Mathematical Tables

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Introduction

Mathematical Tables (MAT) is a small source book of mathematical tables of trigonometric formulae, derivatives, integrals, special functions, series expansions, special formulae, and whatever else turns out to be useful. The idea is not to compete with any of the great source books, but to have a quick reference that is useful to myself and students and whose copyright belongs to me. The entries have been drawn from many sources, but have all been written by me. There is no guarantee of accuracy.

The book is available in electronic form to instructors by request to the author. It is free courseware and can be freely used and distributed, but not used for commercial purposes. MAT is still at a very early stage, but it will grow. User instructors can, of course, add and modify as they list.

Everything is written in plain TEX in my own idiosyncratic style. The entries are referenced in comment statements by the reference I used for them when there is one. This allows me to check the entry and also do searches for it. The references are abbreviated: Hu-3:1 stands for Hudson, 1946, p. 3, equation 1. The codes for references are given in the References just below.

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Contents

References

Hodgman, C. D. 1959, *CRC Standard Mathematical Tables* (Cleveland, Ohio: Chemical Rubber Publishing Company) (Ho)

Hudson, R. G., & Lipka, J. 1946, *A Table of Integrals* (New York: John Wiley & Sons, Inc) (Hu)

Chapter 1 Trigonometric Formulae

Note: The general variable is usually θ, but others such as a and b occur too.

(1) $$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

(2) $$\sin^2\theta = \frac{1}{2}\left[1 - \cos(2\theta)\right]$$

(3) $$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

(4) $$\cos^2\theta = \frac{1}{2}\left[1 + \cos(2\theta)\right]$$

(5) $$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

(6) $$\sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$$

(7) $$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

(8) $$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$ Euler's formula

(9) $$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

(10) $$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Chapter 2 Series

1. Simple Series

(1) $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \qquad \text{binomial theorem}$$

(2) $$S_n = \frac{1-x^{n+1}}{1-x} = \sum_{\ell=0}^{n} x^\ell \qquad \text{finite power series}$$

(3) $$S = \frac{1}{1-x} = \sum_{\ell=0}^{\infty} x^\ell \qquad \text{power series } |x| < 1$$

(4) $$S_0(n) = \sum_{\ell=1}^{n} 1 = n$$

(5) $$S_1(n) = \sum_{\ell=1}^{n} \ell = \frac{n(n+1)}{2}$$

(6) $$S_2(n) = \sum_{\ell=1}^{n} \ell^2 = \frac{n(n+1)(2n+1)}{6}$$

(7) $$S_3(n) = \sum_{\ell=1}^{n} \ell^3 = \frac{n^2(n+1)^2}{4}$$

(8) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}$$

2. Trigonometric Series

(1) $$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - 5040x^7 + \ldots$$

(2) $$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}{}^6 + \ldots$$

3. Riemann Zeta Function

(1) $$\zeta(s) = \sum_{\ell=1}^{\infty} \frac{1}{\ell^s}$$

(2) $$\zeta(2) = \frac{\pi^2}{6}$$

(3) $$\zeta(4) = \frac{\pi^4}{90}$$

(4) $$\zeta(6) = \frac{\pi^6}{945}$$

(5) $$\zeta(8) = \frac{\pi^8}{9450}$$

(6) $$\zeta(10) = \frac{\pi^{10}}{93,555}$$

Chapter 3 Vector Identities

Note: The general vectors are usually $\vec{A}$ and $\vec{B}$.

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

Chapter 4 Derivatives

Note: The general variable is x and general functions of x are u and v. The symbol "log" is used for base 10 logarithm and "ln" for natural logarithm. The natural logarithm base is e of course. All other symbols are constants: e.g., a.

(1) $$\frac{d(a)}{dx} = 0 \qquad \frac{d(x)}{dx} = 1$$

(2) $$\frac{d^n(fg)}{dx^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}} \qquad \text{biderivative formula}$$

(3) $$\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \qquad \text{for} \qquad \sin^{-1} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

(4) $$\frac{d}{dx} \cos^{-1} u = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx} \qquad \text{for} \qquad \cos^{-1} \in [0, \pi]$$

(5) $$\frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx}$$

Chapter 5 Integrals

Note: The general variable is x and general functions of x are u, v, f, and g. The symbol "log" is used for base 10 logarithm and "ln" for natural logarithm. The natural logarithm base is e of course. All other symbols are symbols are constants (e.g., a, b, etc.) unless otherwise noted. For the indefinite integrals the constants of integration C are omitted, except for the fundamental forms.

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1. Fundamental Forms

(1) $$\int f(x)\,dx = \int df = f(x) + C$$

(2) $$d\int f(x)\,dx = f(x)\,dx$$

(3) $$\int \left[\sum_i a_i f_i\right] dx = \sum_i a_i \int f_i\,dx$$

(4) $$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C \qquad \text{for } a \neq 1$$

(5) $$\int x^{-1}\,dx = \ln(x) + C$$

(6) $$\int [f(x)g(x)]'\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx + C$$

(7) $$\int f'(x)g(x)\,dx = \int [f(x)g(x)]'\,dx - \int f(x)g'(x)\,dx + C \qquad \text{integration by parts}$$

(8) $$\int u\,dv = uv - \int v\,du + C \qquad \text{integration by parts}$$

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2. Functions containing $ax^2 + b$

(1) $$\int \frac{dx}{ax^2+b} = \begin{cases} \dfrac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\dfrac{a}{b}}\right) & a>0 \text{ and } b>0; \\ \dfrac{1}{2\sqrt{-ab}}\ln\left(\dfrac{x\sqrt{a}-\sqrt{-b}}{x\sqrt{a}+\sqrt{-b}}\right) & a>0 \text{ and } b<0; \\ \dfrac{1}{2\sqrt{-ab}}\ln\left(\dfrac{\sqrt{b}+x\sqrt{-a}}{\sqrt{b}-x\sqrt{-a}}\right) & a<0 \text{ and } b>0; \\ -\dfrac{1}{ax} & b=0; \\ \dfrac{x}{b} & a=0 \end{cases}$$

(2) $$\int \frac{dx}{(ax^2+b)^2} = \frac{1}{2b}\frac{x}{(ax^2+b)} + \frac{1}{2b}\frac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{a}{b}}\right) \qquad a>0 \text{ and } b>0$$

(3) $$\int \frac{dx}{(ax^2+b)^n} = \frac{1}{2(n-1)b}\frac{x}{(ax^2+b)^{n-1}} + \frac{2n-3}{2(n-1)b}\int \frac{dx}{(ax^2+b)^{n-1}} \qquad n \text{ integer} > 1$$

(4) $$\int \frac{dx}{\sqrt{ax^2+b}} = \frac{1}{\sqrt{a}} \ln\left(x\sqrt{a} + \sqrt{ax^2+b}\right) \qquad a < 0$$

(5) $$\int \frac{dx}{\sqrt{ax^2+b}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(x\sqrt{-\frac{a}{b}}\right) = -\frac{1}{\sqrt{-a}} \cos^{-1}\left(x\sqrt{-\frac{a}{b}}\right) \qquad a < 0$$

(6) $$\int \frac{dx}{(ax^2+b)^{3/2}} = \frac{x}{b\sqrt{ax^2+b}}$$

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3. Functions containing $ax^2 + bx + c$

(1) $$\int \frac{dx}{(ax^2+bx+c)^{3/2}} = -\frac{2(2ax+b)}{(b^2-4ac)\sqrt{ax^2+bx+c}}$$

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4. Functions containing $\sin(ax)$

(1) $$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

(2) $$\int \sin^3(ax)\,dx = -\frac{1}{a}\cos(ax) + \frac{1}{3a}\cos^3(ax)$$

(3) $$\int \sin^5(ax)\,dx = -\frac{1}{5a}\sin^4(ax)\cos(ax) + \frac{4}{15a}\cos^3(ax) - \frac{4}{5a}\cos(ax)$$

(4) $$\int \sin^n(ax)\,dx = -\frac{1}{na}\sin^{n-1}(ax)\cos(ax)\frac{n-1}{n}\int \sin^{n-2}(ax)\,dx$$

(5) $$\int \sin(ax)\sin(bx)\,dx = \frac{\sin[(a-b)x]}{2(a-b)} - \frac{\sin[(a+b)x]}{2(a+b)} \qquad |a| \neq |b|$$

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5. Inverse Trigonometric Functions

(1) $$\int \sin^{-1}(ax)\,dx = x\sin^{-1}(ax) + \frac{1}{a}\sqrt{1-a^2x^2}$$

(2) $$\int \cos^{-1}(ax)\,dx = x\cos^{-1}(ax) - \frac{1}{a}\sqrt{1-a^2x^2}$$

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6. Algebraic and Trigonometric Functions

(1) $$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$$

(2) $$\int x^2\sin(ax)\,dx = -\frac{x^2}{a}\cos(ax) + \frac{2x}{a^2}\sin(ax) + \frac{2}{a^3}\cos(ax)$$

(3) $$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

(4) $$\int x^2\cos(ax)\,dx = \frac{x^2}{a}\sin(ax) + \frac{2x}{a^2}\cos(ax) - \frac{2}{a^3}\sin(ax)$$

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7. Gaussian Function and Integrals

(1) $$G(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$ standard form

(2) $$\int_{-\infty}^{\infty} e^{-\lambda x^2}\,dx = \sqrt{\frac{\pi}{\lambda}}$$

(3) $$\int_{-\infty}^{\infty} x e^{-\lambda x^2}\,dx = \lambda^{-1}$$

(4) $$\int_{-\infty}^{\infty} x^2 e^{-\lambda x^2}\,dx = \frac{\sqrt{\pi}}{2}\frac{1}{\lambda^{3/2}}$$

(5) $$\int_{-\infty}^{\infty} x^3 e^{-\lambda x^2}\,dx = \lambda^{-2}$$

(6) $$\int_{-\infty}^{\infty} x^4 e^{-\lambda x^2}\,dx = \frac{3}{4}\frac{\sqrt{\pi}}{\lambda^{3/2}}$$

(7) $$\int_{-\infty}^{\infty} x^5 e^{-\lambda x^2}\,dx = 2\lambda^{-3}$$

(8) $$\int_{0}^{\infty} x^y e^{-\lambda x^2}\,dx = \frac{1}{2}\frac{[(y-1)/2]!}{\lambda^{(y+1)/2}}$$ for y not an odd negative integer

(9) $$\int_{-\infty}^{\infty} x^y e^{-\lambda x^2}\,dx = \begin{cases} \dfrac{[(y-1)/2]!}{\lambda^{(y+1)/2}}\ , & \text{for } y \text{ an even integer;} \\ 0 & \text{for } y \text{ an odd positive integer;} \\ \text{undefined} & \text{for } y \text{ an odd negative integer;} \\ \text{complex} & \text{for } y \text{ non-integer} \end{cases}$$

(10) $$I_n = \int_0^{\infty} x^n e^{-\lambda x^2}\,dx = \lambda^{-(n+1)/2}\left(\frac{1}{2}\right)\left[\left(\frac{n-1}{2}\right)!\right]$$

(11) $$I_1 = \lambda^{-1}\left(\frac{1}{2}\right)$$

(12) $$I_3 = \lambda^{-2}\left(\frac{1}{2}\right)$$

(13) $$I_5 = \lambda^{-3}$$

(14) $$I_7 = 3\lambda^{-4}$$

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8. Factorial Functions and Integrals

(1) $$z! = \int_0^{\infty} e^{-t} t^z\,dt$$

(2) $z! = z(z-1)!$ for z not a negative integer nor 0

(3) $z! = \dfrac{(z+1)!}{z+1}$ for z not a negative integer

(4) $n! = n(n-1)(n-2)\ldots 1$ for n a non-zero positive integer

(5) $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$ $7! = 5040$

(6) $0! = 1$ $\left(-\frac{1}{2}\right)! = \sqrt{\pi}$

(7) $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$, $\left(\frac{3}{2}\right)! = \frac{3\sqrt{\pi}}{4}$, $\left(\frac{5}{2}\right)! = \frac{15\sqrt{\pi}}{8}$, $\left(\frac{7}{2}\right)! = \frac{105\sqrt{\pi}}{16}$,

(8) $\left(-\frac{3}{2}\right)! = -2\sqrt{\pi}\ , \qquad \left(-\frac{5}{2}\right)! = \frac{4}{3}\sqrt{\pi}\ , \qquad \left(-\frac{7}{2}\right)! = -\frac{8}{15}\sqrt{\pi}\ , \qquad \left(-\frac{9}{2}\right)! = \frac{16}{105}\sqrt{\pi}$

(9) $\frac{f(n,x)}{n!} = \frac{1}{n!}\int_x^\infty e^{-t}t^n\,dt = e^{-x}\frac{x^n}{n!} + \frac{f(n-1,x)}{(n-1)!} = e^{-x}\sum_{\ell=0}^{n}\frac{x^\ell}{\ell!}$ for n non-negative integer

(10) $f(n,x) = n!e^{-x}\sum_{\ell=0}^{n}\frac{x^\ell}{\ell!}$

(11) $g(n,x) = \int_0^x e^{-t}t^n\,dt = n! - f(n,x) = n!\left(1 - e^{-x}\sum_{\ell=0}^{n}\frac{x^\ell}{\ell!}\right)$ for n a non-negative integer

(12) $g(0,x) = 1 - e^{-x}$

(13) $g(1,x) = 1 - e^{-x}(1+x)$

(14) $g(2,x) = 2\left[1 - e^{-x}\left(1 + x + \frac{1}{2}x^2\right)\right]$

(15) $h(n,b,a) = g(n,b) - g(n,a) = n!\left(1 - e^{-x}\sum_{\ell=0}^{n}\frac{x^\ell}{\ell!}\right)\Bigg|_a^b = -n!\left(e^{-x}\sum_{\ell=0}^{n}\frac{x^\ell}{\ell!}\right)\Bigg|_a^b$
for n a non-negative integer

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9. Some Definite Integrals

(1) $$\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{\pi a^2}{4}$$

(2) $$\int_0^a \sqrt{2ax - x^2}\,dx = \frac{\pi a^2}{4}$$

(3) $$\int_0^{2\pi} \sin^2 x\,dx = \pi$$

(4) $$\int_0^{2\pi} \cos^2 x\,dx = \pi$$

(5) $$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \pi$$