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\left[e^{h\nu/(kT)} - 1\right]} ,$$

where $h = 6.6260693(11) \times 10^{-34}\,\mathrm{J\,s}$ is Planck's constant, $c = 2.99792458 \times 10^8\,\mathrm{m/s}$ is the vacuum speed of light, $\nu$ is frequency (in hertz), $k = 1.3806505(24) \times 10^{-23}\,\mathrm{J/K}$ is Boltzmann's constant, and $T$ is temperature in kelvins. What $B_\nu$ is the energy flow per unit time per unit area (perpendicular to the flow direction) per unit frequency per unit solid angle. The energy flow in a particular frequency differential $d\nu$ is $B_\nu\,d\nu$.

Now if you want the wavelength (i.e., $\lambda$) representation, note that

$$\nu = \frac{c}{\lambda} \qquad \text{and} \qquad d\nu = -\frac{c}{\lambda^2}\,d\lambda .$$

For an equivalent energy flow to $B_\nu\,d\nu$ in the wavelength representation, one sets

$$B_\lambda\,d\lambda = B_\nu\,d\nu$$

from which it follows that

$$B_\lambda = B_\nu\frac{d\nu}{d\lambda} = B_\nu\frac{c}{\lambda^2} = \frac{2hc^2}{\lambda^5}\frac{1}{\left[e^{hc/(kT\lambda)} - 1\right]} ,$$

where we got rid of the minus sign for darn good reasons. One should really write $B_\lambda\,d\lambda = B_\nu\,d\nu$ as $B_\lambda(-d\lambda) = B_\nu\,d\nu$ to account for the fact that a differential increase in $\nu$ is a differential decrease in $\lambda$ and we are trying to equate energy flows in a particular band: but no one ever does this since it looks odd: we just know enough to suppress the minus sign.

What is the energy flux $F_\nu$ (energy per unit radiating area per unit time per unit frequency [in the frequency representation]) from a surface radiating like a blackbody? Well imagine a differential patch of surface area $dA$ with outward pointing normal vector: let the angle from normal direction be $\theta$. The amount of area presented by $dA$ perpendicular to a specific intensity beam flowing out at angle $\theta$ is $dA\cos\theta$ which a simple diagram will show. Thus, the differential bit of energy flux emerging from $dA$ in differential solid angle $\sin\theta\,d\theta\,d\phi$ (where $\phi$ is the azimuthal angle) is

$$dF_\nu = B_\nu\cos\theta\sin\theta\,d\theta\,d\phi .$$

Since $B_\nu$ is angle independent we can integrate for $F_\nu$ at once:

$$F_\nu = \int_0^{2\pi}\int_0^{\pi/2} B_\nu \cos\theta \sin\theta \, d\theta \, d\phi = 2\pi \int_0^1 B_\nu \mu \, d\mu = \pi B_\nu = \frac{2\pi h c^2}{\lambda^5} \frac{1}{[e^{hc/(kT\lambda)} - 1]} ,$$

where we have used the transformation $\mu = \cos\theta$ and $d\mu = -\sin\theta \, d\theta$. So the difference between Planck specific intensity and Planck flux is a pesky little factor of $\pi$.

What is the Planck energy density $E_\nu$? Well specific intensity divided by $c$ is the energy density per unit solid angle. The energy density per unit solid angle is $E_\nu/(4\pi)$ since the Planck radiation field is isotropic since it is all a thermodynamic equilibrium radiation field. The division by $c$ can most easily be understood by writing

$$B_\nu = c\frac{E_\nu}{4\pi}$$

and saying (to oneself if no one else) the amount of energy through a bit of area $dA$ perpendicular to the direction of flow in a time $dt$ from a box of volume $dA\,ds$ (where $s$ is the coordinate along the flow direction) is the energy moving in the direction of flow $[E_\nu/(4\pi)]\,dA\,ds$. If one asks for the flow per unit area per unit time (which is just $B_\nu$), one has $E_\nu/(4\pi)]ds/dt$, but photons move at the speed of light and so $ds/dt = c$. So we get the last equation, and so one finds

$$E_\nu = \frac{4\pi}{c} B_\nu = \frac{8\pi h \nu^3}{c^2} \frac{1}{[e^{h\nu/(kT)} - 1]} .$$

In the wavelength representation one has, of course,

$$E_\lambda = \frac{4\pi}{c} B_\lambda = \frac{8\pi h c^2}{\lambda^5} \frac{1}{[e^{hc/(kT\lambda)} - 1]} ,$$

which is just ER-19's equation for the energy density.

Wasn't all that edifying. Now on to the problem.

a) Integrate

$$F_\nu = \frac{2\pi h c^2}{\lambda^5} \frac{1}{[e^{hc/(kT\lambda)} - 1]}$$

over all frequency to find Stefan's Law

$$F = \sigma T^4 ,$$

where $F$ is the fequency-integrated flux and $\sigma = 5.670400(40) \times 10^{-8}\,\mathrm{W/m^2/K^4}$. You should be able to find $\sigma$ in terms of fundamental constants. **HINT:** Change the integration variable to $x = h\nu/(kT)$, remember the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for} \qquad |x| < 1 ,$$

note the factorial function

$$z! = \int_0 t^z e^{-t} \, dt$$

which for $z$ a positive integer $n$ is just $n!$ (Arf-453), and note the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \text{for} \qquad s > 1$$

(Arf-282) gives $\zeta(4) = \pi^4/90$ (Arf-285).

b) Now prove the Wien displacement law:

$$\lambda_{\max} T = \text{constant}$$

where $\lambda_{\max}$ is the maximum of $B_\lambda$ and the Wien constant is $2.8977685(51) \times 10^{-3}\,\mathrm{m\,K}$. Actually the constant cannot be determined exactly analytically. So find a first approximation. **HINT:** Let $x = hc/(kT\lambda)$ and find

$$\frac{dB_\lambda}{d\lambda} = \frac{dB_\lambda}{dx}\frac{dx}{dt} .$$

The maximum of $B_\lambda$ occurs for $dB_\lambda/dx = 0$. Find the maximizing $x$ value to a good first approximation and then the approximate Wien constant.

**SUGGESTED ANSWER:**

a) Behold:

$$\begin{aligned}
F &= \int_0^\infty \frac{2\pi h\nu^3}{c^2} \frac{1}{[e^{h\nu/(kT)} - 1]}\, d\nu = \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty x^3 \frac{1}{[e^x - 1]}\, dx \\
&= \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty x^3 \frac{e^{-x}}{[1 - e^{-x}]}\, dx = \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty x^3 e^{-x} \sum_{n=0}^\infty e^{-nx}\, dx \\
&= \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty x^3 \sum_{n=1}^\infty e^{-nx}\, dx = \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \sum_{n=1}^\infty \int_0^\infty x^3 e^{-nx}\, dx \\
&= \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \sum_{n=1}^\infty \frac{1}{n^4} \int_0^\infty y^3 e^{-y}\, dy = \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \times 3! \times \sum_{n=1}^\infty \frac{1}{n^4} \\
&= \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \times 3! \times \frac{\pi^4}{90} = \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \times \frac{\pi^4}{15} = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3}T^4 \\
&= \sigma T^4 ,
\end{aligned}$$

where we have identified

$$\sigma = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3}$$

which agrees with ER-24.

b) Consider

$$B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{[e^{hc/(kT\lambda)} - 1]} = \frac{2hc^2(kT)^5}{(hc)^5}\frac{x^5}{(e^x - 1)} .$$

Clearly $B_\lambda$ is a minimum at $x = 0$ and $x = \infty$ since it is always positive and it is zero at those points. The $x = 0$ zero can be found from the limit

$$\lim_{x\to 0}\frac{x^5}{(e^x - 1)} = \lim_{x\to 0}\frac{x^5}{(1 + x - 1)} = 0 ,$$

where we have expand the exponential in a Taylor's series to 1st order. Where is it a maximum? It must be where

$$\frac{dB_\lambda}{dx} = 0$$

or where $df/dx = 0$ where

$$f = \frac{x^5}{e^x - 1} .$$

Now

$$\frac{df}{dx} = \frac{5x^4}{e^x - 1} - \frac{x^5 e^x}{(e^x - 1)^2}$$

If we take the limit

$$\begin{aligned}
\lim_{x\to 0}\frac{df}{dx} &= \lim_{x\to 0}\left[\frac{5x^4}{e^x - 1} - \frac{x^5 e^x}{(e^x - 1)^2}\right] \\
&= \lim_{x\to 0}\left(5x^4 e^{-x} - x^5 e^{-x}\right) \\
&= 0 ,
\end{aligned}$$

and so $x = 0$ ($\lambda = \infty$) gives a stationary point, but we've already identified that as a minimum. The value $x = \infty$ ($\lambda = 0$) doesn't solve the equation, the If we take the limit

$$\begin{aligned}
\lim_{x\to\infty}\frac{df}{dx} &= \lim_{x\to\infty}\left[\frac{5x^4}{e^x - 1} - \frac{x^5 e^x}{(e^x - 1)^2}\right] \\
&= \lim_{x\to\infty}\left(5x^3 - x^3\right) \\
&= 0 ,
\end{aligned}$$

and so $x = \infty$ ($\lambda = 0$) gives a stationary point, but we've already identified that as a minimum.

Setting $df/dx = 0$ and rearranging gives.

$$x = 5 \times (1 - e^{-x}) \ .$$

Note $x = 0$ solves this equation, but we've already identified that as a minimum. The value $x = \infty$ doesn't solve the equation, but divisions in the rearrangement mess up the limiting behavior. If we imagine a plot of $y = x$ and $y = 5 \times (1 - e^{-x})$, there must be an intersection since the second curve rises as $5x$ from $x = 0$ which is faster than the first curve, but the second curve eventually plateaus as a constant 5 while the first curve grows to infinity. This intersection is a solution of $x = 5 \times (1 - e^{-x})$ and so is a stationary point and it must be a maximum since the function must have a maximum between the two minima.

Now $x = 5 \times (1 - e^{-x})$ looks like a pretty good iteration equation if we write it as

$$x_i = 5 \times (1 - e^{-x_{i-1}}) \ .$$

If one starts with a good guess for $x_0$ and then iterates, one can expect convergence to $x_\infty$ which is the solution. The reason why it looks like a good iteration equation is that $5(1 - e^{-x})$ is rather slowly varying for $x > 0$ which is the region of interest. As sufficiently slowly varying iteration function usually does the trick. This is not the place to go into the formalism of iteration functions. Let us just guess $x_0 = 0$ and we find $x_1 = 5$ and $x_2 = 5 \times (1 - e^{-5}) \approx 5$. Obviously, 5 is nearly the solution, and so it looks like convergence to about 5 is going to proceed. In any case, our plot $y = x$ and $y = 5(1 - e^{-x})$ suggests that 5 is pretty near the intersection. So let us take $x_{\max} \approx 5$.

Since $x = hc/(kT\lambda)$, we find for the maximizing wavelength that

$$\lambda_{\max} T = \frac{hc}{kx_{\max}} \approx \frac{1.4387751 \times 10^{-2}}{x_{\max}} \approx 0.003$$

using 5 for $x_{\max}$. Thus, we've proven Wien's displacement law and found the Wien constant to within about 4 % though we couldn't know this error without having the modern value given.

Fortran-95 Code

```
      print*
      xmax=5.d0
      print*,'planck,clight,boltzmann in cgs'
      print*,planck,clight,boltzmann
!  6.6260693E-27 2.99792458E+10 1.3806505E-16
      con=(planck*clight/boltzmann)
      con1=con/xmax
      print*,'con,con1 in cgs'
      print*,con,con1
!   1.438775129785083 0.28775502595701663
```

**Redaction:** Jeffery, 2008jan01

# Equation Sheet for Modern Physics

These equation sheets are intended for students writing tests or reviewing material. Therefore they are neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

## 1 Geometrical Formulae

$$C_{\rm cir} = 2\pi r \qquad A_{\rm cir} = \pi r^2 \qquad A_{\rm sph} = 4\pi r^2 \qquad V_{\rm sph} = \frac{4}{3}\pi r^3$$

## 2 Trigonometry

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)] \qquad \sin^2\theta = \frac{1}{2}[1-\cos(2\theta)] \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right] \qquad \sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b)+\sin(a+b)\right]$$

## 3 Blackbody Radiation

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{[e^{h\nu/(kT)}-1]} \qquad B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{[e^{hc/(kT\lambda)}-1]}$$

$$B_\lambda\,d\lambda = B_\nu\,d\nu \qquad \nu\lambda = c \qquad \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$$

$$k = 1.3806505(24)\times10^{-23}\,\mathrm{J/K} \qquad c = 2.99792458\times10^{8}\,\mathrm{m}$$

$$h = 6.6260693(11)\times10^{-34}\,\mathrm{J\,s} = 4.13566743(35)\times10^{-15}\,\mathrm{eV\,s}$$

$$\hbar = \frac{h}{2\pi} = 1.05457168(18)\times10^{-34}\,\mathrm{J\,s}$$

$$hc = 12398.419\,\mathrm{eV\,\AA} \approx 10^4\,\mathrm{eV\,\AA} \qquad E = h\nu = \frac{hc}{\lambda} \qquad p = \frac{h}{\lambda}$$

$$F = \sigma T^4 \qquad \sigma = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3} = 5.670400(40)\times 10^{-8}\,\mathrm{W/m^2/K^4}$$

$$\lambda_{\max} T = \text{constant} = \frac{hc}{kx_{\max}} \approx \frac{1.4387751\times 10^{-2}}{x_{\max}}$$

$$B_{\lambda,\mathrm{Wien}} = \frac{2hc^2}{\lambda^5}e^{-hc/(kT\lambda)} \qquad B_{\lambda,\mathrm{Rayleigh-Jeans}} = \frac{2ckT}{\lambda^4}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\nu = \frac{\omega}{c} \qquad k_i = \frac{\pi}{L}n_i \quad \text{standing wave BCs} \qquad k_i = \frac{2\pi}{L}n_i \quad \text{periodic BCs}$$

$$n(k)\,dk = \frac{k^2}{\pi^2}\,dk = \pi\left(\frac{2}{c}\right)\nu^2\,d\nu = n(\nu)\,d\nu$$

$$\ln(z!) \approx \left(z+\frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \ldots$$

$$\ln(N!) \approx N\ln(N) - N$$

$$\rho(E)\,dE = \frac{e^{-E/(kT)}}{kT}\,dE \qquad P(n) = (1-e^{-\alpha})e^{-n\alpha} \qquad \alpha = \frac{h\nu}{kT}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad f(x-vt) \qquad f(kx-\omega t)$$

## 4 Photons

$$KE = h\nu - w \qquad \Delta\lambda = \lambda_{\mathrm{scat}} - \lambda_{\mathrm{inc}} = \lambda_{\mathrm{C}}(1-\cos\theta)$$

$$\lambda_{\mathrm{C}} = \frac{h}{m_e c} = 2.426310238(16)\times 10^{-12}\,\mathrm{m} \qquad e = 1.602176487(40)\times 10^{-19}\,\mathrm{C}$$

$$m_e = 9.1093826(16)\times 10^{-31}\,\mathrm{kg} = 0.510998918(44)\,\mathrm{MeV}$$

$$m_p = 1.67262171(29)\times 10^{-27}\,\mathrm{kg} = 938.272029(80)\,\mathrm{MeV}$$

$$\ell = \frac{1}{n\sigma} \qquad \rho = \frac{e^{-s/\ell}}{\ell} \qquad \langle s^m\rangle = \ell^m m!$$

## 5 Special Relativity

$$c = 2.99792458\times 10^8\,\mathrm{m/s} \approx 2.998\times 10^8\,\mathrm{m/s} \approx 3\times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns}$$

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \gamma(\beta << 1) = 1 + \frac{1}{2}\beta^2 \qquad \tau = ct$$

| Galilean Transformations | Lorentz Transformations |
|---|---|
| $x' = x - \beta\tau$ | $x' = \gamma(x - \beta\tau)$ |
| $y' = y$ | $y' = y$ |
| $z' = z$ | $z' = z$ |
| $\tau' = \tau$ | $\tau' = \gamma(\tau - \beta x)$ |
| $\beta'_{\text{obj}} = \beta_{\text{obj}} - \beta$ | $\beta'_{\text{obj}} = \dfrac{\beta_{\text{obj}} - \beta}{1 - \beta\beta_{\text{obj}}}$ |

$$\ell = \ell_{\text{proper}}\sqrt{1-\beta^2} \qquad \Delta\tau_{\text{proper}} = \Delta\tau\sqrt{1-\beta^2}$$

$$m = \gamma m_0 \qquad p = mv = \gamma m_0 c\beta \qquad E_0 = m_0c^2 \qquad E = \gamma E_0 = \gamma m_0 c^2 = mc^2$$

$$E = mc^2 \qquad E = \sqrt{(pc)^2 + (m_0c^2)^2}$$

$$KE = E - E_0 = \sqrt{(pc)^2 + (m_0c^2)^2} - m_0c^2 = (\gamma - 1)m_0c^2$$

$$f = f_{\text{proper}}\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{for source and detector separating}$$

$$f(\beta << 1) = f_{\text{proper}}\left(1 - \beta + \frac{1}{2}\beta^2\right)$$

$$f_{\text{trans}} = f_{\text{proper}}\sqrt{1-\beta^2} \qquad f_{\text{trans}}(\beta << 1) = f_{\text{proper}}\left(1 - \frac{1}{2}\beta^2\right)$$

$$\tau = \beta x + \gamma^{-1}\tau' \quad \text{for lines of constant } \tau'$$

$$\tau = \frac{x - \gamma^{-1}x'}{\beta} \quad \text{for lines of constant } x'$$

$$x' = \frac{x_{\text{intersection}}}{\gamma} = x'_{x\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}} \qquad \tau' = \frac{\tau_{\text{intersection}}}{\gamma} = \tau'_{\tau\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}}$$

$$\theta_{\text{Mink}} = \tan^{-1}(\beta)$$