Modern Physics: Physics 305, Section 1

NAME:

Homework 1: Special Relativity Homeworks are due as posted on the course web site. They are **NOT** handed in. The student reports that they are completed and receives one point for each. Solutions are already posted, but students are only permitted to look at the solutions after completion. The solutions are intended to be (but not necessarily are) super-perfect and go beyond a complete answer expected on a test.

---

001 qmult 00010 1 1 5 easy memory: clearly in physics

Extra keywords: Not a serious question.

1. In physics jargon, the word "clearly" means:

   a) clearly. b) unclearly. c) after 4 pages of algebra. d) wrongly. e) all of the above.

   SUGGESTED ANSWER: (e) It seems to me that this is the right answer. When someone says "clearly" in a physics argument, one needs to reflect on their character.

   Wrong answers:
   a) Only sometimes.
   b) Frequently.
   c) Pretty often.
   d) Pretty often too.

   Redaction: Jeffery, 2001jan01

---

001 qmult 00020 1 1 5 easy memory: must be in physics

Extra keywords: Not a serious question.

2. In physics jargon, the phrase "must be" means:

   a) is. b) just accept it that. c) not necessarily so. d) can't be. e) all of the above.

   SUGGESTED ANSWER: (e) It seems to me that this is the right answer. When someone says "must be" in a physics argument, what they mean depends on their stage of desperation.

   Wrong answers:
   a) Only sometimes.
   b) Frequently.
   c) Pretty often.
   d) Pretty often too.

   Redaction: Jeffery, 2001jan01

---

001 qmult 00070 1 4 5 easy deducto-memory: seven samurai

Extra keywords: not a serious question

3. "Let's play *Jeopardy*! For $100, the answer is: In Akira Kurosawa's film *The Seven Samurai* in the misremembering of popular memory, what the samurai leader said when one of the seven asked why they were going to defend this miserable village from a horde of marauding bandits."

   What is "____________," Alex?

   a) For honor. b) It is the way of the samurai. c) It is the Tao. d) For a few dollars more. e) For the fun of it.

   SUGGESTED ANSWER: (e)

   Wrong answers:

d) Actually a few bowls of rise were part of the deal.

Redaction: Jeffery, 2008jan01

---

001 qmult 00090 1 1 3 easy memory: Greene's fabric

4. Brian Greene probably titled his popular book on modern physics *The Fabric of the Cosmos* mostly maybe because:

   a) he's a proponent of superstring theory.
   b) he's **NOT** a proponent of superstring theory.
   c) in imitation of Stephen Toulmin and June Goodfield's *The Fabric of the Heavens*.
   d) in the modern age every book has to have a farfetched metaphorical title like *The God Particle* or *The Snail's Ear*: a title like *A Popular Account of Modern Particle Physics and Cosmology* just doesn't cut it.
   e) of random processes.

   SUGGESTED ANSWER: (a) I'm taking no arguments on this one. Yes, Greene sometimes uses fabric and web for other things than strings (e.g., the spacetime continuum), but I think the strings theme had to be decisive. I could be wrong—in fact the more I read Greene, the more I'm sure I'm wrong–but I'm still taking no arguments.

   Wrong answers:

   c) My vague recollection was that the Toulmin and Goodfield book was titled the *The Fabric of the Cosmos* and I thought that Greene recycled it since titles can't be copyrighted.

   Redaction: Jeffery, 2008jan01

---

001 qmult 00095 1 4 5 easy memory: Brian Greene's ocean
Extra keywords: Gre-580

5. On the back cover of Brian Greene's *The Fabric of the Cosmos* (pocket-size paperback), Brian Greene (one supposes) is in front of an ocean. Which ocean and why?

   a) Greene lives in New York state, and so it's probably the Atlantic Ocean.
   b) It's not an ocean. One can descry Port Colborne, Ontario on the horizon. He's in front of Lake Erie on the New York State side. In fact, he's probably at Angola-on-the-Lake—which is not in Africa whatever you may think.
   c) Well . . .

      . . . like stout Cortez when with eagle eyes
      He star'd at the Pacific—and all his men
      Look'd at each other with a wild surmise—
      Silent, upon a peak in Darien.

   d) Quoting Newton:

      I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

      That's the one.
   e) I've no idea.

   SUGGESTED ANSWER: (e)

   Wrong answers:

   b) My hometown is Port Colborne—and yes I think I can make out my birthplace.
   c) Keats may have known it was Vasco Nuñez De Balboa (1475—1519) who was the first European to see the Pacific from the east side. He may have been invoking the shade of

Cortez (Hernán Cortés [1485–1547]) as a symbol of willpower—these days a conflicted one.

d) I hardly think there could be a copyrighted photo of this one.

Redaction: Jeffery, 2008jan01

---

001 qmult 06030 1 4 5 easy deducto-memory: superstring theory defn.
Extra keywords: Gre-17–18

6. "Let's play *Jeopardy*! For $100, the answer is: In this physical theory (circa 2004 at least), the basic element of matter is a string/filament/little-thingy which vibrates in different ways to make the fundamental particles (e.g., electron, neutrino, quark). The theory requires 9 or 10 space dimensions plus 1 time dimension and thus 10 or 11 spacetime dimensions. The higher numbers are for the version called M-theory."

What is ____________, Alex?

a) Aristotelian physics b) Newtonian physics c) Einsteinian relativistic physics
d) quantum mechancis e) superstring theory

SUGGESTED ANSWER: (e)

Wrong answers:

a) As Lurch would say AAAARGH.

Redaction: Jeffery, 2008jan01

---

001 qmult 06040 2 3 3 moderate math: Planck units
Extra keywords: Gre-17

7. The Planck units are quantities constructed by dimensional analysis from 5 fundamental constants:

$$\begin{cases} c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} & \text{the vacuum light speed;} \\ G = 6.67428(67) \times 10^{-11}\,\mathrm{J\,m/kg^2} \approx 7 \times 10^{-11}\,\mathrm{J\,m/kg^2} & \text{the gravitational constant which can also be given the units } \mathrm{m^5/(J\,s^4)}; \\ \hbar = 1.054571628(53) \times 10^{-34}\,\mathrm{J\,s} \approx 1 \times 10^{-34}\,\mathrm{J\,s} & \text{which is h-bar or Dirac's constant or Planck's constant divided by } 2\pi; \\ k_{\mathrm{C}} = \dfrac{1}{4\pi\epsilon_0} = 8.987551787 \times 10^9\,\mathrm{J\,m/C^2} \approx 9 \times 10^9\,\mathrm{J\,m/C^2} & \text{the Coulomb force constant where } \epsilon_0 = 8.854187817 \times 10^{12}\,\mathrm{F/m} \text{ is the vacuum permittivity;} \\ k = 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} \approx 1.4 \times 10^{-23}\,\mathrm{J/K} & \text{which is Boltzmann's constant,} \end{cases}$$

where the values have been taken from Wikipedia (2007oct21). The Planck units (originally proposed by Max Planck) are based only on general universal physics and not arbitrary human choices. They should have some fundamental significance and are often ingredients in advanced theory. Its helpful in constructing Planck units to note that G/c^4 has units of m/J and G/c^5 has units of s/J (which incidentally makes it the inverse of the Planck power).

Brian Greene (Gre-17), in the customary arcane jargon of grand high theorists, refers to a length "some hundred billion billion times smaller than a single atomic nucleus." Atomic nuclei have a size scale of order 10^{-15} m. Evidently, he is referring to the Planck length. Construct the Planck length from the above constants and evaluate it approximately.

a) $\sqrt{\hbar c^5/G} \approx 1.8 \times 10^9\,\mathrm{m}$ b) $\sqrt{\hbar G/c^5} \approx 5 \times 10^{-44}\,\mathrm{m}$

c) $\sqrt{\hbar G/c^3} \approx 1.6 \times 10^{-35}\,\mathrm{m}$

d) $\sqrt{\hbar c/G} \approx 2 \times 10^{-8}\,\mathrm{m}$

e) $\sqrt{c^5/(\hbar G)} \approx 2 \times 10^{43}\,\mathrm{m}$

SUGGESTED ANSWER: (c) Well $\hbar G/c^5$ has units of s^2, and so multiplying $\hbar G/c^5$ by c^2 and taking the square root is the answer.

Wrong answers:

a) Planck energy in joules.
b) Planck time in seconds.
d) Planck mass in kg.
e) Planck frequency in hertz.

Redaction: Jeffery, 2008jan01

---

001 qmult 08010 1 1 1 easy thinking: coming of age

8. Nowadays takeoffs on cultural detritus of all kinds frequently occur without any acknowledgement. The section title *Coming of Age in Space and Time* of Greene's chapter 1, section 8 is probably a takeoff on:

a) both of (b) and (c) maybe.
b) *Coming of Age in the Milky Way* (1989) by Timothy Ferris (1944–).
c) *Coming of Age in Somoa* (1928) by Margaret Mead (1901–1978).
d) *The Waning of the Middle Ages* (1924) by Johan Huizinga (1872–1945).
e) *Coming of Nonage in Fermullan* by Thomas Caskey, Sr. (1883–1964 or so).

SUGGESTED ANSWER: (a) I think Ferris popularized the takeoff on Mead and Greene was thinking of Ferris, but was aware of Mead.

Wrong answers:

e) Grandad probably read little and wrote less. It's shameful that I don't know his dates exactly.

Redaction: Jeffery, 2008jan01

---

038 qmult 00100 1 1 2 easy memory: Galilean transformations

9. Classical Newtonian laws of physics are inertial frame covariant under:

a) the Lorentz transformations. b) the Galilean transformations. c) no transformations at all. d) the Keplerian transformations. e) the Maxwellian transformations.

SUGGESTED ANSWER: (b)

Wrong answers:

a) Exactly wrong out the choice of two real transformations listed.

Redaction: Jeffery, 2001jan01

---

038 qmult 00110 1 4 1 easy deducto-memory: Heinrich Hertz

10. Heinrich Hertz (1857–1894) experimentally verified the existence of:

a) electromagnetic waves. b) light. c) an upper limit to all physical velocities. d) gravity. e) gravitational radiation.

SUGGESTED ANSWER: (a)

Wrong answers:

e) Ah, circa 2001, the LIGO detector is set to try to verify the existence of these critters predicted by general relativity. Probably it won't do it. Any few orders of magnitude of sensitivity is probably needed.

Redaction: Jeffery, 2001jan01

---

038 qmult 00200 1 1 1 easy memory: discovery special relativity
Extra keywords: physci

11. Einstein first published his theory of special relativity in:

a) 1905. b) 1879. c) 1955. d) 1776. e) 1066.

SUGGESTED ANSWER: (a)

Wrong answers:

b) Year of Einstein's birth.
c) Year of Einstein's death.
d) Year of the American Revolution.
e) The year of the Norman conquest of England and the Battle of Hastings: 1066 and all that.

Redaction: Jeffery, 2001jan01

---

038 qmult 00300 1 1 2 easy memory: 2 special relativity postulates
Extra keywords: physci KB-94-19

12. Einstein developed special relativity starting from ____________ basic postulates—at least in the accounts of most textbooks—which may be thinking of the ideal Einstein rather than of the historical Einstein.

a) zero b) two c) three d) four e) infinite

SUGGESTED ANSWER: (b) The relativity postulate and the vacuum speed of light postulate

Wrong answers:

e) As Lurch would say: "Aaaarh."

Redaction: Jeffery, 2001jan01

---

038 qmult 00310 1 1 3 easy memory: relativity postulate 1
Extra keywords: physci KB-92-11, KB-94-19

13. According to the principle (or postulate) of relativity of special relativity, the laws of physics or, more exactly, the expressions of physical law are the same in:

a) all frames of reference.
b) most frames of reference.
c) all inertial frames (i.e., fundamentally unaccelerated frames) of reference.
d) all non-inertial frames (i.e., fundamentally accelerated frames) of reference.
e) no frames of reference

SUGGESTED ANSWER: (c) The answer without the "more exactly" phrase is the way the relativity principle is usually expressed in elementary books (KB-81; HRW-921). And as a shorthand so I guess it is adequate. But it makes me a bit squimish in that it implies the laws of physics are different in non-inertial frames which is not true. One should say that physical law expressions should be the same in all inertial frames (La-5). Non-inertial frames can be treated too by making appropriate adjustments. But that is longwinded and, perhaps, obscuring for elementary students. In fact, in general relativity the relativity postulate is that the expressions of physical law are the same in all frames of reference. But I'm not sure if this is definitely proven.

Wrong answers:

e) All things are wrong.

Redaction: Jeffery, 2001jan01

---

038 qmult 00330 1 1 4 easy memory: light speed postulate
Extra keywords: physci KB-94-19

14. The vacuum speed of light has the same value, labeled by c, in all directions in:

    a) some inertial frames. b) frames moving with the Earth only. c) no frames. d) all inertial frames. e) no inertial frames.

SUGGESTED ANSWER: (d) See La-6.

Wrong answers:

e) Exactly wrong.

Redaction: Jeffery, 2001jan01

---

038 qmult 00400 1 4 2 easy deducto-memory: revising Newtonian mechanics
Extra keywords: physci

15. Einstein's two special relativity postulates implied that classical Newtonian physics:

    a) needed no revision. b) needed revision. c) was wildly wrong in all cases of interest. d) was not correct now, but would be in the future. e) was completely useless.

SUGGESTED ANSWER: (b)

Wrong answers:

a) It sure did need revision. New transformations (i.e., the Lorentz transformations), new rules for kinetic energy, momentum, and force, a 4-vector formulation, and other things. Of course, for much of the human realm the revisions are of negligible.

c) C'mon. It works fine for all low velocity, macroscopic cases outside of strong gravitational fields.

d) Huh?

e) No way. Classical Newtonian mechanics has a vast realm relevant to human and natural activities in which it is entirely adequate or almost so. Relativistic and quantum corrections in this realm are negligible. This realm includes macroscopic engineering and most celestial mechanics.

Redaction: Jeffery, 2001jan01

---

038 qmult 00500 1 1 5 easy memory: Lorentz transformations

16. In special relativity, the spacetime coordinates in different inertial frames are related by the ____________ transformations.

    a) Galilean b) Keplerian c) Newtonian d) Maxwellian e) Lorentz

SUGGESTED ANSWER: (e)

Wrong answers:

a) This is the classical or non-relativistic result. It is an excellent approximation when $v/c << 1$.

Redaction: Jeffery, 2001jan01

---

038 qmult 00610 1 4 1 easy deducto-memory: Fitzgerald contraction

17. The proper or rest-frame length L_0 parallel to the direction of motion of an object moving uniformly relative to an inertial frame S is related to the length L measured by an observer at rest in S by the Fitzgerald contraction formula:

    a) $L = L_0\sqrt{1-\beta^2}$. b) $L = L_0/\sqrt{1-\beta^2}$. c) $L = L_0$. d) $L = 1/L_0$. e) $L = L_0^2$.

SUGGESTED ANSWER: (a)

Wrong answers:
b) Exactly wrong. It is easy to be exactly wrong in relativistic problems.
c) This is the classical or non-relativistic result. It is an excellent approximation when $v/c << 1$.
d) Dimensionally wrong: i.e., the units are wrong.
e) Dimensionally wrong.

Redaction: Jeffery, 2001jan01

---

038 qmult 00700 1 4 4 easy deducto-memory: relativistic mass increase
Extra keywords: physci KB-92-12

18. As an object's velocity increases relative to some observer, the mass of the object as measured by that observer:

a) goes to zero. b) decreases. c) stays constant. d) increases and approaches infinity as the speed approaches the vacuum light speed. e) is infinite.

SUGGESTED ANSWER: (d)

Wrong answers:
e) As Lurch would say: "Aaaarh."

Redaction: Jeffery, 2001jan01

---

038 qmult 00800 1 1 2 easy memory: time dilation mnemonic
Extra keywords: physci

19. The mnemonic for the time dilation effect is "moving clocks:

a) run fast." b) run slow." c) are stopped." d) are right twice a day." e) run backward."

SUGGESTED ANSWER: (b)

Wrong answers:
e) You can make a mechanical clock run backwards, of course, but you can't really make time run backward. Entropy always increases in a closed system and time hurrying chariot doesn't step into the same stream twice.

Redaction: Jeffery, 2001jan01

---

038 qmult 00900 1 4 3 easy deducto-memory: twins paradox
Extra keywords: physci

20. In the twins paradox, the twin who ages least is the one who is:

a) not accelerated. b) not Fitzgerald contracted. c) accelerated. d) Fitzgerald contracted. e) hermetically sealed in a time capsule.

SUGGESTED ANSWER: (c)

Wrong answers:
a) Exactly wrong.
b) Either twin can see the other as Fitzgerald contracted. And both are compared to observers in any other inertial frame.
e) Hermetically means air-tight. Either twin could be sealed in a time capsule for all the good it would do.

Redaction: Jeffery, 2001jan01

---

038 qmult 01000 1 4 3 easy deducto-memory: Minkowski diagrams

21. "Let's play *Jeopardy*! For $100, the answer is: It can be divided into Past, Future, Now, and Elsewhen, and on it world-lines evolve."

   a) What is ____________, Alex?

      a) a conformal mapping b) a Feynman diagram c) a Minkowski diagram d) history e) life

   SUGGESTED ANSWER: (c)

   Wrong answers:
   a) Say what?
   b) Very useful in physics, not that I know anything about them.
   d) There's no Elsewhen in history.
   e) Not the best answer in the context of a physics class.

   Redaction: Jeffery, 2001jan01

---

038 qmult 01010 1 4 1 easy deducto-memory: world line

22. A world line in a Minkowski diagram is:

   a) the trajectory of an object in spacetime.
   b) the trajectory of an object in space alone.
   c) trajectory of an object in time alone.
   d) the arrow of time.
   e) a telephone that is a party line for the entire world—recommended for exhibitionists.

   SUGGESTED ANSWER: (a)

   Wrong answers:
   d) In physics jargon the arrow of time is entropy.
   e) Maybe so, but not the best answer in the context of a physics test.

   Redaction: Jeffery, 2001jan01

---

038 qmult 01020 1 5 5 easy thinking: light cone Alpha Centauri

23. We know already that a tricky political situation will arise on Alpha Centauri 6 (4 light-years distant) in a election 3 years hence. In our Terracentric way, we the people of Earth are going to make our views known to the Alpha Centaurs. What is our chance of influencing their decision?

   a) It's possible: Alpha Centauri 3 years hence is **INSIDE** our light cone. A radio signal can be sent and received and the Alpha Centaurs will conclude that there is life out there—arguably intelligent.
   b) So-so.
   c) Immense: nothing is more cherished than unasked for advice from those incompetent to give it.
   d) Nil: Alpha Centaurs would rather take their own bad advice, than someone else's good advice: they're pretty normal that way—explaining the horse's hind end is another matter.
   e) Nil: Alpha Centauri 3 years hence is **OUTSIDE** of our light cone. Nothing we do now can have the slightest effect on them then. Jump up and down, scream, shout, giggle inopportunely—nothing can help.

   SUGGESTED ANSWER: (e)

   Wrong answers:
   a) Wrong, they are outside our light cone. Of course, the message will eventually arrive and then the will conclude that there is life out there—arguably intelligent.
   b) Well no.
   c) Right if you recognize sarcasm.

d) Not the best answer in context. They will never have the chance to contemplate our advice—however momentarily—before going on to do whatever they will anyway.

Redaction: Jeffery, 2001jan01

---

038 qmult 01100 1 1 5 easy memory: time travel

24. If you could travel instantly over a finite distance in any inertial reference frame, you could:

a) do anything. b) write the great American novel. c) play a didjeridu. d) ride a unicycle and spit backwards at the same time. e) time travel.

SUGGESTED ANSWER: (e) It's easiest to see this on a Minkowski diagram. Say you traveled intantly some distance to the right in the inertial frame (i.e., traveled along a line of simultaneity) for which diagram was drawn and then accelerated to an inertial frame moving to the right relative to the first one. The second frame has lines of simultaneity that are lines of positive slope when plotted on the diagram. Since you can travel instantly in any inertial frame, you can now travel to the left and downward along a second-frame line of simultaneity until you are at the same space point from which you started. There you decelerate back to the first inertial frame. You are now where you started in space, but earlier in time: you have time traveled into the past. You could now alter the course of history—and perhaps erase yourself from reality—as Oliver Hardy (1892–1957) would say "That's another fine mess you've gotten us into."

Wrong answers:

c) "The didjeridu is a traditional Australian aboriginal instrument that consists of a long cylindrical tube, that is played by 'blowing' in a similar way that a trumpet or other brass instrument is played. Didjeridu's are traditionally made of a hollowed out log, although there are a variety of novel, and more modern, materials used to create these instruments. There are lots of relevant sites . . .

d) I know you all can do this. But in the context of the question, it's implied that its something you could only do if you could travel instantly over a finite distance in an inertial reference frame.

Redaction: Jeffery, 2001jan01

---

038 qmult 01200 1 1 5 easy memory: Einstein's equation presented

Extra keywords: physci

25. Einstein's equation (or mass-energy equivalence equation) is

a) $E = mc^3$. b) $E = m/c^3$. c) $E = m/c^2$. d) $E = mc^4$. e) $E = mc^2$.

SUGGESTED ANSWER: (e)

Wrong answers:

a) C'mon.

Redaction: Jeffery, 2001jan01

---

038 qmult 01210 2 1 5 moderate memory: Einstein's equation in general

Extra keywords: physci

26. Einstein's equation $E = mc^2$:

a) applies only to rest mass.
b) applies only to electromagnetic radiation.
c) applies only to nuclear reactions.
d) is only valid in non-inertial frames.
e) is general. All forms of energy have inertial mass and a gravitational effect. Rest mass is itself a special form of energy like electromagnetic field, thermal, or kinetic energies.

SUGGESTED ANSWER: (e) La46 confirms this general formulation. Some elementary books like HRW avoid stating it clearly leaving the impression that only rest mass energy has inertial mass.

Wrong answers:
a) No although it is often used in rest mass calculations.
d) C'mon. Special relativity goes on endlessly about inertial frames

Redaction: Jeffery, 2001jan01

---

038 qmult 01220 1 3 1 easy math: E=mc**2 applied to electron
Extra keywords: physci KB-93-33

27. The mass of an electron is 9.1×10^{-31} kg. The energy equivalent of this mass is approximately:

a) 10^{-13} J. b) 10^{-47} J. c) 3×10^{-22} J. d) 3×10^{-38} J. e) 1 J.

SUGGESTED ANSWER: (a) Behold

$$E = mc^2 \approx 10^{-30} \times (3 \times 10^8) \approx 10^{-30} \times 10^{17} = 10^{-13}\,\mathrm{J} .$$

Wrong answers:
b) You probably have the wrong sign on the exponent of the c^2 value.

Redaction: Jeffery, 2001jan01

---

038 qfull 00100 2 3 0 moderate math: relativistic identities

28. Do the following.

a) Given

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} ,$$

find β as a function of the Lorentz factor γ.

b) Given $p = \gamma m_0 v = \gamma m_0 c\beta$, find β as a function of $p/(m_0 c)$.

c) Given $E = \gamma m_0 c^2$, show that $E = \sqrt{p^2c^2 + (m_0c^2)^2}$.

d) Given $E = \gamma m_0 c^2$, find β as a function of $E/(m_0c^2)$.

e) Given the result of part (d), find β as a function of kinetic energy KE.

f) If you know Taylor's series, verify that $\beta = \sqrt{2KE/(m_0c^2)}$ to 1st order in $KE/(m_0c^2)$. The 1st order expression is, of course, the classical (i.e., non-relativistic result) result.

SUGGESTED ANSWER:

a) Behold

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} , \qquad 1-\beta^2 = \frac{1}{\gamma^2} , \qquad \text{and} \qquad \beta = \sqrt{1-\gamma^{-2}} .$$

b) Behold

$$p = \gamma m_0 c\beta , \qquad \frac{p}{m_0c} = \frac{\beta}{\sqrt{1-\beta^2}} , \qquad \left(\frac{p}{m_0c}\right)^2 (1-\beta^2) = \beta^2 ,$$

and

$$\beta = \frac{\dfrac{p}{m_0c}}{\sqrt{1+\left(\dfrac{p}{m_0c}\right)^2}} .$$

c) Behold

$$E = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-\beta^2}} = \frac{m_0 c^2}{\sqrt{1-\frac{x^2}{1+x^2}}} = \frac{m_0 c^2}{\sqrt{\frac{1}{1+x^2}}} = \sqrt{1+x^2}\, m_0 c^2$$
$$= \sqrt{1+\left(\frac{p}{m_0 c}\right)^2}\, m_0 c^2 = \sqrt{p^2c^2+(m_0c^2)^2} \ ,$$

we let $x = p/(m_0 c)$ for some intermediate steps. Note that this derived expression for E is actually more general than the original $E = \gamma m_0 c^2$ expression though we have not proven this. The derived expression applies to photons which have no rest mass: i.e., for photons $E = pc$.

d) Behold

$$E = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-\beta^2}} \ , \qquad 1-\beta^2 = \left(\frac{m_0 c^2}{E}\right)^2 \ ,$$

and

$$\beta = \sqrt{1-\left(\frac{m_0 c^2}{E}\right)^2} \ .$$

e) Well

$$E = m_0 c^2 + KE \ ,$$

and so

$$\beta = \sqrt{1-\left(\frac{m_0 c^2}{m_0 c^2 + KE}\right)^2} \ .$$

f) Well

$$\beta = \sqrt{1-\left[\frac{1}{1+KE/(m_0c^2)}\right]^2} \approx \sqrt{1-\left(1-2\frac{KE}{m_0c^2}\right)} = \sqrt{2\frac{KE}{m_0c^2}} \ .$$

where the last expression (which is 1st order in $KE/(m_0c^2)$) is the classical relation between β and kinetic energy.

Redaction: Jeffery, 2001jan01

---

038 qfull 00200 2 5 0 moderate thinking: Stacy Fitzgerald contraction

29. Stacy Dragila, Pocatello's pride and 2000 Olympic gold-medal winner, is barrelling along at a relativistic speed with her pole. The pole is 15 ft long in rest length. She runs through a barn 10 ft long in rest length and open at both ends. Stacy's going so fast that to an observer at rest in the ground inertial frame, her pole is Fitzgerald contracted to less than 10 ft. Thus the ground-frame observer sees Stacy and pole entirely vanish into the barn for a brief time. But to Stacy it's the barn that is Fitzgerald contracted and she and pole are never entirely inside the barn. Resolve the paradox. **HINT:** The answer isn't very long.

SUGGESTED ANSWER: Oh, Stacy certainly sees that at some times each of the ends of her pole are in the barn. But for her those times are never simultaneous. For the ground-frame observer some of those times are simultaneous. Remember simultaneity is inertial frame dependent.

Redaction: Jeffery, 2001jan01

---

038 qfull 00300 2 3 0 moderate math: alternate Fitzgerald contraction

30. The Fitzgerald contraction formula is

$$\ell = \ell_0\sqrt{1-\beta^2} \ ,$$

where ℓ_0 is the proper length of object (i.e., length measured in the rest frame of the object) moving at speed β relative to some other inertial frame (let's call it the lab frame) and ℓ is the corresponding length in the lab frame: note the lengths are both parallel to the direction of motion. The formula is derived from the Lorentz transformations assuming that the measurement of the location of the ends of the length was simultaneous in the lab frame. But there is another way one can measure the length of the moving object in the lab frame: measure the time $\Delta\tau$ (i.e., the time in the "length form" equal to $c\Delta t$) between the front end of the object passing a fixed point in the lab frame and the rear end passing the same point, and multiply by the speed of the object. Call this length ℓ_*, where

$$\ell_* = \beta\Delta\tau \ .$$

Using the Lorentz transformations determine ℓ_* as a function of ℓ_0. **HINT:** The ends of the object passing the fixed point constitute two events with different, but related, spacetime coordinates. Also, observers in both frames see the measurements happening on the end points of the object. In the frame of the object, the measurement events are on an object at rest.

SUGGESTED ANSWER: Well the Lorentz transformations between two events confined to only one spatial dimension are

$$\Delta x' = \gamma(\Delta x - \beta\Delta\tau)$$

and

$$\Delta\tau' = \gamma(\Delta\tau - \beta\Delta x) \ ,$$

where we have written time in the "length" form: i.e., $\tau = ct$. Note these equations assume that the positive x and x' directions are in the direction of motion as measured from the unprimed frame: i.e., the lab frame. In the present case the spatial separation in the lab frame between the events is $\Delta x = 0$ and the time separation is $\Delta\tau = \ell_*/\beta$. Thus

$$\Delta x' = -\gamma\ell_*$$

or

$$\ell_* = -\Delta x'\sqrt{1-\beta^2} \ .$$

Now in the rest frame of the object, the magnitude of the displacement between the events is "clearly" ℓ_0: the two events are at the end points of the object in all frames, of course. However the second event happens further toward the negative direction: thus $\Delta x' = -\ell_0$. So finally

$$\ell_* = \ell_0\sqrt{1-\beta^2} \ .$$

This is the same formula for length that the lab observer obtains for simultaneous measurement of the two ends. It's a nice consistency that these two procedures for measuring the length of an object will yield the same result.

Redaction: Jeffery, 2001jan01

---

038 qfull 00400 3 5 0 hard thinking: colliding star ships

31. Federation star ships Egregious and Execrable are on a head-on collision course. Egregious is moving at velocity $\beta_1 = 0.6$ and Execrable at velocity $\beta_2 = -0.8$: both velocities are relative to the more-or-less inertial frame of the Galaxy.

    a) What is the velocity β_2' of Execrable in the inertial frame of the Egregious?

    b) Good ship Execrable has proper length 500 m. What are it's lengths in the Galaxy frame and in the Egregious frame?

    c) In the Galaxy frame Egregious and Execrable start off a light-year apart ($1\,\text{lyr} = 9.460 \times 10^{15}\,\text{m}$). How long in **YEARS** till they collide in the Galaxy frame? **HINT:** Calculate their relative velocity as seen in the Galaxy frame.

d) In part (c), you obtained a superluminal velocity, but in special relativity we say c (speed of light in vacuum) is the ultimate physical speed. Both the part (c) answer and saying c is the ultimate physical speed are correct. Resolve the paradox? **HINT:** Note the word "physical" before speed.

e) By the simplest means calculate the time to collision in the rest frame of the Egregious.

f) Now for the challenge. Calculate the time to collision in the rest frame of the Egregious, using velocities, lengths, and times as measured in the Egregious frame. Is the answer the same as that in part (e)? **HINT:** First, find Execrable's spacetime coordinates in the Egregious frame.

SUGGESTED ANSWER:

a) Well the Egregious frame is moving at $\beta_1 = 0.6$ with respect to the Galaxy frame and this is the frame-transforming velocity from the Galaxy frame. The velocity of Execrable is $\beta_2 = -0.8$ relative to the Galaxy frame and this is the velocity to be transformed. From the velocity transformation rule

$$\beta_2' = \frac{\beta_2 - \beta_1}{1 - \beta_2\beta_1} = \frac{-0.8 - 0.6}{1 - (-0.8) \times 0.6} = \frac{-1.4}{1.48} \approx -0.95$$

to about 2-digit accuracy: to higher accuracy

$$\beta_2' = -0.9459 \ .$$

b) Recall the Fitzgerald contraction rule for lengths parallel to the direction of motion

$$L = L_0\sqrt{1-\beta^2} \ ,$$

where L_0 is the proper length and L is the length measured in the frame with respect to which the object is moving at velocity β. Execrable with respect to the Galaxy frame has

$$L = L_0\sqrt{1-0.8^2} = 500 \times \sqrt{0.36} = 300\,\text{m}$$

and with respect the Egregious frame,

$$L = L_0\sqrt{1-0.95^2} \approx 500 \times \sqrt{0.1} \approx 170\,\text{m} \ .$$

The 1st value is accurate to the precision of the initial values and the 2nd is about 2-digit accurate. To higher accuracy the 2nd value is 162.2 m.

c) The relative speed between the two star ships as seen in the Galaxy frame is a superluminal $|\beta_2 - \beta_1| = 1.4$. Let x_2 be the distance from Egregious to Execrable in the Galaxy frame. Thus in the Galaxy frame the time to collision is

$$\Delta t_{\text{col}} = \frac{x_2}{c|\beta_2 - \beta_1|} \approx \frac{10^{16}}{4.2 \times 10^8} \approx 2.5 \times 10^7\,\text{s} \approx 0.8\,\text{yr} \ ,$$

to about 1-digit accuracy. Now if I were smarter than the average bear, I'd have just said that light travels at a speed of $c = 1\,\text{lyr/yr}$, and thus the time to collision is

$$\frac{1\,\text{lyr}}{1.4 \times 1\,\text{lyr/yr}} \approx 0.7\,\text{yr} \ ,$$

to about 1-digit accuracy. The two values agree to the accuracy that I've calculated them. To higher accuracy, the time to collision is 0.7143 yr.

d) The speed of light in vacuum c is the ultimate physical speed. This means no information can propagate faster than that speed. But what are called geometrical velocities can certainly exceed c. For example, in the part (c) answer the star ships approach each other at a relative speed greater than c in the Galaxy frame, but no physical signal is traveling at greater than c relative to the Galaxy frame. Of course, two colliding objects can never have relative geometrical speed exceeding $2c$ because their physical speeds can't exceed c: head-on colliding light pulses will have a geometrical relative speed of $2c$. However, higher geometrical speeds can be found: there is really no limit. Consider the geometrical rotational speed of a remote galaxy at 10^6 lyr about the Earth taken as being at rest. That galaxy goes around us every day at a geometrical speed of

$$\frac{2\pi r}{\Delta t} \approx \frac{6 \times 10^{22}\,\mathrm{m}}{10^5\,\mathrm{s}} = 6 \times 10^{17}\,\mathrm{m/s} \approx 2 \times 10^9 \times c\ .$$

But again no physical signal propagates at that speed.

e) Recall the time dilation rule

$$\Delta t' = \Delta t\sqrt{1-\beta^2}\ ,$$

where $\Delta t'$ is the proper time of a moving clock and Δt is the time measured by clocks in the frame with respect to which the moving clock is moving at velocity β. In this case $\Delta t = 0.7143\,\mathrm{yr}$ and the Egregious has $\beta_1 = 0.6$. Thus in the frame of the Egregious, time to collision is

$$\Delta t'_{\mathrm{col}} = \frac{x_2}{c|\beta_2-\beta_1|}\sqrt{1-\beta_1^2} \approx 0.7143 \times \sqrt{1-0.36} \approx 0.57\,\mathrm{yr}\ ,$$

to about 1-digit accuracy. To higher accuracy, the time to collision is $0.5714\,\mathrm{yr}$.

f) Hm, tricky. You might be tempted to use the Fitzgerald contraction to the Egregious frame on the Galaxy-frame separation distance to obtain the Egregious-frame separation distance and then divide that by the Egregious-frame speed of Execrable and call that the time to collision. That's wrong, wrong, wrong.

The case is actually rather subtle. When the Galaxy observer sees the two star ships $x_2 = 1\,\mathrm{lyr}$ apart, that constitutes two simultaneous observations for the Galaxy observer, but those observations are not simultaneous in the Egregious frame. In the Egregious frame, the Execrable is not $x'_2 = x_2/\sqrt{1-\beta_1^2} = \gamma_1 x_2$ away at Egregious-frame time zero: it is x'_2 away at some other Egregious-frame time. Here I take Egregious-frame time zero to coincide with Galaxy-frame time zero at the Egregious position. Galaxy-frame time zero is, of course, when the ships are x_2 apart in the Galaxy frame. We are free to make this choice.

We must now find the spacetime coordinates of the Execrable in the Egregious frame when the two star ships are $1\,\mathrm{lyr}$ apart simultaneously in the Galaxy frame. From the Lorentz transformations

$$x' = \gamma(x-\beta\tau) \qquad \text{and} \qquad \tau' = \gamma(\tau-\beta x)\ ,$$

we obtain

$$x'_2 = \gamma_1 x_2 \qquad \text{and} \qquad \tau'_2 = -\gamma_1\beta_1 x_2\ ,$$

where β_1 is the transformation velocity, γ_1 is the Lorentz factor for β_1, $(x_2, 0)$ is the Galaxy-frame spacetime coordinates of the Execrable, and (x'_2, τ'_2) are the Egregious-frame spacetime coordinates of the Execrable. One must interpret (x'_2, τ'_2) correctly: what they mean that at τ'_2 from the time origin in the Egregious frame, the Execrable is spatially at x'_2. Thus at the time origin in the Egregious frame, the Execrable is spatially at $x'_2 - \tau'_2\beta'_2$: we are subtracting off the distance the Execrable has traveled in time τ'_2: since $\tau'_2 < 0$ and $\beta'_2 < 0$, the Execrable is closer to the Egregious at the Egregious frame time zero than it was at (x'_2, τ'_2). The time to collision in the Egregious frame is

$$\begin{aligned}\Delta t'_{\rm col} &= \frac{1}{c}\left|\frac{x'_2-\tau'_2\beta'_2}{\beta'_2}\right| = \frac{\gamma_1 x_2}{c}\left|\frac{1+\beta_1\beta'_2}{\beta'_2}\right| = \frac{\gamma_1 x_2}{c}\left|\frac{1}{\beta'_2}+\beta_1\right| \\ &= \frac{\gamma_1 x_2}{c}\left|\frac{1-\beta_2\beta_1}{\beta_2-\beta_1}+\beta_1\right| = \frac{\gamma_1 x_2}{c}\left|\frac{1-\beta_1^2}{\beta_2-\beta_1}\right| \\ &= \frac{x_2}{c|\beta_2-\beta_1|}\sqrt{1-\beta_1^2} = 0.5714\,{\rm yr}\ .\end{aligned}$$

This is indeed the value we got in part (e). Note also that the final formula for $\Delta t'_{\rm col}$ is exactly the formula for $\Delta t'_{\rm col}$ that we found in the part (e) answer. This is one of those everything-works-out-consistently-in-relativity situations.

Another way to think of the calculation is to find the absolute Egregious-frame time of collision: this is

$$t'_{\rm col} = \frac{1}{c}\left[\frac{x'_2}{(-\beta'_2)}+\tau'_2\right] = \frac{1}{c}\left[\frac{x'_2-\beta'_2\tau'_2}{(-\beta'_2)}\right] = \frac{1}{c}\left|\frac{x'_2-\beta'_2\tau'_2}{\beta'_2}\right|\ .$$

This is the same expression as for $\Delta t'_{\rm col}$ in the last paragraph. The reason is that we asked how long is it to the collision from the Egregious-frame time origin: hence $\Delta t'_{\rm col} = t'_{\rm col}$.

The foregoing seems all correct, but I'm still trying to find the right mental formula to make it clear to me that it is all correct.

Fortran Code

```
      print*
      beta1=.6          ! Egregious velocity
      beta2=-.8         ! Execrable velocity
      xl2=500.          ! Execrable length
      xdist=1.          ! Initial separation
*                       ! of the ships in light years.
      beta2p=(beta2-beta1)/(1.-beta2*beta1)
      xl2g=xl2*sqrt(1.-beta2**2)
      xl2p=xl2*sqrt(1.-beta2p**2)
      delt=xdist/abs(beta2-beta1)
      deltp=delt*sqrt(1-beta1**2)
      print*,beta2p,xl2g,xl2p,delt,deltp
* -0.945945953  299.999992  162.162151  0.714285696
*                                        0.571428544
```

Redaction: Jeffery, 2001jan01

---

038 qfull 00500 2 3 0 moderate math: Smaug, Frodo, and time dilation

32. You and your loyal hound Frodo are traveling through space in your star ship Smaug. You are limited to going at less than the ultimate physical speed $c = 1\,{\rm lyr/yr}$ or $\beta = 1$, of course. But other than that Smaug can go at any speed and you and Frodo are real tough and can stand any acceleration.

   a) Say Frodo wants to travel from Earth to Sirius (distance $d = 8.7\,{\rm lyr}$) and you want to make the trip in 1 year for **YOU**. At what β value would Smaug have to go? **HINTS:** You will have to do some algebra. Recall the Galaxy-frame travel time is $d/(c\beta)$.

   b) Stirred by your first conquest of space, you and Frodo—well Frodo is less eager—now want to cross the bulge to the far end of the disk of Galaxy: distance $100,000\,{\rm lyr}$. But you want to live to see the journey's end. What physical effect makes it possible to survive the journey?

c) How much **LESS** than the speed of light (in units of c: i.e., in terms of β) do you need to go at to make the pan-Galactic journey of $100,000\,$lyr in 1 year for **YOU**? **HINT:** Recall the first order Taylor expansion

$$\frac{1}{\sqrt{1+x}} \approx 1 - \frac{1}{2}x$$

which is valid for $x << 1$.

d) How long are each of your the trips in dog years for Frodo? And why does Frodo want to stay at Sirius?

SUGGESTED ANSWER:

a) Ah, the old time dilation effect means you can slow down your proper clocks (e.g., wrist watch, heart beats, biorhythms) by going fast enough. In this case, the Galaxy frame time to Sirius must $t = d/(c\beta)$, where β is your velocity. From the time dilation formula

$$t' = t\sqrt{1-\beta^2} = \frac{d}{c\beta}\sqrt{1-\beta^2} \ ,$$

where t' is the proper time which you want to be 1 year. We now solve for β:

$$\beta = \frac{1}{\sqrt{1+(ct'/d)^2}} \approx 1 - \frac{1}{2}\left(\frac{ct'}{d}\right)^2 \approx 1 - \frac{1}{160} \approx 0.993 \ ,$$

to about 3-digit accuracy. Note that I've used the first order Taylor expansion which should be pretty accurate here since $ct'/d << 1$.

b) Again, that old time dilation effect means you can do it if you only go fast enough.

c) Well from the formula from the part (a) answer and using the Taylor expansion formula, we have

$$\beta \approx 1 - \frac{1}{2}\left(\frac{ct'}{d}\right)^2 \ ,$$

and thus you must travel slower than the speed of light by

$$\frac{1}{2}\left(\frac{ct'}{d}\right)^2 \approx 0.5 \times 10^{-10}$$

in units of the speed of light. That is pretty fast—but any old light beam will beat you.

d) That's 7 years in dog years for both.

Sirius is the Dog Star since it is the brightest star in Canis Major, the larger of Orion's dogs. To the ancient Egyptians Sirius was Anubis, the jackal-head god. In fact Sirius is the brightest star seen from Earth. As you all know, the Sun moves eastward relative to the celestial sphere of the fixed stars in the course of a year. When the Sun is very close to a star, the star is never seen: it is always lost in the daytime sky. As the Sun moves just east of star, that star can be seen rising just before sunrise before being lost in the daytime sky. This first appearance of a bright star after being lost in the Sun—the heliacal rising—was considered a good omen in ancient times—the cosmos was still working as it had. The heliacal risings often were used to mark the beginning of seasons. The heliacal rising of Sirius circa mid-July (but even dozens of websites fail to say exactly when—it's varied a bit over the millennia) marks the beginning of the Nile floods and the long, hot, dry days of summer—the Dog days. Sirius from the Greek Seirios means scorcher. "By the Dog" was Socrates's favorite oath. Hesiod (circa 700 BC) tells us:

Oh when thistle bursts and cicada,

hid in his tree, shrill and timeless,
sings his song—timeless,
then summer swoons and goat is fat
and wine is good, and maids are riggish,
but burnt are streams and men—burnt dry
by Sirius teaming with the Sun—but I
in the Dog days love a rocky shade
and Biblos from the vine.

Redaction: Jeffery, 2001jan01

---

038 qfull 00510 2 5 0 moderate thinking: Frodo and the black hole

33. Once again you and your trusty hound Frodo are voyaging through Galactic space on your star ship Smaug.

a) Unbeknownst to you, Frodo has an inner life in which he in comtemplates the universe. He asks himself—since it's no use asking you–"if the gravitational force is always attractive and dominates large-scale structure, why doesn't the whole Galaxy of stars collapse to form a super-massive blackhole?" Assuming Frodo answers himself correctly, what is his answer?

b) While Frodo is musing, you are doing a few calculations. Your speed is a lazy 3.000×10^4 km/s—kilometers per second mark you—relative to the more-or-less inertial frame of the Galaxy and Smaug's rest mass is 1.000×10^5 kg. What is Smaug's **NON-RELATIVISTIC** kinetic energy?

c) Calculate the kinetic energy of Smaug in special relativity. Does the special relativistic calculation of kinetic energy yield a significantly different value (say 0.1 % different) from the non-relativistic calculation?

d) Just as Frodo is begging to go for a space walk, you realize from your space chart that you are on a straight line path for a black hole—your controls are jammed—could be Frodo was gnawing on some vital connection—the mass of the black hole is 1 solar mass (1 solar mass is 1.9891×10^{30} kg) and it is 1.5×10^{11} m away. From a Newtonian calculation at what distance will you reach the speed of light! **NOTE**: Because of special and general relativistic effects this calculation is not really valid, but it does lead to some insight into what must happen.

e) There's only one thing left to do—get the last word. So when your Lorentz factor reaches 10^{10}—just before your gravitational shielding (a pure plot device) fails and your atoms become permanently uncooperative—you send a proper time 3 minute video message to all your friends. Just from special relativistic physics, how long will your video message run in **YEARS** in the more-or-less inertial frame of the Galaxy? **NOTE**: General relativity will also affect time and in reality you won't be at a constant velocity.

SUGGESTED ANSWER:

a) Arf, arf, aaarf—Translation: The stars have kinetic energy that they can't lose much of by dissipative forces in the near vacuum of space. This kinetic energy resists gravity. In more detail, the stars are flying around trying to go in straight lines, but gravity keeps pulling them off straight line paths into orbits around the Galactic center more or less. At a more advanced level, one has to say that angular momentum is a conserved quantity in a pure central force problem. The problem isn't exactly central force, but it approximates that to a high degree. The Galaxy is quasi-eternal. Actually there are some dissipative effects and long, long in the future the Galaxy might coalesce into a super-massive black hole.

b) Behold

$$KE = \frac{1}{2}m_0 v^2 = 4.5 \times 10^{19}\ \mathrm{J}\ .$$

c) Behold

$$KE = (\gamma - 1)m_0c^2 \approx \left(1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 - 1\right) m_0c^2 \approx \frac{1}{2}m_0v^2\left(1 + \frac{3}{4}\beta^2\right) ,$$

where we have used Taylor's series to 2nd order in β^2 and where the last expression is written with a relativistic correction factor 2nd order good in β. Now

$$\beta = \frac{v}{c} \approx 0.1 .$$

Thus the correction factor is about 1.0075 or the relativistic kinetic energy is 0.75 % larger than the classical result. So the relativistically correct calculation yields what I would call a marginally significantly different result. It is certainly larger than 0.1 % level of significance the question suggests. Of course, a significance difference depends on your measurement accuracy and/or on how big a difference will affect what you are doing.

d) One can use conservation of energy:

$$\frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}mv_2^2 - \frac{GMm}{r_2} ,$$

and thus

$$r_2 = \frac{GM/c^2}{\beta_2^2 - \beta_1^2 + GM/(c^2 r_1)} .$$

If we set $\beta_2 = 1$, then

$$r_2 = \frac{GM/c^2}{1 - \beta_1^2 + GM/(c^2 r_1)} = 2.984 \times 10^3\,\mathrm{m} = 2.984\,\mathrm{km} .$$

Note this is nearly at the Schwarzschild radius, the radius of the event horizon—the point of no return even for light:

$$r_{\rm sch} = \frac{GM}{c^2} = 2.954\sqrt{\frac{M}{M_\odot}}\,\mathrm{km} ,$$

where $M_\odot$ is the solar mass. In the calculation, you reach the speed of light above the event horizon because at infinity you had some initial kinetic energy.

e) The students have to recognize that γ is the Lorentz factor. They should know, that 10^{10} is not β which for a physical velocity must always obey $\beta \leq 1$. From the time dilation formula,

$$\tau = \tau_{\rm proper}\gamma = 1.80 \times 10^{12}\,\mathrm{s} \approx 60,000\,\mathrm{yr} ,$$

where I made use of the fact that 1 year is $\pi \times 10^7$ to within 0.5 %: the year is 0.5 % longer. Yes your video message will drone on interminably. But don't worry, there won't be a sad ending. I'm going use a time warp to save Smaug, Frodo, and "you" for further adventures in time and space.

Fortran Code

```
      print*
      rsch=2.*grav*solmass/(clight**2)
      beta1=3.e+9/clight    ! v in cm/s
      r1=1.5e+13 ! in cm
      rfin=rsch/(1.-beta1**2+rsch/r1)
      print*,'beta1,rsch,rfin'
      print*,beta1,rsch,rfin
* 0.100069229  295369.965 cm  298357.669 cm
```

Redaction: Jeffery, 2001jan01

---

038 qfull 00600 1 3 0 easy math: fast protons

34. The proton mass in energy units is 938.272029 MeV or approximately 1 GeV. The proton synchrotron at Fermilab near Chicago can (or could) accelerate a proton to total energy (i.e., kinetic energy plus rest mass energy) of about 500 Gev.

    a) What is the kinetic energy of the proton to the same order of accuracy that the total energy was quoted to?

    b) What is the Lorentz factor of the proton?

    c) What is the velocity β of the proton?

SUGGESTED ANSWER:

a) The rest energy is very small compared to the total energy. So the energy is mostly kinetic energy, and to the accuracy of the quote the kinetic energy is also about 500 GeV.

b) Well total energy to rest mass energy is related by

$$E = \gamma E_0 \ ,$$

where E is total energy, E_0 is rest mass energy, and γ is the Lorentz factor. In this case the Lorentz factor is about 500.

c) Well the formula is

$$\beta = \sqrt{1 - \gamma^{-2}} \ .$$

Expanding to 2nd order in small γ^{-2} gives

$$\beta \approx 1 - \frac{1}{2}\gamma^{-2} \approx 1 - 2 \times 10^{-6} \ .$$

Thus the proton is traveling at near the speed of light. But, of course, as a massive particle it can never reach that speed.

Redaction: Jeffery, 2001jan01

Equation Sheet for Modern Physics

These equation sheets are intended for students writing tests or reviewing material. Therefore they are neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

---

1 Geometrical Formulae

$$C_{\rm cir} = 2\pi r \qquad A_{\rm cir} = \pi r^2 \qquad A_{\rm sph} = 4\pi r^2 \qquad V_{\rm sph} = \frac{4}{3}\pi r^3$$

---

2 Trigonometry

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)] \qquad \sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)] \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right] \qquad \sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$$

---

3 Blackbody Radiation

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{[e^{h\nu/(kT)} - 1]} \qquad B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{[e^{hc/(kT\lambda)} - 1]}$$

$$B_\lambda\,d\lambda = B_\nu\,d\nu \qquad \nu\lambda = c \qquad \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$$

$$k = 1.3806505(24)\times 10^{-23}\,\text{J/K} \qquad c = 2.99792458\times 10^{8}\,\text{m}$$

$$h = 6.6260693(11)\times 10^{-34}\,\text{J s} = 4.13566743(35)\times 10^{-15}\,\text{eV s}$$

$$\hbar = \frac{h}{2\pi} = 1.05457168(18) \times 10^{-34}\,\mathrm{J\,s}$$

$$hc = 12398.419\,\mathrm{eV\,Å} \approx 10^4\,\mathrm{eV\,Å} \qquad E = h\nu = \frac{hc}{\lambda} \qquad p = \frac{h}{\lambda}$$

$$F = \sigma T^4 \qquad \sigma = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3} = 5.670400(40) \times 10^{-8}\,\mathrm{W/m^2/K^4}$$

$$\lambda_{\max} T = \text{constant} = \frac{hc}{kx_{\max}} \approx \frac{1.4387751 \times 10^{-2}}{x_{\max}}$$

$$B_{\lambda,\mathrm{Wien}} = \frac{2hc^2}{\lambda^5} e^{-hc/(kT\lambda)} \qquad B_{\lambda,\mathrm{Rayleigh-Jeans}} = \frac{2ckT}{\lambda^4}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\nu = \frac{\omega}{c} \qquad k_i = \frac{\pi}{L}n_i \quad \text{standing wave BCs} \qquad k_i = \frac{2\pi}{L}n_i \quad \text{periodic BCs}$$

$$n(k)\,dk = \frac{k^2}{\pi^2}\,dk = \pi\left(\frac{2}{c}\right)\nu^2\,d\nu = n(\nu)\,d\nu$$

$$\ln(z!) \approx \left(z + \frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \ldots$$

$$\ln(N!) \approx N\ln(N) - N$$

$$\rho(E)\,dE = \frac{e^{-E/(kT)}}{kT}\,dE \qquad P(n) = (1 - e^{-\alpha})e^{-n\alpha} \qquad \alpha = \frac{h\nu}{kT}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad f(x - vt) \qquad f(kx - \omega t)$$

---

4 Photons

$$KE = h\nu - w \qquad \Delta\lambda = \lambda_{\mathrm{scat}} - \lambda_{\mathrm{inc}} = \lambda_{\mathrm{C}}(1 - \cos\theta)$$

$$\lambda_{\mathrm{C}} = \frac{h}{m_e c} = 2.426310238(16) \times 10^{-12}\,\mathrm{m} \qquad e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$m_e = 9.1093826(16) \times 10^{-31}\,\mathrm{kg} = 0.510998918(44)\,\mathrm{MeV}$$

$$m_p = 1.67262171(29) \times 10^{-27}\,\text{kg} = 938.272029(80)\,\text{MeV}$$

$$\ell = \frac{1}{n\sigma} \qquad \rho = \frac{e^{-s/\ell}}{\ell} \qquad \langle s^m \rangle = \ell^m m!$$

---

5 Special Relativity

$$c = 2.99792458 \times 10^8\,\text{m/s} \approx 2.998 \times 10^8\,\text{m/s} \approx 3 \times 10^8\,\text{m/s} \approx 1\,\text{lyr/yr} \approx 1\,\text{ft/ns}$$

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \gamma(\beta << 1) = 1 + \frac{1}{2}\beta^2 \qquad \tau = ct$$

Galilean Transformations	Lorentz Transformations
$x' = x - \beta\tau$	$x' = \gamma(x - \beta\tau)$
$y' = y$	$y' = y$
$z' = z$	$z' = z$
$\tau' = \tau$	$\tau' = \gamma(\tau - \beta x)$
$\beta'_{\text{obj}} = \beta_{\text{obj}} - \beta$	$\beta'_{\text{obj}} = \dfrac{\beta_{\text{obj}} - \beta}{1 - \beta\beta_{\text{obj}}}$

$$\ell = \ell_{\text{proper}}\sqrt{1-\beta^2} \qquad \Delta\tau_{\text{proper}} = \Delta\tau\sqrt{1-\beta^2}$$

$$m = \gamma m_0 \qquad p = mv = \gamma m_0 c\beta \qquad E_0 = m_0 c^2 \qquad E = \gamma E_0 = \gamma m_0 c^2 = mc^2$$

$$E = mc^2 \qquad E = \sqrt{(pc)^2 + (m_0 c^2)^2}$$

$$KE = E - E_0 = \sqrt{(pc)^2 + (m_0 c^2)^2} - m_0 c^2 = (\gamma - 1) m_0 c^2$$

$$f = f_{\text{proper}}\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{for source and detector separating}$$

$$f(\beta << 1) = f_{\text{proper}}\left(1 - \beta + \frac{1}{2}\beta^2\right)$$

$$f_{\text{trans}} = f_{\text{proper}}\sqrt{1-\beta^2} \qquad f_{\text{trans}}(\beta << 1) = f_{\text{proper}}\left(1 - \frac{1}{2}\beta^2\right)$$

$$\tau = \beta x + \gamma^{-1}\tau' \quad \text{for lines of constant } \tau'$$

$$\tau = \frac{x - \gamma^{-1} x'}{\beta} \quad \text{for lines of constant } x'$$

$$x' = \frac{x_{\text{intersection}}}{\gamma} = x'_{x \text{ scale}} \sqrt{\frac{1-\beta^2}{1+\beta^2}} \qquad \tau' = \frac{\tau_{\text{intersection}}}{\gamma} = \tau'_{\tau \text{ scale}} \sqrt{\frac{1-\beta^2}{1+\beta^2}}$$

$$\theta_{\text{Mink}} = \tan^{-1}(\beta)$$