Modern Physics: Physics 305, Section 1

NAME:

Homework 2: Blackbody Radiation Homeworks are due as posted on the course web site. They are **NOT** handed in. The student reports that it is completed and receives one point for this. Solutions are already posted, but students are only permitted to look at the solutions after completion. The solutions are intended to be (but not necessarily are) super-perfect and go beyond a complete answer expected on a test.

1. "Let's play *Jeopardy*! For \$100, the answer is: A thought experiment (which can actually be done) that has been used for arguing for absolute space as a physically active thing."

   What is ____________, Alex?

   a) Newton's bucket experiment b) Maxwell's demon experiment
   c) Einstein's elevator experiment d) Bohr's microscope experiment
   e) Schrödinger's cat experiment

2. Ersnt Mach (1838–1916), a Czech-Austrian physicist, is noted for Mach's principle (first so called by Einstein). This principle, which is really vague hypothesis, is that inertia is determined somehow by the universal distribution of mass. How it does so and what formulae apply have never adequately established. Mach is also famous:

   a) for Mach number. This is the speed of an object relative to a fluid medium in units of the sound speed in that medium.
   b) in ornithology. He is the eponym of the Machingbird.
   c) for work in pyrology. You've heard of machsticks.
   d) the invention of the mach: a kind of cooking pan.
   e) for his forceful character whence "macho", "machismo", and "Sado-Machocism".

3. "Let's play *Jeopardy*! For \$100, the answer is: It is the vague hypothesis that the bulk distribution of matter in the universe determines the inertial mass of bodies."

   What is ____________, Alex?

   a) Zeno's paradox b) Fermat's last theorem c) Mach's principle
   d) Poincaré's conjecture e) the Merton thesis

4. The blackbody radiation spectrum depends only on the:

   a) density of the receiver. b) density of the emitter. c) temperature of the receiver.
   d) temperature of the emitter. e) the color of the emitter.

5. The density of wavenumber states in wavenumber space (or k-space) per space space volume V in the continuum limit for the box-quantization system (or particle-in-a-box system) is:

   a) linear $1/V$. b) independent of V. c) linear in V. d) quadratic in V.
   e) cubic in V.

6. "Let's play *Jeopardy*! For \$100, the answer is: The person who first proposed that energy states of microscopic systems could form a discrete (or quantized) set instead of a continuum."

   Who is ____________, Alex?

   a) Thomas Young (1773–1829) b) Lord Rayleigh (1842–1919)
   c) Max Planck (1858–1947) d) Wilhelm Wien (1864–1928) e) James Jeans (1877–1946)

7. You have an incompressible fluid of density ρ in a bucket. The bucket is rotating at angular speed ω. Due to viscosity the fluid has come to a steady state where it is co-rotating with the bucket: i.e., each fluid element is rotating with angular speed ω. In the rotating frame of the bucket the fluid is at rest. The bucket frame is, of course, non-inertial, but it can be treated as an inertial frame by introducing the centrifugal force (which is categorized as an inertial or fictitious force). Since fluid is at rest in the

rotating frame, the Coriolis force (another inertial force) is zero (Fr-523). The external air pressure is assumed to be a constant p_{ext}.

a) Why can we assume the external air pressure is a constant?

b) In a general sense, what do you expect the shape of the surface of the water to be. **HINT:** Try swirling water in a clear glass. In a rough qualitative way it only matters that the water is rotating something like a rigid rotator: the cup doesn't have to be.

c) What is the centrifugal force per unit mass in a rotating frame (treated as an inerital frame)? Take r to be the horizontal coordinate measured from axis of rotation. **HINT:** In the rotating frame, the centrifugal force is the force that cancels the centripetal force on any body at rest in the rotating frame. The centrifugal force is always present in the rotating frame no matter what other forces are acting.

d) Derive the expression for static equilibrium pressure in the vertical direction y taking y positive as upward. Leave the constant of integration undetermined for the moment: it will depend on r and p_{ext}.

e) Derive the expression for static equilibrium pressure in the horizontal direction r. This is done treating the rotating frame as a static reference frame with a centrifugal force. Leave the constant of integration undetermined for the moment: it will depend on y and p_{ext}. **HINT:** Consider the inner and outer horizontal pressures on a differential hollow cylinder of thickness dr and mean surface area dA centered on the rotation axis.

f) Set the zero level of y to be the height of the fluid at the $r = 0$ point. Now combine the two pressure expressions with the boundary condition $p(r = 0, y = 0) = p_{\text{ext}}$, to find pressure as a function of r and y. **HINT:** Pressure must have a unique value for each location (r, y).

g) From the general pressure expression found in the part (f) answer, determine the formula for the height y of the surface as a function of r.

8. The standard wave equation in 1 dimension is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} ,$$

where y is the oscillating quantity, x is the 1 space dimension, t is time, and v is the constant phase speed of wave propagation (WA-710). Because this differential equation has more than one independent variable (it has x and t as independent variables), it is a partial differential equation.

a) Verify that $f(x - vt)$ is a general traveling wave solutions of the wave equation where $f(x)$ is any function. What is the initial condition of the solution? What is the direction of propagation of the solution? Consider the wave system as nonrelativistic.

b) In quantum mechanics, it is traditional to write the argument of a 1-dimensional wave as $kx - \omega t$ (rather than $x - vt$), where k is the wavenumber and ω is the angular frequency. The ω is always taken as positive and the sign of k determines the direction of a traveling wave: $k > 0$ gives travel in the positive direction and $k < 0$ gives travel in the negative direction.

Since the wave equation is a linear equation, any two solutions can be added to give another solution. You are given two traveling wave solutions $A\sin(kx - \omega t)$ and $A\sin(-kx - \omega t)$, where A is a constant amplitude, k is a positive wave number, ω is angular frequency, and $\omega/|k| = v$, the phase speed. What is the superposition of the waves (i.e., what is their sum) and what does this superposition amount to physically. **HINT:** The trivial answer is not an answer.

9. Prove the Stirling's approximation version

$$\ln(N!) \approx \left(N + \frac{1}{2}\right)\ln(N) - N + \frac{3}{2} - \frac{3}{2}\ln\left(\frac{3}{2}\right) + \frac{1}{8N} ,$$

where N is an integer greater than or equal to 1. For very large N (as in most of statistical mechanics), one usually uses the simpler and more memorable approximation

$$\ln(N!) \approx N\ln(N) - N .$$

Actually, there is a more exact Stirling's approximation. This is the real Stirling's series given by Arf-464 and WA-542. Both our Stirling approximation and the Stirling's series become more accurate as N increases. **HINT:** Write $\ln(N!)$ as a sum and approximate the sum by an analytical integral. A sketch comparing the sum in histogram form and the integrand curve helps to get the best simple choices for the integration boundaries. You will also need to a Taylor's series expansion of a form $\ln(1+x)$ for small x.

10. There are several different ways of presenting the Planck or blackbody spectrum. They are all equivalent in a sense, but each is most useful in some special case. The commonest one in astrophysical radiative transfer circles is probably the frequency representation of the Planck specific intensity:

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{[e^{h\nu/(kT)} - 1]} ,$$

where $h = 6.6260693(11) \times 10^{-34}\,\mathrm{J\,s}$ is Planck's constant, $c = 2.99792458 \times 10^8\,\mathrm{m/s}$ is the vacuum speed of light, ν is frequency (in hertz), $k = 1.3806505(24) \times 10^{-23}\,\mathrm{J/K}$ is Boltzmann's constant, and T is temperature in kelvins. What B_ν is the energy flow per unit time per unit area (perpendicular to the flow direction) per unit frequency per unit solid angle. The energy flow in a particular frequency differential $d\nu$ is $B_\nu\,d\nu$.

Now if you want the wavelength (i.e., λ) representation, note that

$$\nu = \frac{c}{\lambda} \qquad \text{and} \qquad d\nu = -\frac{c}{\lambda^2}\,d\lambda \ .$$

For an equivalent energy flow to $B_\nu\,d\nu$ in the wavelength representation, one sets

$$B_\lambda\,d\lambda = B_\nu\,d\nu$$

from which it follows that

$$B_\lambda = B_\nu \frac{d\nu}{d\lambda} = B_\nu \frac{c}{\lambda^2} = \frac{2hc^2}{\lambda^5} \frac{1}{[e^{hc/(kT\lambda)} - 1]} ,$$

where we got rid of the minus sign for darn good reasons. One should really write $B_\lambda\,d\lambda = B_\nu\,d\nu$ as $B_\lambda(-d\lambda) = B_\nu\,d\nu$ to account for the fact that a differential increase in ν is a differential decrease in λ and we are trying to equate energy flows in a particular band: but no one ever does this since it looks odd: we just know enough to suppress the minus sign.

What is the energy flux F_ν (energy per unit radiating area per unit time per unit frequency [in the frequency representation]) from a surface radiating like a blackbody? Well imagine a differential patch of surface area dA with outward pointing normal vector: let the angle from normal direction be θ. The amount of area presented by dA perpendicular to a specific intensity beam flowing out at angle θ is $dA\cos\theta$ which a simple diagram will show. Thus, the differential bit of energy flux emerging from dA in differential solid angle $\sin\theta\,d\theta\,d\phi$ (where ϕ is the azimuthal angle) is

$$dF_\nu = B_\nu \cos\theta \sin\theta\,d\theta\,d\phi \ .$$

Since B_ν is angle independent we can integrate for F_ν at once:

$$F_\nu = \int_0^{2\pi}\int_0^{\pi/2} B_\nu \cos\theta\sin\theta\,d\theta\,d\phi = 2\pi\int_0^1 B_\nu\mu\,d\mu = \pi B_\nu = \frac{2\pi hc^2}{\lambda^5}\frac{1}{[e^{hc/(kT\lambda)} - 1]} ,$$

where we have used the transformation $\mu = \cos\theta$ and $d\mu = -\sin\theta\,d\theta$. So the difference between Planck specific intensity and Planck flux is a pesky little factor of π.

What is the Planck energy density E_ν? Well specific intensity divided by c is the energy density per unit solid angle. The energy density per unit solid angle is $E_\nu/(4\pi)$ since the Planck radiation field is isotropic since it is all a thermodynamic equilibrium radiation field. The division by c can most easily be understood by writing

$$B_\nu = c\frac{E_\nu}{4\pi}$$

and saying (to oneself if no one else) the amount of energy through a bit of area dA perpendicular to the direction of flow in a time dt from a box of volume $dA\,ds$ (where s is the coordinate along the flow direction) is the energy moving in the direction of flow $[E_\nu/(4\pi)]\,dA\,ds$. If one asks for the flow per unit area per unit time (which is just B_ν), one has $E_\nu/(4\pi)]ds/dt$, but photons move at the speed of light and so $ds/dt = c$. So we get the last equation, and so one finds

$$E_\nu = \frac{4\pi}{c} B_\nu = \frac{8\pi h\nu^3}{c^2} \frac{1}{[e^{h\nu/(kT)} - 1]} \ .$$

In the wavelength representation one has, of course,

$$E_\lambda = \frac{4\pi}{c} B_\lambda = \frac{8\pi hc^2}{\lambda^5} \frac{1}{[e^{hc/(kT\lambda)} - 1]} \ ,$$

which is just ER-19's equation for the energy density.

Wasn't all that edifying. Now on to the problem.

a) Integrate

$$F_\nu = \frac{2\pi hc^2}{\lambda^5} \frac{1}{[e^{hc/(kT\lambda)} - 1]}$$

over all frequency to find Stefan's Law

$$F = \sigma T^4 \ ,$$

where F is the fequency-integrated flux and $\sigma = 5.670400(40) \times 10^{-8}\,\mathrm{W/m^2/K^4}$. You should be able to find σ in terms of fundamental constants. **HINT:** Change the integration variable to $x = h\nu/(kT)$, remember the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for} \qquad |x| < 1 \ ,$$

note the factorial function

$$z! = \int_0 t^z e^{-t}\,dt$$

which for z a positive integer n is just $n!$ (Arf-453), and note the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \text{for} \qquad s > 1$$

(Arf-282) gives $\zeta(4) = \pi^4/90$ (Arf-285).

b) Now prove the Wien displacement law:

$$\lambda_{\max} T = \text{constant}$$

where $\lambda_{\max}$ is the maximum of B_λ and the Wien constant is $2.8977685(51) \times 10^{-3}\,\mathrm{m\,K}$. Actually the constant cannot be determined exactly analytically. So find a first approximation. **HINT:** Let $x = hc/(kT\lambda)$ and find

$$\frac{dB_\lambda}{d\lambda} = \frac{dB_\lambda}{dx}\frac{dx}{dt} \ .$$

The maximum of B_λ occurs for $dB_\lambda/dx = 0$. Find the maximizing x value to a good first approximation and then the approximate Wien constant.

Equation Sheet for Modern Physics

These equation sheets are intended for students writing tests or reviewing material. Therefore they are neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

1 Geometrical Formulae

$$C_{\rm cir} = 2\pi r \qquad A_{\rm cir} = \pi r^2 \qquad A_{\rm sph} = 4\pi r^2 \qquad V_{\rm sph} = \frac{4}{3}\pi r^3$$

2 Trigonometry

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)] \qquad \sin^2\theta = \frac{1}{2}[1-\cos(2\theta)] \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right] \qquad \sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$$

3 Blackbody Radiation

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{[e^{h\nu/(kT)} - 1]} \qquad B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{[e^{hc/(kT\lambda)} - 1]}$$

$$B_\lambda\, d\lambda = B_\nu\, d\nu \qquad \nu\lambda = c \qquad \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$$

$$k = 1.3806505(24)\times 10^{-23}\,\mathrm{J/K} \qquad c = 2.99792458\times 10^{8}\,\mathrm{m}$$

$$h = 6.6260693(11)\times 10^{-34}\,\mathrm{J\,s} = 4.13566743(35)\times 10^{-15}\,\mathrm{eV\,s}$$

$$\hbar = \frac{h}{2\pi} = 1.05457168(18)\times 10^{-34}\,\mathrm{J\,s}$$

$$hc = 12398.419\,\mathrm{eV\,Å} \approx 10^4\,\mathrm{eV\,Å} \qquad E = h\nu = \frac{hc}{\lambda} \qquad p = \frac{h}{\lambda}$$

$$F = \sigma T^4 \qquad \sigma = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3} = 5.670400(40) \times 10^{-8}\,\mathrm{W/m^2/K^4}$$

$$\lambda_{\mathrm{max}} T = \mathrm{constant} = \frac{hc}{kx_{\mathrm{max}}} \approx \frac{1.4387751 \times 10^{-2}}{x_{\mathrm{max}}}$$

$$B_{\lambda,\mathrm{Wien}} = \frac{2hc^2}{\lambda^5} e^{-hc/(kT\lambda)} \qquad B_{\lambda,\mathrm{Rayleigh-Jeans}} = \frac{2ckT}{\lambda^4}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\nu = \frac{\omega}{c} \qquad k_i = \frac{\pi}{L} n_i \quad \text{standing wave BCs} \qquad k_i = \frac{2\pi}{L} n_i \quad \text{periodic BCs}$$

$$n(k)\,dk = \frac{k^2}{\pi^2}\,dk = \pi\left(\frac{2}{c}\right)\nu^2\,d\nu = n(\nu)\,d\nu$$

$$\ln(z!) \approx \left(z + \frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \ldots$$

$$\ln(N!) \approx N\ln(N) - N$$

$$\rho(E)\,dE = \frac{e^{-E/(kT)}}{kT}\,dE \qquad P(n) = (1 - e^{-\alpha})e^{-n\alpha} \qquad \alpha = \frac{h\nu}{kT}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad f(x - vt) \qquad f(kx - \omega t)$$

4 Photons

$$KE = h\nu - w \qquad \Delta\lambda = \lambda_{\mathrm{scat}} - \lambda_{\mathrm{inc}} = \lambda_{\mathrm{C}}(1 - \cos\theta)$$

$$\lambda_{\mathrm{C}} = \frac{h}{m_e c} = 2.426310238(16) \times 10^{-12}\,\mathrm{m} \qquad e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$m_e = 9.1093826(16) \times 10^{-31}\,\mathrm{kg} = 0.510998918(44)\,\mathrm{MeV}$$

$$m_p = 1.67262171(29) \times 10^{-27}\,\mathrm{kg} = 938.272029(80)\,\mathrm{MeV}$$

$$\ell = \frac{1}{n\sigma} \qquad \rho = \frac{e^{-s/\ell}}{\ell} \qquad \langle s^m \rangle = \ell^m m!$$

5 Special Relativity

$$c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 2.998 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns}$$

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \gamma(\beta << 1) = 1 + \frac{1}{2}\beta^2 \qquad \tau = ct$$

Galilean Transformations	Lorentz Transformations
$x' = x - \beta\tau$	$x' = \gamma(x - \beta\tau)$
$y' = y$	$y' = y$
$z' = z$	$z' = z$
$\tau' = \tau$	$\tau' = \gamma(\tau - \beta x)$
$\beta'_{\text{obj}} = \beta_{\text{obj}} - \beta$	$\beta'_{\text{obj}} = \dfrac{\beta_{\text{obj}} - \beta}{1 - \beta\beta_{\text{obj}}}$

$$\ell = \ell_{\text{proper}}\sqrt{1-\beta^2} \qquad \Delta\tau_{\text{proper}} = \Delta\tau\sqrt{1-\beta^2}$$

$$m = \gamma m_0 \qquad p = mv = \gamma m_0 c\beta \qquad E_0 = m_0c^2 \qquad E = \gamma E_0 = \gamma m_0 c^2 = mc^2$$

$$E = mc^2 \qquad E = \sqrt{(pc)^2 + (m_0c^2)^2}$$

$$KE = E - E_0 = \sqrt{(pc)^2 + (m_0c^2)^2} - m_0c^2 = (\gamma - 1)m_0c^2$$

$$f = f_{\text{proper}}\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{for source and detector separating}$$

$$f(\beta << 1) = f_{\text{proper}}\left(1 - \beta + \frac{1}{2}\beta^2\right)$$

$$f_{\text{trans}} = f_{\text{proper}}\sqrt{1-\beta^2} \qquad f_{\text{trans}}(\beta << 1) = f_{\text{proper}}\left(1 - \frac{1}{2}\beta^2\right)$$

$$\tau = \beta x + \gamma^{-1}\tau' \quad \text{for lines of constant } \tau'$$

$$\tau = \frac{x - \gamma^{-1}x'}{\beta} \quad \text{for lines of constant } x'$$

$$x' = \frac{x_{\text{intersection}}}{\gamma} = x'_{x\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}} \qquad \tau' = \frac{\tau_{\text{intersection}}}{\gamma} = \tau'_{\tau\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}}$$

$$\theta_{\text{Mink}} = \tan^{-1}(\beta)$$