

# CAPACITORS, CAPACITANCE, AND DIELECTRICS

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## ABSTRACT

Lecture notes on what the title says.

*Subject headings:* capacitors — capacitance — dielectrics

## 1. INTRODUCTION

Capacitors are simple circuit devices that have immense practical utility. e.g., for tuning radio receivers, as power supplies, etc.

They're everywhere—but we won't go into applications much.

An older name for capacitor is **CONDENSER** which almost never occurs in modern usage, but it turns up in older works.

## 2. DEFINITION OF CAPACITOR AND CAPACITANCE

A capacitor is a device consisting of two conductors called **PLATES** (which sometimes are plates or rolled up plates) separated usually by a dielectric (which is a term for an insulator when viewed as electrically active and which we discuss in § 6), but sometimes by air or vacuum (which air approximates).

We'll begin with vacuum separated plates (which are not much different than air separated plates) because they are conceptually easiest to introduce. Vacuum separated-plates do have special uses (Wikipedia: Capacitor).

Almost always the **PLATES** are metals—and so metal capacitors are really all one thinks of.

In their conventional operation, the **PLATES** carry equal and opposite charges:  $Q$  and  $-Q$ . Capacitors are **UNSIMPLE** dipoles.

The capacitor charge is defined to  $Q$  which formally is always positive. The capacitor charge can be negative in cases where one plate is defined as the positive plate for some derivational or practical reason and this plate happens to acquire a negative charge (e.g., see § 5.5).

In electrostatic equilibrium, the plates are **EQUIPOTENTIALS**. Actually, even in non-equilibrium case, the plates are usually assumed to be equipotentials and this is usually a good approximation.

The potential difference  $V$  between the **PLATES** is the capacitor potential: it is the positive plate potential minus the negative plate potential. The capacitor potential is always positive except in cases where the defined positive plate happens to have a negative charge and therefore a negative potential (e.g., see § 5.5).

**CAPACITANCE** is defined to be

$$C = \frac{Q}{V} . \tag{1}$$

In words, capacitance is how much charge a capacitor can hold per capacitor voltage (i.e., how many coulombs per volt). The capacitor potential is often imposed by some voltage source. The intrinsic capacitance is the capacitance when no outside forces perturb the charge distribution. We will always assume that we are dealing with the intrinsic capacitance which

is usually a good approximation for real capacitors.

For easy reference, we repeat equation (1) and gives its immediate to the consequences

$$C = \frac{Q}{V}, \quad Q = CV, \quad V = \frac{Q}{C}. \quad (2)$$

Since equation (1) is exact by definition, it leads to the definition of the unit of capacitance. This unit is the farad (F) named for Michael Faraday (1791–1867):

$$1 \text{ F} = 1 \text{ C/V}. \quad (3)$$

It turns out that 1 farad is a bloody big **CAPACITANCE**.

So typical small device capacitors are often in microfarads ( $\mu\text{F}$  which often written confusingly mF which are not millifarads), nanofarads (nF), or picofarads (pF which are sometimes called puffs (e.g., Wolfson & Pasachoff 1990, p. 621)).

Ideally, **CAPACITANCE** is actually independent of  $Q$  and  $V$ —which seems odd given its official definition.

**CAPACITANCE** does depend on the geometry (in which we include size scale) of the plates, and the **DIELECTRICS** in which the plates are embedded.

To understand, imagine a capacitor with charge  $Q$  and potential  $V$ . Any bit of charge on the capacitor is held in stable static equilibrium by the forces of all the other charges and the force of the conductor surface which we regard as an impenetrable wall. Remember

Fig. 1.— Generic capacitor and capacitor symbol  $\text{--}||\text{--}$ .

in electrostatic equilibrium all the net and/or separated electric charge is on a conductor surface. For the moment we assume the capacitor is embedded in vacuum, and so don't need to worry about dielectric effects. The capacitor is also isolated from all external charges and forces.

(The conductor surface force is actually a short-range electrostatic force caused by the nature of the conductor. But we idealize this force as that of an impenetrable wall.)

Now electrostatic electric field and potential just depend linearly on charge:

$$\vec{E}(\vec{r}) = \sum_i \frac{kq_i}{|\vec{r} - \vec{r}_i|^3}(\vec{r} - \vec{r}_i) \quad \text{and} \quad V(\vec{r}) = \sum_i \frac{kq_i}{|\vec{r} - \vec{r}_i|}, \quad (4)$$

where the sum is over all charge of the capacitor charge distribution with point charges labeled by  $i$ .

So if one just instantaneously scales up all charges by a common factor  $f$ , then electric field and potential at any point just scale up by  $f$  too.

But does any charge move after an instantaneous scale-up?

No. The net electric force on any charge scales up  $f^2$ , and so the direction of the electric force stays the same. But the charge stays in electrostatic equilibrium since the conductor surface forces scale up to keep the charge from moving—remember the surface is an impenetrable wall.

So nothing moves. The distribution of charges stays the same after the scale-up. The system is still in stable static equilibrium. So charges don't start moving after the scale-up

Fig. 2.— Generic capacitor with forces on a single charge indicated.

and the scaled-up electric field and potential don't start changing just after the instantaneous scale up.

Thus, the ratio  $Q/V$  should be unaffected by the scale-up:  $(fQ)/(fV) = Q/V$ . The ratio is just the capacitance  $C$  itself. So capacitance is actually independent of  $Q$  and  $V$ . Impose a charge  $Q$  on a capacitor and there will be a potential  $Q/C$ . Impose a potential  $V$  on a capacitor and there will be a charge  $CV$ .

One also has to add that for an isolated ideal capacitor with a given capacitor charge  $Q$ , the charge distribution and potential are unique—or so I believe. This means that capacitance (i.e., intrinsic capacitance) is unique.

Embedding the capacitor in an ideal dielectric (or less ideally multiple dielectrics) does not change the result. Ideally, the effect of the dielectric is proportional to the (free) charge of the capacitors, and so just scales up with that charge. (Dielectrics can have separated bound charge in them: they are polarizable insulators—all insulators are polarizable actually I believe.)

So the argument with dielectrics present is virtually unchanged from what we have given. But the capacitance  $C$  will depend on the dielectrics through their dielectric constants  $\kappa$ . We introduce the dielectric constant in § 6.

Just one more thing—as Colombo used to say—take another perspective on our scaling argument. Instead of scaling charges, let's scale all lengths by  $f$ . Now all potentials scale down by  $1/f$ . But still no free charges move from their relative positions. The free charge electrostatic forces on them scale down by  $1/f$ , but they stay in stable static equilibrium since the surfaces forces scale down to match them. Now the ratio  $Q/V$  changes to  $Q/(V/f) = fQ/V$ . So the capacitance of capacitor scales with the linear size of the capacitor. We note that the dimension of the capacitance is that of length over the dimension of the Coulomb

constant:

$$[C] = \frac{L}{[k]} = L[4\pi\epsilon_0], \quad (5)$$

where  $[\ ]$  is the dimension function and  $L$  the length dimension.

What does this mean? It just takes less energy to create a charge distribution, the farther the charges are apart overall. The farther apart the weaker the inter-charge forces.

## 2.1. Charging a Capacitor

There are many ways that a capacitor can be charged. But mostly they amount to much the same thing. So let's just consider charging with a battery for concreteness.

One attaches by conducting wires the defined positive plate to the positive terminal of the battery and the defined negative plate to the negative terminal of the battery. Usually, the choice of defined positive and negative plate is arbitrary since most capacitors work symmetrically.

Initially after connection, the battery creates an electric field that extends through the wires into the capacitor plates. That field that pulls electrons off the positive plate (which is like pushing positive charge onto it) and pushes electrons onto the negative plate.

This is the charging process itself

The charging continues until each terminal and its plate become equipotential. When

Fig. 3.— Charging a capacitor.

this happens there is no electric field in the conductors to move charge.

The capacitor potential recall is the potential difference between the plates. When fully charged (i.e., then the charging process has stopped), the capacitor potential is the battery potential (i.e., the potential between the battery terminals).

What actually goes on in the battery is beyond our scope—it's chemistry. But schematically, chemical forces create electrical forces that transfer the charge between the plates and transform chemical energy into the electrical potential energy  $PE$  of the separated charge. The battery does work on the charge.

We consider the potential energy of capacitors in § 5.

Does the capacitor have to end up neutral overall with  $Q$  on the positive plate and  $-Q$  on the negative plate? And is the capacitance of the capacitor in the battery system the same as if it were just an isolated set of plates?

Well in general no to both questions.

But in most practical applications the answers are yes to some approximation.

Say the system was overall neutral.

The ideal battery and approximately real batteries should stay overall macroscopically neutral in all states. Internally, there is a charge separation effected by chemical forces which makes the positive terminal positive and the negative terminal negative, but overall the battery should stay neutral ideally.

Thus, the capacitor and wires should stay overall neutral approximately just by themselves.

The battery transfers positive charge from the negative plate and connecting wire to the positive plate and connecting wire in the charging process without collecting any for itself

ideally.

So the net charges on the positive plate and connecting wire and negative plate and connecting wire should be equal in magnitude.

Now because of the strong attraction between positive and negative charge, one expects the final stable capacitor and connecting wire charge distribution to be one where nearly all the positive charge is on the positive plate facing nearly all the negative charge on the negative plate. Probably very little, charge is on other faces of the conductors or on wires.

So usually the capacitor should end up nearly overall neutral by itself with charges of equal magnitude and opposite sign on the plates.

The battery and capacitor can in general perturb their final stable equilibrium states relative to what they would be in isolation. But often this is slight. The two are often relatively remote, and so their respective electric fields will not much influence the other's behavior.

Thus, the capacitor capacitance in a circuit is usually not too different than if it were isolated.

But one must remark that all of the above could change if the battery-capacitor system is not overall neutral, if the battery and capacitor do perturb each other, and if there are outside perturbations.

In many electrical applications, you don't have to worry about these complications. But if they are strong, you may have too. Or if you are working to the highest level of accuracy.

We won't worry about such complications much hereafter.



### 3. CALCULATING CAPACITANCE

In intro physics classes, one typically thinks of parallel-plate capacitors for doing examples. But there are lots of other geometries. Here we look at a few with no dielectrics.

#### 3.1. Spherical Capacitors

Imagine two concentric conducting spherical shells. These are our plates. The outer radius of the inner shell is  $r_1$  and that of the outer is  $r_2$ . We put a charge  $Q$  on plate 1 and  $-Q$  on plate 2. From Gauss's law and integration of the electric field, the potential **BETWEEN** the shells is

$$V(r) = \frac{kQ}{r} , \tag{6}$$

where we have chosen the arbitrary constant of the potential to be **ZERO**. Note that the potential is zero (relative to infinity) everywhere outside plate 2.

All the charge on the plates must be facing surfaces by the following argument in which strict spherical symmetry is assumed. Remember in electrostatic equilibrium  $\vec{E} = 0$  inside conductors and all charge must be on their surfaces. But if any charge were inner surface of the inner plate, the E-field would not be zero inside the conductor. So all the charge of the inner plate is on its outer surface. For the outer plate if there were any net charge inside of a Gaussian spherical surface in its interior, the E-field would be non-zero. Since the E-field must be zero there, all the charge on the outer plate must be on its inner surface. We note that the only region of non-zero electric field is between the plates.

The potential difference—which is the capacitor potential between the plates is

$$V = kQ \left( \frac{1}{r_1} - \frac{1}{r_2} \right) . \tag{7}$$

Thus, the capacitance is

$$\begin{aligned}
 C = \frac{Q}{V} &= \frac{1}{k(1/r_1 - 1/r_2)} = \frac{r_1 r_2}{k(r_2 - r_1)} \\
 &= \frac{r_1 r_2}{k r_1 (1 - r_1/r_2)} \\
 &= \frac{r_1 (r_1 + \Delta r)}{k \Delta r} ,
 \end{aligned} \tag{8}$$

where  $\Delta r = r_2 - r_1$ .

The result is independent of  $Q$  and  $V$  as we proved in general it should be in § 2.

As a reference example, what is the capacitance for  $r_1 = 1$  m and  $r_2 = 2$  m? Well

$$C = \frac{r_1}{k(1 - r_1/r_2)} \approx \frac{1}{10^{10} \times (1/2)} = 2 \times 10^{-10} \text{ F} = 0.2 \text{ nF} \tag{9}$$

which is doesn't seem like a lot for such big object on the human scale. There are ways of getting capacitances of this size with much smaller capacitors.

We note that capacitance scales up with the length scale of the capacitor. This is just an explicit formula example of the result we proved in § 2: capacitance scales up with the length scale of the capacitor.

We also note that capacitance increases as  $\Delta R$  decreases. This result just reflects the fact that potential difference between separated charge tends to decrease with decreasing charge separation. This in turn reflects the fact that it takes less applied work to separate charge from the neutral state, the smaller the distance one has to push opposite to the

Fig. 4.— A spherical capacitor.

attractive electric force between the separating charge. This increase of capacitance with decreasing plate separation is just a strong tendency. It obviously depends on the path between the plates and the nature of the electric field. Usually, by separation one means the shortest distance.

In analyzing capacitor behavior one should keep in mind the two capacitance trends: (1) linearly increasing capacitance with increasing overall capacitor scale and (2) the tendency for capacitance to increase with decreasing plate separation.

If  $r_2 \rightarrow \infty$ , one has in a sense a one-plate capacitor with capacitance

$$C = \frac{r_1}{k} = 4\pi\epsilon_0 r_1 . \quad (10)$$

We note that capacitance scales up with  $r_1$  since the charge on the plate is being more widely separated.

What does a one-plate capacitor mean?

Just what it seems.

You take  $Q$  from infinity where one thinks of point charges making up  $Q$  as being infinitely far apart and at zero potential difference with respect to each other and everything else.

You move this  $Q$  to your plate and leave  $-Q$  infinitely far away. But infinity doesn't notice the loss of  $Q$  because it's an infinite reservoir of charge of both flavors.

Now “infinity” is a idealization.

But say you assemble a net charge of  $Q$  on a relatively isolated conductor. This can be done in practice.

The  $-Q$  left behind is somewhere, but it might be dispersed in some very large remote reservoir—effectively infinity.

### 3.2. Cylindrical Capacitors

We consider two conducting con-axial cylindrical conducting shells. The inner one has outer radius  $R_1$  and outer one has inner radius  $R_2$ . We treat them with the approximation of exact cylindrical symmetry in order to have simple expressions for electric field, potential, and charge distribution. This is an approximation since their length is in fact  $\ell$ , and so they can't have exact cylindrical symmetry. These are the plates of a cylindrical capacitor. We give the inner plate a charge  $Q$  and the outer plate a charge  $-Q$ . The charge per unit length on the inner plate is  $\lambda = Q/\ell$  and on the outer plate is  $-\lambda = -Q/\ell$ .

From Gauss's law, one has

$$\vec{E} = \frac{2k\lambda}{R} \hat{R} \quad (11)$$

for the electric field between the plates where  $R$  is the cylindrical radius coordinate, and where  $\hat{R}$  is the cylindrical coordinate radius unit vector. The electric field is zero everywhere else by essentially the same reasoning as in § 3.1.

Now we integrate to get the potential difference between  $\vec{r}_0$  to  $\vec{r}$  where we continue to use small  $r$  to denote general displacement from a general origin on the axis of symmetry and reserve The integration is

$$V(r) = - \int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{s} = - \int_{R_0}^R E_R dR = - \int_{R_0}^R \frac{2k\lambda}{R} dr = -2k\lambda \ln \left( \frac{R}{R_0} \right) . \quad (12)$$

We define  $R_0$  to the cylindrical coordinate of zero potential.

The potential difference between the plates with linear charge densities  $\lambda$  on the inner plate and  $-\lambda$  on the outer plate is

$$V = -2k\lambda \left[ \ln \left( \frac{R_1}{R_0} \right) - \ln \left( \frac{R_2}{R_0} \right) \right] = 2k\lambda \ln \left( \frac{R_2}{R_1} \right) \quad (13)$$

which is greater than zero note. The linear charge density  $\lambda = Q/\ell$ .

Thus, in the exact cylindrical symmetry approximation, the capacitance is

$$C = \frac{Q}{V} = \frac{\ell}{2k \ln(R_2/R_1)} \quad (14)$$

and the capacitance per unit length is

$$\frac{C}{\ell} = \frac{1}{2k \ln(R_2/R_1)} . \quad (15)$$

Another version of the cylindrical-plate capacitor formula can be obtained by introducing the plate separation  $\Delta R$ :

$$\Delta R = R_2 - R_1 . \quad (16)$$

With this definition, the capacitance is

$$C = \frac{\ell}{2k \ln(1 + \Delta R/R_1)} . \quad (17)$$

Equation (17), exhibits the two capacitance trends discussed in § (3.1): (1) linearly increasing capacitance with increasing overall capacitor scale (determined by  $\ell$  in eq. 17)) and (2) the tendency for capacitance to increase with decreasing plate separation (determined by  $\Delta R$  in eq. 17)).

### 3.3. Parallel-Plate Capacitors

The parallel-plate capacitor is actually the usual example capacitor in intro physics. It's sort of the prototype simplest capacitor.

Fig. 5.— A cylindrical capacitor.

Let's consider the ideal infinite parallel-plate capacitor which actual parallel-plate capacitors approximate. The plates are conductors and have finite thicknesses.

On one plate we put surface charge density  $\sigma$  and on the other we put  $-\sigma$ . However the charge is arranged on the plates, it is arranged in layers that have infinite planar symmetry. This means that outside of the plates all the planes of positive and negative charge add up to give zero electric field. Since the surface electric field is proportional to the surface charge density for conductors: there is no surface charge density on the outer surfaces of the plates. Thus, all the charge must be on the inner surfaces of the plates. So  $\sigma$  is on one inner surface and  $-\sigma$  is on the other. The electric field between the plates is uniform and points from the positive plate to the negative plate and has value

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{z} + \frac{(-\sigma)}{2\epsilon_0} (-\hat{z}) = \frac{\sigma}{\epsilon_0} \hat{z} , \quad (18)$$

where we have used the infinite plane of charge result twice and  $\hat{z}$  is a unit vector pointing from the positive to the negative plate.

We now integrate from the negative to the positive plate by any path to find the potential difference between the two plates. We find

$$V = - \int_{\vec{r}_-}^{\vec{r}_+} \vec{E} \cdot d\vec{s} = - \int_{\vec{r}_-}^{\vec{r}_+} \vec{E}_z d\vec{z} = \frac{\sigma d}{\epsilon_0} , \quad (19)$$

where  $r_+$  the location of the positive plate,  $r_-$  the location of the positive plate and  $d$  is the distance between the plates.

Let's say we have finite plates of area  $A$  that we treat in the infinite plate approximation. Note  $\sigma = Q/A$ , where  $Q$  is the capacitor charge, of course. The capacitance is

$$C = \frac{Q}{V} = \frac{Q\epsilon_0}{\sigma d} = \frac{\epsilon_0 A}{d} . \quad (20)$$

Equation (20), exhibits the two capacitance trends discussed in § (3.1): (1) linearly increasing capacitance with increasing overall capacitor scale (determined by  $A/d$  in eq. 20))

and (2) the tendency for capacitance to increase with decreasing plate separation (determined by  $d$  in eq. 20)).

One can't decrease  $d$  to vanishing thickness, however, and make an arbitrarily large capacitance. For a given potential  $V$ , the electric field between the plates is  $E = V/d$ . Thus, decreasing  $d$  increases the E-field between the plates and at some point there will be a charge flow. If the gap between the plates is a vacuum, the E-field will at some point pull charge off the plates and allow conduction. If there is a dielectric between the plates, the dielectric will at some point break down and conduct.

### 3.3.1. Example 1: Fiducial Parallel-Plate Capacitance

Say we have fiducial values  $d = 1 \text{ m}$  and  $A = 1 \text{ m}^2$ , then

$$C = \frac{\epsilon_0 A}{d} = 8.854 \dots \times 10^{-12} \text{ F} \times \left( \frac{A}{1 \text{ m}^2} \right) \left( \frac{1 \text{ m}}{d} \right) = 8.854 \dots \text{ pF} \times \left( \frac{A}{1 \text{ m}^2} \right) \left( \frac{1 \text{ m}}{d} \right) . \quad (21)$$

Now I know what you are thinking.

The above fiducial values are a lousy choice.

We've assumed that one can make the infinite plate approximation. But that is only valid if the plates are large compared to the separation distance—and the plates are only comparable to the separation distance.

A better choice for the fiducial separation distance is  $d = 1 \text{ mm} = 0.001 \text{ m}$ .

$$C = \frac{\epsilon_0 A}{d} = 8.854 \dots \times 10^{-9} \text{ F} \times \left( \frac{A}{1 \text{ m}^2} \right) \left( \frac{1 \text{ mm}}{d} \right) = 8.854 \dots \text{ nF} \times \left( \frac{A}{1 \text{ m}^2} \right) \left( \frac{1 \text{ mm}}{d} \right) . \quad (22)$$

### 3.3.2. Practical Capacitors

There are all kinds of practical capacitors. It is beyond our scope to say much about them—all that’s for other courses or practical work. See, however,

<http://www.uoguelph.ca/~antoon/gadgets/caps/caps.html> .

But a typical design is two long strips of aluminum foil for plates with a layer of dielectric plastic between them and some other second dielectric (but being used just for electrical insulation) on the outside of one of the foils. The strips are rolled up for compactness and sealed in insulating cover which is probably usually plastic (e.g., Wolfson & Pasachoff 1990, p. 621). The second dielectric prevents the two foils from touching in the rolling up: if they touched, they would be an equipotential in any static situation. Wire attached to the plates can connect the two plates to the outside environment.

Such a rolled-up capacitor approximates a set of con-axial cylindrical capacitors, but it can’t be exactly that since each turn of the roll does not close. However, it is also still approximately a parallel-plate capacitor. Recall one of our formulae for the cylindrical capacitor in the exact cylindrical symmetry approximation (eq.. (17) in § 3.2):

$$C = \frac{\ell}{2k \ln(1 + \Delta R/R_1)} , \quad (23)$$

where  $\ell$  is the capacitor length,  $\Delta R$  is the plate separation in cylindrical radius coordinate, and  $R_1$  is the inner plate cylindrical radius. If we Taylor’s series expand around  $\Delta R/R_1 = 0$  and truncate after the 1st order term, we obtain

$$C \approx \frac{\ell}{2k(\Delta R/R_1)} = \frac{2\pi\epsilon_0\ell R_1}{\Delta R} \quad (24)$$

which leads to

$$C \approx \frac{\epsilon_0 A_{\text{cyl}}}{\Delta R} \quad (25)$$



where  $A_{\text{cyl}} = 2\pi R_1 \ell$  is the area of the inner cylinder. The last equation is just the parallel-plate capacitor formula equation (20) all over again. Remember that formula is derived for the exact planar symmetry, and so is itself an approximation for real capacitors.

So a rolled-up capacitor where the outer turns (which have  $\Delta R/R_1$  small) dominate the capacitance should be approximately a parallel-plate capacitor with capacitance given approximately by equation (20). The  $d$  value is the 1st dielectric thickness. The area of the plates is the whole area of the strips which is the sum of the turn areas  $A_{\text{cyl}}$ . If the initial unrolled capacitor was long and thin, then the outer turns should dominate. I'd guess this is the usual case.

The capacitance of the capacitor is usually marked on the outside. The dielectric between the plates increases the capacitance (§ 6). Dielectrics with high dielectric constants  $\kappa$  can greatly increase the capacitance. The dielectric can also increase the breakdown field of the capacitor and prevent conduction for high E-fields. And also it is probably easier to construct structurally sturdy capacitors if there is no air or vacuum gap between the plates, but a solid material. However, capacitors with vacuum between their plates have special uses: see

[http://en.wikipedia.org/wiki/Capacitor#Capacitor\\_types](http://en.wikipedia.org/wiki/Capacitor#Capacitor_types) .

### 3.4. Zero Capacitance Capacitors

This odd entity is what an ideal **OPEN CIRCUIT** is: typically this is an unconnected wire. For example, say you have two wires connected to battery terminals, but not connected to each other and left dangling. The wires have the battery potential difference between them since each wire and its terminal are equipotentials in electrostatic equilibrium. But the wires

Fig. 6.— Parallel-plate capacitor.

Fig. 7.— Zero and infinite capacitance capacitors.

should be neutral ideally. Thus, ideally

$$C = \frac{Q}{V} = 0 . \quad (26)$$

Actually, this can't be quite true—the battery should be able to effect some charge separation on the wires—but typically dangling wires have low capacitance—but usually this is low enough to be approximated as zero.

Of course, not just two dangling wires form a capacitor. The contacts of a switch form a capacitor that ideally should have zero capacitance. A single dangling wire at some potential can be viewed as the plate of a capacitor with other plate being the whole environment. Ideally, this should also have zero capacitance.

Small accidental capacitances are, in fact, all over and in very detailed work probably need to be accounted for.

### 3.5. Infinite Capacitance Capacitors

Now what the devil are these? You can put charge on them without a potential. Actually, current flowing through an ideal conductor is like this.

For example, consider a short segment of a wire. A current of positive charge flows into the point and the same current flows out of the point. This can go on endlessly loading one side of the segment with positive charge while the other side of the segment is getting negatively charged. Or so one can imagine it. For an ideal wire there be no potential difference across the segment (as we'll discuss in the lecture **DIRECT CURRENT**), but in reality there will be a small one. For the ideal case, for any amount of mythical charge  $Q$  on the segment one finds

$$C = \frac{Q}{V} = \infty . \quad (27)$$

Now this way of regarding segments of wires may seem a bit useless. But when one deals with circuit formulae with capacitors and one wonders what the case is with capacitor taken out, but circuit connected where the capacitor was, the appropriate mathematical operation is to let  $C$  go to infinity (e.g., Weber & Arfken 2004, p. 327).

#### 4. CAPACITORS IN COMBINATIONS

Capacitors in combinations occur in circuits. One usually thinks of circuits as having current flowing. They are a network of conductors with various electrical devices in which current flows. In fact, the basic capacitor behavior is the same for current-flowing and electrostatic cases since the electrostatic properties of conductors apply even for current-flowing cases to high accuracy at least as far as capacitors are concerned. Thus, all the results given above and below carry over to current-flowing cases to high accuracy.

We'll need circuit diagrams in this section. Circuit diagrams are standardized schematic diagrams of circuits in which the wires are represented by straight lines that are usually only horizontal and vertical and circuit symbols stand for standard circuit elements or devices. The capacitor symbol is  $\parallel$ .

Note the ideal battery has a fixed potential difference between its terminals no matter what: if current is flowing or not.

Real batteries can't quite do this when current is flowing since they have internal re-

Fig. 8.— Circuit symbols for battery, capacitor, open switch and closed switch.

sistance which lowers the potential difference when current flows, but batteries are close to ideal in the electrostatic case.

#### 4.1. Capacitors in Parallel

Say one has a set of conducting devices each device with two leads: one a positive lead and the other a negative lead. The positive leads are all connected and constitute an equipotential and the negative leads are all connected and constitute an equipotential. The devices are described as being in parallel and as having the same potential change across them.

Now say one had an assembly of capacitors in parallel. This could be effected for example by connecting the defined positive plates to a positive battery terminal and the defined negative plates to a negative battery terminal. When fully charged each positive plate is at the battery positive potential and each negative plate is at the battery negative potential in an electrostatic situation.

Now again say one had an assembly of capacitors in parallel with potential difference  $V$ . For capacitor  $i$ , one has

$$V = \frac{Q_i}{C_i}, \quad (28)$$

where  $C_i$  is the capacitance and  $Q_i$  is the charge of capacitor  $i$ .

Thus

$$Q_i = C_i V \quad (29)$$

and the total charge  $Q$  on the capacitors collectively is

$$Q = \sum_i Q_i = \sum_i C_i V. \quad (30)$$

Thus

$$\frac{Q}{V} = \sum_i C_i \tag{31}$$

Making use of the definition of capacitance (eq. (1) in § 2), we identify

$$C = \sum_i C_i \tag{32}$$

as the capacitance of the assembly.

This is a nice simple, memorable result. The capacitance of capacitors in parallel is the sum of the individual capacitor capacitances.

Note that

$$C \geq \max(C_i) . \tag{33}$$

## 4.2. Capacitors in Series

To have devices in series in circuit theory means that a set of conducting devices with two terminals are connected one after another in such a way that a common current flows through them—if there is any current flowing.

Say we have an assembly of capacitors in series.

We charge the capacitor assembly by a battery, for example. We connect the positive battery terminal to the defined positive end of the series and the negative battery to the

Fig. 9.— Capacitors in parallel with a battery put across them.

defined positive end of the series. Usually the choice of negative and positive ends of the series is arbitrary.

Initially, the battery causes a current that pulls negative charge off the positive plate of the first capacitor (the highest plate) and pushes negative charge onto the negative plate of the last conductor (the lowest plate). But this process sets up a concatenation of current flows internal to the series.

The charging process continues until the highest and lowest plates are equipotentials with the terminals they are connected too.

The arrangement of charge inside the series can be complex in general, but in practice it's usually pretty simple. The charge tries to self-neutralize as much as possible, that happens if the individual capacitors in the series are neutral because that is where charge can get close together on opposing plates. Any excess net charge on one capacitor will tend to result in a repulsion of charge from that capacitor to somewhere else or an attraction of neutralizing charge from somewhere else. Perturbations between the capacitors and outside perturbations can change things, but often in practice the capacitors should end nearly neutral overall and the connecting wires neutral in detail. We'll assume this is exactly the case.

But remember any two connected plates the series (which are on different capacitors note) and their connecting wire must be overall neutral by charge conservation (assuming they were neutral to begin with).

Given the neutrality considerations above, one must end up with the same capacitor charge for each capacitors. Say the highest plate has charge  $Q$ . The other plate of its capacitor must have charge  $-Q$ . But the positive plate of the next capacitor must then have charge  $Q$  by charge conservation. This capacitor's other plate must have charge  $-Q$ , since the individual capacitors are neutral as we have argued. And so on. All the capacitors have

charge  $Q$ .

Now the potential drop as one goes across each capacitor  $i$  must be

$$V_i = \frac{Q}{C_i}, \quad (34)$$

where the  $Q$  is common to all the capacitors as argued above.

Remember the plates connected by wires and the connecting wires are equipotentials and the sum of potential difference between the battery terminal must be the same for any path between the terminals. Thus, the battery potential difference must equal the sum of the potential differences of the capacitors:

$$V = \sum_i V_i = \sum_i \frac{Q}{C_i}. \quad (35)$$

Regarding the series as a black box, a charge  $Q$  was moved from the negative end to the positive end.

Thus, from an outside perspective, the series treated as a single capacitor has charge  $Q$ .

The capacitance of the overall assembly of capacitors is

$$C = \frac{Q}{V} = \frac{Q}{\sum_i Q/C_i}, \quad (36)$$

from which we obtain

$$\frac{1}{C} = \sum_i \frac{1}{C_i} \quad (37)$$

as the equation of the assembly's overall capacitance.

Note that

$$\frac{1}{C} \geq \max\left(\frac{1}{C_i}\right), \quad (38)$$

and thus

$$C \leq \min(C_i). \quad (39)$$



So connecting capacitors in series gives a smaller capacitance than any single capacitor alone.

Why would one use capacitors in series? The only practical reason I can think of is to construct a capacitor of the size needed for some purpose from a set of standard sized capacitors. Maybe there other reasons I know not of. Capacitances in series might turn up accidentally

#### 4.2.1. Example: Capacitors of Equal Capacitance in Series

Say that all the capacitors in series have equal capacitance  $C_0$  and there are  $N$  of them. Then the equivalent capacitance is given by

$$\frac{1}{C} = \frac{N}{C_0} \quad \text{or} \quad C = \frac{C_0}{N} . \quad (40)$$

This case gives some insight into why capacitance goes down by putting capacitors in series. Viewing the series capacitors as hidden in a black box, they have separated charge

$$Q = CV \quad (41)$$

but inside the box they have separated charge  $NCV$  which is just the charge one capacitor of  $C_0$  alone would have separated since

$$NCV = N \frac{C_0}{N} V = C_0 V . \quad (42)$$

Of course, that separation inside the black box ideally has no consequence in the outside world.

Fig. 10.— Capacitors in series with a battery put across them.

#### 4.2.2. Example: Two Capacitors in Series

In this case

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \quad (43)$$

which immediately gives

$$C = \frac{C_1 C_2}{C_1 + C_2} . \quad (44)$$

## 5. POTENTIAL ENERGY STORED IN A CAPACITOR

What is the potential energy stored in a capacitor?

Consider an isolated capacitor.

It has zero net charge and the electrical potential energy relative to infinity is zero.

If we moved charge from the defined negative plate to the defined positive plate, we build up a separated charge distribution and the potential energy increases.

We could do it with a battery as discussed in § 2.1. But the details of the process do not matter to the state of the plates. The same plate  $PE$  is created for the same amount of charge transfer no matter how it is done. The capacitor capacitance is unaffected by the movement process. Recall, the electrostatic electric force is conservative. One doesn't have to use a battery at all, but that is usually how a capacitor is charged in simple demonstrations.

So we don't have to consider any detailed process. We can just imagine transferring the charge abstractly through the potential difference between the plates of a capacitor of capacitance  $C$ .

Say the potential difference between the plates was  $V'$  at some time in the transfer

process. The  $PE$  change transferring a differential amount of charge  $dq$  at that time is

$$dPE = V' dq = \frac{q}{C} dq , \quad (45)$$

where  $q$  is the capacitor charge at time of transfer and  $V' = q/C$  follows from the definition of capacitance. We can now just integrate to get total  $PE$  for a transfer of total charge  $Q$ :

$$PE = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C} . \quad (46)$$

All the variations on the  $PE$  formula follow using the definition of capacitance  $C = Q/V$ :

$$PE = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV . \quad (47)$$

Note that the  $PE$  for fixed  $Q$  increases with decreasing capacitance and for fixed  $V$  increases with increasing capacitance.

We see that capacitor can be used to store energy as well as charge. The energy is extracted from the capacitor by letting the charge on the capacitor flow back to the neutral situation (e.g., through wires and resistors).

There are actually practical limits to storing charge and energy in a capacitor. If the potential difference gets too large (which implies a large electric field), charge will start to flow between the plates. It can be pulled off the surface of the plates if the capacitor has vacuum between the plates and if there is a dielectric between the plates (which is usual), then the dielectric can break down (i.e., start to conduct).

Unlike batteries capacitors do not naturally release energy at a constant rate (i.e., do not produce constant power). This is a problem for their use in many applications. But if you need a fast release of energy and do not care much about constant power, then capacitors may be what you need. Some flash cameras use capacitors to store up energy for flashes. Capacitors have much longer lifetimes than chemical batteries. They are just two metal

plates with dielectric between them. They can be recharged a quasi-infinity of times. Very high-capacitance capacitors (Wikipedia: Electric double-layer capacitor) may even be used in place of batteries in electric cars—it’s one of those battle of competing technologies things.

### 5.1. A Bit of History

In the 18th century people thought of electricity as a sort of fluid—which is actually not such a bad idea. That idea somehow led to the idea that electricity could be contained in water in a jar. And that idea led to the Leyden jar which is a capacitor consisting of metal foils on the inside and outside of a glass jar—the water turned out to be unnecessary. The foils are the plates and the glass is the dielectric that separates the plates. The Leyden jar is a cylindrical capacitor (Wikipedia: Leyden jar).

Large charge separations could be built up in Leyden jars—sometimes leading to large shocks. Ben Franklin realized that there was no need for a jar and used foils on plates of glass: i.e., he created a parallel-plate capacitor. For one experiment, he built up a large charge and tried to use it—one can picture him fiendishly cackling to himself in his laboratory—to kill a turkey—he probably would have had it for dinner—but ended up shocking himself: “I tried to kill a turkey, but nearly succeeded in killing a goose” (e.g., Tipler & Mosca 2008, p. 803).

Fig. 11.— Ben Franklin and the turkey.

## 5.2. Another Derivation of the Capacitor Potential Energy: Optional

Recall

$$PE = \frac{1}{2} \int \rho(\vec{r})V(\vec{r}) d\mathcal{V} \quad (48)$$

is the potential energy of a localized charge distribution where  $\rho(\vec{r})$  is the charge density,  $V(\vec{r})$  is the potential relative to zero potential at infinity, and the integral is over all space which we idealize as only containing the charge distribution.

For the case of a set of isolated electrostatic conductors, the potential energy formula specializes to

$$PE = \frac{1}{2} \sum_i q_i V_i , \quad (49)$$

where the sum is over all conductors  $i$ ,  $q_i$  is the charge on conductor  $i$ , and  $V_i$  is the potential of conductor  $i$ . Recall conductors are equipotentials in an electrostatic case.

Say we just have two conductors that are overall neutral. Without loss of generality let  $q_1 > 0$ . Then  $q_2 = -q_1$  and we have

$$PE = \frac{1}{2} \sum_i q_i V_i = \frac{1}{2} q_1 (V_1 - V_2) . \quad (50)$$

The system is just a capacitor with capacitor charge  $q_1$  and capacitor potential  $V_1 - V_2$ . Since  $C = Q/V$ , we recover our earlier results for capacitor potential energy (eq. (47 in § 5)).

Since the potential energy to charge a capacitor is known to be greater than zero from the simple consideration of moving bits of charge from one plate to another, we know that  $V_1 - V_2 \geq 0$ .

We might ask what are  $V_1$  and  $V_2$  individually. Yours truly thinks this is the most interesting aspect of this development. The answer is that there is no one answer. The individual  $V_1$  and  $V_2$  values depend on the geometry of the system.

For example, say the conductors were identical spherically symmetric balls that didn't magically overlap. Symmetry then dictates that

$$V_2 = -V_1 , \tag{51}$$

and then that capacitor potential is

$$\frac{1}{2}(V_1 - V_2) = V_1 . \tag{52}$$

On the other hand, say we had two concentric spherical shells for conductors. The potential outside is zero everywhere since the charge distribution is spherically symmetric and neutral. So the outer shell is at zero potential and the inner shell differs in potential by  $|V_1 - V_2|$  from the outer shell. If the inner shell is the positive one, then  $V_1 = kq/r_1 > 0$  (where the inner shell radius is  $r_1$ ) and  $V_2 = 0$ . If the inner shell is the negative one, then  $V_2 = -kq/r_2 < 0$  (where the inner shell radius is  $r_2$ ) and  $V_1 = 0$ .

I don't think that the positive shell can ever have less than zero potential and the negative shell more, but I can't think of simple decisive argument at the moment.

### 5.3. The Energy of the Electric Field

We mentioned in the lecture **ELECTRIC POTENTIAL** that electric potential is the energy of the electric field. It is probably true in all real contexts that potential energies are actually describable as other forms of energies. The term potential energy just means that one can associate each position in space with an amount of some form of energy. It does not in itself specify the nature of that energy any further.

We can derive the formula for the electric field energy with some simple assumptions. Consider an ideal parallel-plate capacitor with no dielectric:

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d} , \tag{53}$$

where  $A$  is the plate area and  $d$  is the plate separation. For, an ideal parallel-plate capacitor, the E-field exists only between the plates, is uniform, and satisfies  $V = Ed$ . Thus

$$PE = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\varepsilon_0 A}{d}(Ed)^2 = \frac{1}{2}\varepsilon_0 E^2 Ad . \quad (54)$$

Now  $Ad$  is the volume of the region between the plates. If we assume that the  $PE$  is the electric field energy and this energy has a density which is uniform for a uniform E-field, then the energy density  $u$  is plausibly

$$u = \frac{PE}{Ad} \quad (55)$$

which gives the formula

$$u = \frac{1}{2}\varepsilon_0 E^2 . \quad (56)$$

Equation (56) is indeed correct for vacuum E-fields. Proving the generality of the result is beyond our scope though. The formula is actually exactly true in classical physics at the microscopic level even in the presence of dielectrics. At the macroscopic level inside of matter, the vacuum permittivity  $\varepsilon_0$  is replaced by the matter permittivity  $\varepsilon$  (Wikipedia: Electric field):

$$u = \frac{1}{2}\varepsilon E^2 . \quad (57)$$

Note the E-field energy density is proportional to the square of  $\vec{E}$  and is always a positive quantity.

Note any electric field has the energy density given by equation (56) or equation (57) whether or not a potential energy, and therefore a potential, can be defined. Faraday law induced electric fields do not give conservative forces and do not allow potential to be defined. Electric fields for moving charge do not allow potential to be defined accept in the low speed limit (i.e., speed much less than the vacuum light speed), but, in fact, in many natural and human contexts the charge is in the low speed limit. Even in the low speed limit, potential can in general only at instants in time, unless the system is somehow in steady. But the

low-speed steady state case is what most circuits are in, and so it is a very important special case where potential can be defined.

#### 5.4. Further Considerations with Electric Field Energy

It is probably most fundamental (i.e., most true) to view electrical  $PE$  as being actually the energy of the electric field which has vacuum energy density

$$u = \frac{1}{2}\epsilon_0 E^2 . \quad (58)$$

One must, in fact, use electric field energy in cases where potential energy cannot be defined for the electric field: e.g., for Faraday's law induced electric fields, for electromagnetic radiation, and in general relativity (e.g., Griffiths 1999, p. 96). But electric field energy does require some further considerations even at our level.

First, consider a thin spherical shell of charge with total charge  $Q$  and radius  $R$  and uniform surface charge density (it is not necessarily a conductor). As we well know the potential at the shell surface relative to infinity is

$$V = \frac{kQ}{R} . \quad (59)$$

as we obtain from Gauss's law. The general formula for the self energy (i.e., self-potential energy) of a continuous charge distribution is

$$PE = \frac{1}{2} \int \rho(\vec{r})V(\vec{r}) d\mathcal{V} , \quad (60)$$

Fig. 12.— Thin spherical shell of uniform surface charge density.



where the integral is over the whole volume. In the present case,

$$PE = \frac{1}{2} \int \rho(\vec{r})V(\vec{r}) d\mathcal{V} = \frac{1}{2} \int \frac{Q}{4\pi R^2} \frac{kQ}{R} dA = \frac{1}{2} \frac{kQ^2}{R} . \quad (61)$$

This is the  $PE$ , but where is the  $PE$ ? What is the  $PE$ ? Maybe the question doesn't need an answer in many applications. But since we claim that the energy is the electric field, we better be able to calculate it from the electric field. The E-field for the shell is

$$\vec{E} = \begin{cases} \frac{kQ}{r^2} \hat{r} & r \geq R; \\ 0 & r < R. \end{cases} \quad (62)$$

from Gauss's law. If we now integrate up the electric field energy density over all space, we obtain

$$PE = \int_0^\infty \frac{1}{2} \varepsilon_0 E^2 (4\pi r^2) dr = \int_R^\infty \frac{1}{2} \varepsilon_0 \frac{k^2 Q^2}{r^4} (4\pi r^2) dr = \frac{1}{2} (4\pi \varepsilon_0) \frac{k^2 Q^2}{R} = \frac{1}{2} \frac{kQ^2}{R} , \quad (63)$$

where we have used

$$k = \frac{1}{4\pi \varepsilon_0} . \quad (64)$$

So we have agreement between the potential energy approach and the E-field energy density approach. That's satisfying.

But the E-field energy density approach always gives us a positive energy and we know that there are negative potential energy cases: e.g., the potential energy of a negative and positive point charge pair. And the resolution is that the potential energy approach did not count up the energy of assembling the point charges. In calculating the energy from the E-field energy density approach, we get the total energy of the assembling the charge distribution. The potential energy approach only gives the differences in energy from having the charge assemblies at different relative locations. Charge assemblies of opposite sign have their  $PE$  reduced by bringing them together from infinity, and thus the  $PE$  change is negative. If you add that in the energy of assembling the assemblies, then one would that

one would get agreement again. Proving this in general though is beyond our scope—and my current knowledge.

But there is a big problem. What the devil is the energy of a point charge? If we take our charged shell result and let  $R \rightarrow 0$ , the energy goes to infinity. One resolution may be that there are no real point charges. The proton for example is not really point-like: it has an electromagnetic RMS radius of  $\sim 0.8 \times 10^{-15}$  m (e.g. Enge 1966, p. 35) and quark structure it is believed. But on the other hand, the electron maybe be truly point-like or maybe not: perhaps modern **STRING THEORY** will eventually solve this point. There is no resolution in classical physics. In any case, the electron does not have infinite energy. From the Einstein equation  $E = mc^2$  and the electron mass of  $9.109 \times 10^{-31}$  kg, we know its energy is 0.511... MeV and we can create electrons using this amount of energy. The paradoxes of true point charges have not yet been resolved in modern physics (e.g., Griffiths 1999, p. 95).

What do we do about microscopic particle charges in classical physics? Well we just don't count the energy of assembling those charges when we count the energy of charge distributions made up of the particles. We don't count the self energy of electrons, protons, and nuclei. Sometimes we count changes in self energy of atoms and ions, but let's leave that to chemistry.

But what about continuous charge distributions with which we often approximate the point charges with? In those cases, we spread out the point charges into continuous charge densities. In the lecture **ELECTRIC POTENTIAL**, we showed that the self energy of point charges into spread into little volume cells made a negligible contribution to the self-energies of 3-dimensional and 2-dimensional continuous charge distributions. The self energies of the whole macroscopic distributions are dominated by long-range contributions between the cells. Thus counting those cell self-energies in counting up the self energy of a

continuous charge distribution was effectively like not counting the self energy of the actual point charges. It's paradoxical that point charges have infinite self energy, but spread that charge out into a little cell and for a macroscopic charge distribution the cell self energy is negligible, but that's the way it is.

The continuous charge distribution approach is a valid approximation for 3 and 2 dimensions for counting self energy of a macroscopic system without counting the self energy of actual point charges.

A continuous 1-dimensional charge distribution actually has infinite energy as we also showed in the lecture **ELECTRIC POTENTIAL**. We know this because the potential at infinite line of charge is infinite relative to infinity or any finite point: so adding charge to the distribution to build it up gives infinite energy. So a continuous 1-dimensional charge distribution is not a valid approach for calculating the self-energy of a macroscopic charge distribution without counting the self energy of actual point charges. We simply cannot use the 1-dimensional continuous charge distribution as an approximation for calculating the self energy of a macroscopic system. We can use it for calculating other things like electric field and potential in the vicinity of a thin line of charge.

Just one more elucidation—as Colombo used to say. Using continuous charge distributions, we can create and destroy gross charge, and so create and destroy self energy and electric field. We don't ever create or destroy net charge. The creations and destructions are only at the macroscopic level in many ordinary technological and natural contexts. In these contexts, the microscopic particles (electrons, protons, nuclei) are not created or destroyed or changed, and so the self energy associated with them does not have to be counted.

There are contexts where the microscopic particles are created or destroyed or changed. These are high energy events which include radioactive decay process. Those contexts usually don't come up in ordinary electromagnetic technology and they are beyond the scope of this

course.

### 5.5. Example: Capacitors Switched from Isolated to Parallel

This example is a generalization of an example of SJ-733. The generalization is actually clearer than the special case in my view. It often happens that way at least for comparatively simple systems: thinking generally from the start leads to a simple general solution from which the special cases follow.

Say we had a collection of isolated capacitors with capacitances  $C_i$ , charges  $Q_i$ , and potentials  $V_i$ : note  $Q_i = C_i V_i$  of course. We then order them with the fiducial positive plates all on the left say. If a plate happens to be actually negative, then its  $Q_i$  and  $V_i$  are negative. We now connect the capacitors in parallel using ideal conducting wires and allow the overall system to come to electrostatic equilibrium. What we expect to happen is the charge to rearrange itself so that the connected positive plates are an equipotential and the connected negative plates are an equipotential too. The wires are ideal and so should have no capacitance—there will be no net charge on them in equilibrium.

From an energetic point of view, the electric force must do work on the charges to rearrange them. Thus, electric potential energy gets converted into kinetic energy. If there were no way to dissipate this energy, then the charges would not settle down to equilibrium, but would be bouncing around. The situation is analogous to a perfectly elastic ball dropped

Fig. 13.— A collection of capacitors before and after being connected in parallel.

on a perfectly rigid surface: it will bounce perpetually. However, we assume that the kinetic energy of the moving charge gets dissipated to waste heat due to resistance in the capacitors and in the wires (which are not ideal in this regard). One could also imagine using the charge kinetic energy to do work. The loss of charge kinetic energy implies that the final  $PE$  must be lower than the initial  $PE$ . The transformation to the final equilibrium state is a typical example a system impelled by forces to a lower energy state where the freed up energy is either used to do work or gets dissipated to a random form (e.g., waste heat) which is the natural thing to happen to energy and which is a manifestation the 2nd law of thermodynamics that entropy increases (i.e., randomness increases) in closed systems to a maximum.

Now for some math. When connected in parallel, the net capacitance of the system is

$$C = \sum_i C_i \quad (65)$$

as we know from § 4.1. The net charge is, of course,

$$Q = \sum_i Q_i . \quad (66)$$

Thus equilibrium potential of the net capacitor is

$$V = \frac{Q}{C} = \frac{\sum_i C_i V_i}{\sum_i C_i} . \quad (67)$$

For SJ-733's example, there are only two capacitors and  $V_2 = -V_1$ . Thus,

$$V = \left( \frac{C_1 - C_2}{C_1 + C_2} \right) V_1 . \quad (68)$$

Now for energy. The initial potential energy of the system is

$$PE_{\text{initial}} = \sum_i \frac{1}{2} C_i V_i^2 . \quad (69)$$

Let's define

$$\delta V_i = V_i - V . \quad (70)$$

Now note that

$$V = \frac{\sum_i C_i V_i}{\sum_i C_i} = \frac{\sum_i C_i (\delta V_i + V)}{\sum_i C_i} = \frac{\sum_i C_i \delta V_i}{\sum_i C_i} + V \quad (71)$$

which implies that

$$\sum_i C_i \delta V_i = 0 . \quad (72)$$

Thus,

$$\begin{aligned} PE_{\text{initial}} &= \sum_i \frac{1}{2} C_i (\delta V_i + V)^2 = \sum_i \frac{1}{2} C_i (\delta V_i^2 + 2\delta V_i V + V^2) \\ &= \sum_i \frac{1}{2} C_i \delta V_i^2 + \frac{1}{2} C V^2 \\ &= \sum_i \frac{1}{2} C_i \delta V_i^2 + PE_{\text{final}} , \end{aligned} \quad (73)$$

where we recognized  $PE_{\text{final}} = (1/2)CV^2$ . Since the first term in the last equation is always greater than or equal to zero, we find that

$$PE_{\text{initial}} \geq PE_{\text{final}} \quad (74)$$

as we expected from our qualitative analysis. The equality only holds if all the  $\delta V_i = 0$  which means that the initial potentials were all equal to each other and the final potential. In this case, no charge would flow and nothing would change when the isolated capacitors were connected.

The final potential energy can be expressed in a couple of ways:

$$PE_{\text{final}} = \frac{1}{2} C V^2 = \frac{1}{2} \frac{(\sum_i C_i V_i)^2}{\sum_i C_i} . \quad (75)$$

From either form, we can get the the special case result of SJ-733's example. From the first form,

$$PE_{\text{final}} = \frac{1}{2} C V^2 = \frac{1}{2} (C_1 + C_2) \left( \frac{C_1 - C_2}{C_1 + C_2} \right)^2 V_1^2 = \left( \frac{C_1 - C_2}{C_1 + C_2} \right)^2 PE_{\text{initial}} . \quad (76)$$

From the second form,

$$PE_{\text{final}} = \frac{1}{2} \frac{(C_1 - C_2)^2 V_1^2}{C_1 + C_2} = \left( \frac{C_1 - C_2}{C_1 + C_2} \right)^2 \left( \frac{1}{2} \right) (C_1 + C_2) V_1^2 = \left( \frac{C_1 - C_2}{C_1 + C_2} \right)^2 PE_{\text{initial}} . \quad (77)$$

## 6. CAPACITORS WITH DIELECTRICS

**DIELECTRIC** is almost a synonym for **INSULATOR**. The two can often be used interchangeably.

But **DIELECTRIC** is used when one considers the insulator as interacting with electromagnetic fields.

We only consider the simplest kind of dielectrics which are linear dielectrics (e.g., Jackson 1975, p. 146–152; Barger & Olson 1987, p. 156; Griffiths 1999, p. 179): which we will just call dielectrics for short. Say in the absence of dielectric, the electric field is  $\vec{E}_0$ . In the presence of dielectric (i.e., a linear dielectric that is homogeneous) that fills all space (except for regions of ideal conductor), the field is changed to

$$\vec{E} = \frac{\vec{E}_0}{\kappa}, \quad (78)$$

where  $\kappa$  is the dielectric constant or relative permittivity (e.g., Griffiths 1999, p. 180). (That  $\kappa$  is constant with respect to  $\vec{E}_0$  is the defining property of linear dielectrics (e.g., Barger & Olson 1987, p. 156).) The dielectric constant  $\kappa$  is a property of a dielectric and is a dimensionless quantity: it does depend on temperature and pressure and perhaps other environmental conditions to some degree. The dielectric constant  $\kappa$  is always greater than or equal to 1: it is 1 for vacuum and infinite for an ideal conductor in electrostatic cases: recall in an electrostatic case,  $\vec{E} = 0$  inside an ideal conductor.

Equation (78) does not strictly apply if dielectric is not uniform or if there are regions of different dielectrics or no dielectrics (other than ideal conductors) (e.g., Jackson 1975, p. 146). In those cases, one must solve for electric field taking the dielectrics explicitly into account. The one simple case seems to be if one has exact planar symmetry where again one can just use equation (78) directly to correct an already known E-field solution.

Table 1 gives a few dielectric constants.

Table 1. Approximate Dielectric Constants and Dielectric Strengths<sup>a</sup>

Material	Dielectric Constant $\kappa$	Dielectric Constant Strength <sup>b</sup> ( $10^6$ V/m)
Vacuum	1	—
Air (dry, 20° C, 1 atm)	1.00054	3
Paper	3.7	16
Pyrex glass	5.6	14
Diamond	5.7	?
Water	80	—
Strontium titanate	233	8
Ideal conductor	$\infty$	0

<sup>a</sup>The dielectric constant values are at about room temperature unless otherwise noted. The values are drawn from Griffiths (1999, p. 180) and SJ-736. If one searches the web for dielectric constants, one finds many extensive tables.

<sup>b</sup>The dielectric strength is the maximum electric field that that dielectric can withstand without breaking down and becoming a conductor. The dielectric strength is actually very dependent on the impurities and flaws in a material. The values in this table are all approximate or typical.

Note. — Water is actually a weak conductor, but for short periods of it can be considered an approximate insulator (e.g. Wolfson & Pasachoff 1990, p. 631). The vacuum is actually a strange case. If there is no charge at all in it, it should not conduct. But on the other hand, if the electric field becomes sufficiently strong, charge can be pulled out materials and make the former vacuum conduct. The vacuum can actually conduct (without injected charges) if the electric becomes high enough: charged particles get created. This is a quantum mechanical effect at very, very high electric field strength and I think nearly impossible in ordinary devices (Wikipedia: Virtual particle: Pair production).



The physical explanation of dielectrics we leave largely to § 8, but we need say a word here. An applied external field  $\vec{E}_0$  causes the dielectric to become polarized at the microscopic level. In planar symmetry (which is all we will consider in this section), the internal polarization effects largely cancel, but at the surface this is not the case: there will regions with the positive dipole ends are on the surface and regions where the negative dipole ends will be on the surface: these are induced surface charges. In planar symmetry, these induced surface charges can be dealt with at least in some cases such as the example below in § 6.1.

How does using a dielectric in a capacitor affect the capacitor? Say you have an isolated capacitor with charge  $Q$ . Initially, the capacitor is embedded in vacuum (or air which is nearly vacuum for dielectric properties) and has potential  $V_0$ . The capacitance is

$$C_0 = \frac{Q}{V_0} . \tag{79}$$

Since the capacitor is isolated the charge cannot change. Now magically you embed the capacitor in a uniform dielectric with dielectric constant  $\kappa$ . The dielectric extends through all the previously empty space in which there was any electric field. Since the dielectric is everywhere outside of the capacitor where there was an electric field and is uniform, we get the simple result that electric field gets reduced by  $1/\kappa$  (e.g., Jackson 1975, p. 146). Since this is a scaling down by a common factor, the charge distribution should not change (i.e., have charge flows). But the potential should scale down by  $1/\kappa$ : remember the potential can be found by integrating the electric field along a path between the plates and if the electric

Fig. 14.— Parallel-plate capacitor with a dielectric in the gap between the plates.

field scales down everywhere, so does the potential. Let

$$V = \frac{V_0}{\kappa} . \tag{80}$$

The capacitance is now

$$C = \frac{Q}{V} = \kappa \frac{Q}{V_0} = \kappa C_0 . \tag{81}$$

So the capacitance scales up by  $\kappa$ .

On the other hand, what if we hold the capacitor potential fixed at  $V$  (say by attaching a battery across it that causes no other perturbation) as we change from no dielectric to embedded in dielectric? Let the original capacitor charge be  $Q_0$ . The electric field should stay constant, but the charge distribution should scale up by a factor of  $\kappa$  everywhere in order to compensate for the scaling down of the electric field everywhere by a factor of  $\kappa$  that would happen if the charge didn't scale up. The new charge is, of course, in an ideal sense moved from the negative to the positive plate. The capacitor charge is changed to  $Q = \kappa Q_0$ . In this case,

$$C = \frac{Q}{V} = \kappa \frac{Q_0}{V} = \kappa C_0 . \tag{82}$$

The result is the same—which it had to be—since the ratio of the charge to potential should be independent of either ideally as we discussed in § 2.

We see that going from vacuum environment to all-dielectric environment increases the capacitance by  $\kappa$ :

$$C = \kappa C_0 , \tag{83}$$

where  $C$  is the capacitance with the dielectric and  $C_0$  is the capacitance without. This result is a bit of an idealization since in reality one usually just puts the dielectric between the plates of a capacitor and not throughout the environment. However, for a good capacitor the electric field is mainly between the plates (and ideally totally so), and so result should a very good approximation if one is filling the whole region between the plates with dielectric.

Why use dielectrics in capacitors instead of just embedding the plates in air or, with more difficulty, vacuum (which has nearly the same dielectric constant)? There are three obvious reasons:

1. To increase capacitance.
2. To increase voltage limit of operation above that of air. The vacuum voltage limit is actually very high. The voltage limit is when the electric field reaches the dielectric strength of the embedding material and the capacitor starts to conduct.
3. Just to give structural support between the plates. If the distance between the plates is very small (which tends to give high capacitance), it might hard to keep them from touching (and thus conducting) without some insulating material in between.

Capacitors with vacuum—not air, vacuum—between the plates have special uses: they can withstand very high potentials (and thus electric fields) without breaking down and conducting (Wikipedia: Capacitor: Dielectric materials). This is because conduction can only happen if the E-field is strong enough to pull charge out of the plates or to create charge in the vacuum which is a quantum mechanical effect at very, very high electric field strength and I think nearly impossible in ordinary devices (Wikipedia: Virtual Particle: Pair production).

### **6.1. Example: Parallel-Plate Capacitor with the Gap Partially Filled by Dielectric**

This example subsumes SJ-742–743’s examples with a simpler more general approach.

Imagine an ideal parallel-plate capacitor with vacuum in the space between the plates which we will call the gap for brevity.

Since parallel-plate capacitor is ideal, the E-field is entirely between the plates and is uniform there and the vacuum capacitance is

$$C_0 = \frac{\varepsilon_0 A}{d} \quad (84)$$

where  $A$  is the plate area and  $d$  is the plate separation.

Now we insert a layer of dielectric in the gap. The layer has the same area  $A$  as the plates and the same shape in the plane of the plates, has thickness  $fd$  where  $f$  is factor between 0 and 1, and is inserted so that its surfaces are parallel to the planar symmetry directions. We can insert it anywhere in the gap as long as planar symmetry is maintained.

There now are four parallel planes of charge that we treat as being infinite insofar as the E-fields they produce. (Recall the ideal parallel-plate capacitor is one treated in the infinite limit.) The four planes of charge are the inner sides of the plates and the outer sides of the dielectric where induced charge appears. By planar symmetry, the charge density on each of these planes is uniform. Say, the plates have charge density  $\sigma$  and  $-\sigma$ : they must sum to zero by the neutrality condition. Say, the dielectric surfaces have charge density  $\sigma_{\text{di}}$  and  $-\sigma_{\text{di}}$ : they must sum to zero by the neutrality condition. Going from left to right we order the plane charge densities  $-\sigma$ ,  $\sigma_{\text{di}}$ ,  $-\sigma_{\text{di}}$ , and  $\sigma$ . We just define the left-hand plate as the negative one and the right-hand plate as the positive one. The induced charge densities have the ordering that the polarization of the dielectric in the applied field dictates: the negative charges in the dielectric are attracted to the positive plate and the positive charges in the dielectric are attracted to the negative plate.

Fig. 15.— Parallel-plate capacitor with the gap partially filled by dielectric.

From the Gauss's law result of an infinite plane of uniform charge,

$$\vec{E} = \frac{\sigma}{2\varepsilon_0} \hat{n} , \quad (85)$$

where here  $\sigma$  is general and  $\hat{n}$  points outward from the plane on either side. For our capacitor, let  $\hat{x}$  point to the left—and I mean  $\hat{x}$ , not  $\hat{n}$ —and I know what I'm talking about. Now for our capacitor in the left gap vacuum region,

$$\vec{E}_{\text{left}} = \frac{(-\sigma)}{2\varepsilon_0} \hat{x} + \frac{\sigma_{\text{di}}}{2\varepsilon_0} (-\hat{x}) + \frac{(-\sigma_{\text{di}})}{2\varepsilon_0} (-\hat{x}) + \frac{\sigma}{2\varepsilon_0} (-\hat{x}) = -\frac{\sigma}{\varepsilon_0} \hat{x} . \quad (86)$$

In the middle dielectric region,

$$\vec{E}_{\text{di}} = \frac{(-\sigma)}{2\varepsilon_0} \hat{x} + \frac{\sigma_{\text{di}}}{2\varepsilon_0} \hat{x} + \frac{(-\sigma_{\text{di}})}{2\varepsilon_0} (-\hat{x}) + \frac{\sigma}{2\varepsilon_0} (-\hat{x}) = -\left(\frac{\sigma}{\varepsilon_0} - \frac{\sigma_{\text{di}}}{\varepsilon_0}\right) \hat{x} . \quad (87)$$

In the right gap vacuum region,

$$\vec{E}_{\text{right}} = \frac{(-\sigma)}{2\varepsilon_0} \hat{x} + \frac{\sigma_{\text{di}}}{2\varepsilon_0} \hat{x} + \frac{(-\sigma_{\text{di}})}{2\varepsilon_0} \hat{x} + \frac{\sigma}{2\varepsilon_0} (-\hat{x}) = -\frac{\sigma}{\varepsilon_0} \hat{x} . \quad (88)$$

We see that  $\vec{E}_{\text{left}} = \vec{E}_{\text{right}}$  and both are exactly what the E-field would be in the absence of the dielectric, but the same left and right charge densities  $-\sigma$  and  $\sigma$ , respectively. The effect of the dielectric on the field external to the dielectric cancels out in this case. Let's just call the vacuum field  $\vec{E}$ . As above, the dielectric field is  $\vec{E}_{\text{di}}$  which by earlier specifications equals  $\vec{E}/\kappa$ .

If we calculate the potential between the plates by moving on a straight line from the negative to the positive plate, we obtain

$$E(1-f)d + \frac{E}{\kappa}fd = V . \quad (89)$$

Note that the charge on the positive plate is

$$Q = A\sigma . \quad (90)$$

The capacitance we now find to be

$$C = \frac{Q}{V} = \frac{A\sigma}{E(1-f)d + (E/\kappa)fd} = \frac{\varepsilon_0 A}{d} \frac{\kappa}{f + \kappa(1-f)} = C_0 \frac{\kappa}{f + \kappa(1-f)}, \quad (91)$$

where  $C_0$  is again the capacitance without any dielectric. This is the result of SJ-743.

We can now investigate special cases for the capacitor:

$$C = \begin{cases} C_0 \frac{\kappa}{f + \kappa(1-f)} & \text{in general;} \\ C_0 & \text{for } f = 0 \text{ or } \kappa = 1; \\ C_0 \kappa & \text{for } f \rightarrow 1; \\ C_0 \frac{1}{1-f} & \text{for } \kappa \rightarrow \infty \\ & \text{as for an ideal conductor;} \\ \infty & \text{for } f \rightarrow 1 \text{ and } \kappa \rightarrow \infty. \end{cases} \quad (92)$$

## 7. ELECTRIC DIPOLES IN ELECTRIC FIELDS

Consider a simple electric dipole consisting of two point charges  $q$  and  $-q$  separated by a distance  $2a$ . The dipole moment  $\vec{p}$  is defined by

$$\vec{p} = 2aq\hat{p}, \quad (93)$$

where  $\hat{p}$  points from the negative to the positive charge.

### 7.1. Torque on a Dipole in a Uniform Electric Field

Now let's say that the dipole is in a uniform electric field  $\vec{E}$ . The net external force on the dipole is

$$\vec{F}_{\text{net,ext}} = q\vec{E} - q\vec{E} = 0, \quad (94)$$

and so the center of mass of the dipole will not accelerate. If we think of the dipole charges as fixed on a massless rigid rod  $2a$  apart, then the charges will not accelerate away from each

other to infinity.

There is, however, a net torque on the charges. Let's put the origin at the midpoint between the two charges. The net external torque on the dipole about an axis through this midpoint and perpendicular to  $\vec{E}$  is

$$\vec{\tau}_{\text{net,ext}} = a\hat{p} \times q\vec{E} + a(-\hat{p}) \times (-q)\vec{E} = 2aq\hat{p} \times \vec{E} = \vec{p} \times \vec{E}. \quad (95)$$

This result is general, but it isn't probably very useful unless the charges are bound into an at least a semi-rigid dipole structure since otherwise the  $a$  and the dipole magnitude would change under an applied electric force. In fact, there are many cases where molecules (which are not simple dipoles) can be considered to have a permanent dipole moment. (Atoms can have permanent dipole moments too, but not much is said about that it seems.) The molecules with permanent dipole moments are polar molecules (see below).

The effect of the net torque is try to rotate the dipole moment (vector) into alignment with the electric field (vector). When alignment is achieved, the torque is zero: it is a stable equilibrium situation for the dipole acted on by the electric field alone. The torque can be regarded as a restoring torque just forces that create a stable equilibrium are called restoring forces.

If  $\vec{p}$  and  $\vec{E}$  are anti-aligned, then the torque is also zero and the dipole is in equilibrium. But not stable equilibrium: any perturbation from exact anti-alignment will cause a net torque that tries to align  $\vec{p}$  and  $\vec{E}$ .

Fig. 16.— An electric dipole in a uniform electric field.

## 7.2. General Proof of the Electric Dipole Torque Formula: Optional

The general definition of the dipole moment of a discrete set of charges is

$$\vec{p} = \sum_i \vec{r}_i q_i , \quad (96)$$

where the sum is over all charges in the distribution,  $\vec{r}_i$  is the charge displacement relative to some origin, and  $q_i$  is the charge on charge  $i$  (e.g., Griffiths 1999, p. 149).

Before we begin with the main event, first an opening factoid. Say we write

$$\vec{r}_i = \vec{r}'_i + \vec{r}'_0 , \quad (97)$$

where  $\vec{r}'_0$  is a new origin and  $\vec{r}'_i$  is charge  $i$ 's position relative to the new origin. Now

$$\vec{p} = \sum_i \vec{r}_i q_i = \sum_i (\vec{r}'_i + \vec{r}'_0) q_i = \sum_i \vec{r}'_i q_i + \vec{r}'_0 \sum_i q_i = \vec{p}' + \vec{r}'_0 \sum_i q_i , \quad (98)$$

where  $\vec{p}'$  is the dipole moment relative to the new origin. If the charge distribution is overall neutral, then  $\sum_i q_i = 0$  and the dipole moment is origin independent. Dipole moments are usually most important for overall neutral systems, and so the origin-independence of the dipole moments of such systems is useful property in their treatment.

Now for what we really wanted to find out. Say we put a system of discrete charges in an external uniform electric field  $\vec{E}$ . The net external force is given by

$$\vec{F}_{\text{net,ext}} = \sum_i q_i \vec{E} . \quad (99)$$

If the net charge is zero, the center of mass of the system of charges will not accelerate. On the other hand, the net external torque is

$$\vec{\tau}_{\text{net,ext}} = \sum_i \vec{r}_i \times q_i \vec{E} = \left( \sum_i \vec{r}_i q_i \right) \times \vec{E} = \vec{p} \times \vec{E} . \quad (100)$$

Thus, the torque result we found above is general:

$$\vec{\tau}_{\text{net,ext}} = \vec{p} \times \vec{E} . \quad (101)$$



This result is most useful if the system is overall neutral since then the torque is independent of the origin. If the charge system is also a rigid body, then the analysis of the motion of the system is straightforward.

### 7.3. The Potential Energy of a Dipole in a Uniform Electric Field

If we do assume a dipole is a rigid body, then we can find the potential energy of its orientation in the electric field. From the lecture **ROTATIONAL DYNAMICS**, we know for a rigid body that if the work done by a torque depends only on the end points of rotational displacement, then a potential energy can be defined.

In this case, we define the E-field direction as  $\theta = 0$  which we also label the positive  $x$  direction. We now calculate the work done by the electric field torque for a displacement from  $\pi/2$  (chosen as for zero-point for potential energy for reasons that you'll soon see and also the  $y$  direction) to a general angle  $\theta$ . Adapting a work expression from the lecture **ROTATIONAL DYNAMICS**, we find

$$\begin{aligned} W(\theta) &= \int_{\pi/2}^{\theta} \vec{\tau} \cdot \hat{z} d\theta' = \int_{\pi/2}^{\theta} (\vec{p} \times \vec{E}) \cdot \hat{z} d\theta' \\ &= \int_{\pi/2}^{\theta} (-pE \sin \theta') d\theta' = pE \cos \theta' \Big|_{\pi/2}^{\theta} = pE \cos \theta \\ &= \vec{p} \cdot \vec{E} , \end{aligned} \tag{102}$$

where direction of  $\hat{z}$  is determined by the right-hand rule from the  $x$  and  $y$  directions and the minus sign arises since the torque points in the  $-z$  direction for  $\theta > 0$  and in the  $z$  direction for  $\theta < 0$ . Since the work done by the electric field torque depends only on the initial and final displacement and not on the path, a potential energy can be defined.

As usual, the potential energy is defined by

$$PE = -W = -\vec{p} \cdot \vec{E} , \tag{103}$$

where we see that our choice of fiducial starting point for the work expression leads to a potential energy expression with no additive constant. We in fact take  $\theta = \pi/2$  to be the zero point for potential, and thus no additive constant is needed for the potential energy expression. Recall the zero-point of a potential energy can be chosen for convenience since only potential energy differences have physical significance.

The usual  $PE$  formula  $PE = -W$  (where  $W$  is the work done by the force or torque for which we are defining the potential energy) just means that when the conservative force or torque does positive work, the system loses  $PE$  and when it does negative work, the system gains  $PE$ .

#### 7.4. Example: Oscillation of an Electric Dipole in a Uniform Electric Field

From lecture **ROTATIONAL DYNAMICS**, we know that the rotational Newton's 2nd law (AKA the equation of motion) for a rigid body about a single fixed axis is

$$\tau_z = I\alpha \tag{104}$$

where  $\tau_z$  is the net external torque about the  $z$  axis (which is conventional designation for the single fixed axis),  $I$  is the rotational inertia of the rigid body about the  $z$  axis, and  $\alpha$  is the angular acceleration.

We consider a rigid electric dipole in a uniform electric field whose direction is the  $x$  direction. We choose the  $z$  axis to be normal to the dipole moment vector and the electric field. The dipole will not be accelerated in a uniform E-field, and so we can consider a frame in which it is at rest and the  $z$  axis passes through the dipole's center of mass.

Thus, the  $z$  axis is the single fixed axis of rotation of the system.

For the dipole moment with E-field causing the only external torque  $\vec{\tau}$ , the equation of

motion is

$$\vec{\tau}_z = \vec{\tau} \cdot \hat{z} = -pE \sin \theta = I\alpha = I \frac{d^2\theta}{dt^2} , \quad (105)$$

where we measure  $\theta$  counterclockwise (as viewed from positive  $z$  direction) from the E-field direction, the minus sign arises since the torque points in the  $-z$  direction for  $\theta > 0$  and in the  $z$  direction for  $\theta < 0$ . The equation of motion can be rewritten in a standard form:

$$\frac{d^2\theta}{dt^2} = -\frac{pE}{I} \sin \theta = -\omega_0^2 \sin \theta , \quad (106)$$

where

$$\omega_0 \equiv \sqrt{\frac{pE}{I}} \quad (107)$$

is not angular velocity  $\omega = d\theta/dt$ , but is an angular frequency parameter.

Equation (106) is the differential equation that we solve for rotational motion of the dipole. Though pretty innocuous looking, equation (106) has no exact analytical solution—that transcendental function sine spoils things.

But we can make the small-angle approximation to get an approximate equation of motion that can be solved. The small-angle approximation is

$$\sin \theta \approx \theta . \quad (108)$$

This approximation is first order good in small  $\theta$  and the error grows as  $|\theta^3/6|$ . Even at  $30^\circ$  the small-angle approximation is in error by only about 4.7%.

With the small-angle approximation, the equation of motion becomes

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \theta . \quad (109)$$

At this point you should recognize equation (109) as the simple harmonic oscillator equation: our dipole in the small-angle approximation is a simple harmonic oscillator.

By inspection and over 30 years of experience, the general solution is known to be

$$\theta = A \cos(\omega_0 t) + B \sin(\omega_0 t) , \quad (110)$$

where  $A$  and  $B$  are constants determined by the initial conditions. This solution is the simple harmonic oscillator solution or simple harmonic motion solution.

The period  $P$  of the oscillation is

$$P = \frac{1}{f} = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{I}{pE}} , \quad (111)$$

which is independent of the amplitude of the oscillation: a remarkable feature of simple harmonic motion. We see that the period increases with rotational inertia and decreases with  $pE$  (which is the maximum value of the restoring torque magnitude).

If at time zero we start the dipole from rest at  $\theta = A > 0$  and  $B = 0$  (since otherwise  $d\theta/dt \neq 0$  at time zero), then  $A$  is the amplitude and the solution is

$$\theta = A \cos(\omega_0 t) . \quad (112)$$

The motion is perpetual if there is no dissipation of energy.

If we do not make the small-angle approximation, the solution is still oscillatory if there is no dissipation. If the dipole mechanical energy is smaller than the maximum potential energy, the dipole moment will oscillate between positions of maximum  $|\theta|$  (which is the amplitude) where all of its energy is potential and it is instantaneously at rest. The maximum

Fig. 17.— Electric dipole undergoing simple harmonic motion.

$|\theta|$  positions are called the turning points. In this case, the period does depend on the amplitude (e.g., Arfken 1970, p. 273–274).

If the mechanical energy of the dipole is greater than the maximum potential energy, the dipole will rotate in one direction with a varying speed. The direction is determined by the initial conditions. In this case, when the dipole is at  $\theta = \pi$ , it still has kinetic energy and non-zero angular momentum and that angular momentum will keep it moving in the same direction. There are no turning points.

## 8. ATOMIC EXPLANATION OF DIELECTRICS

Neutral molecules are overall neutral, of course. But there is charge separation in them that cannot be eliminated: the laws of quantum mechanics (QM) forbid it.

Such molecules are actually rather complex with a swarm of electrons, some tightly bound to the nuclei of the atoms and some in states that are essentially shared between the atoms: these are the valence electrons. An electron actually is not in one place, but is in a superposition of places at one time—but we won't go deeper into QM. And, of course, electron density is static in isolation. But molecules are seldom/never in perfect isolation—perturbations from the environment cause a continual adjustment of the structure.

The remarks above (and below) can also be made about atoms *mutatis mutandis*, but we are mostly thinking of molecules for dielectrics—although not absolutely always—in physics almost every remark takes a qualification and the chain of qualifications is quasi-infinite.

Because there is charge separation, a complex electric field exists in the vicinity of a molecule. This field can be expanded in series of terms of which the first for a neutral molecule is a dipole field that decays with distance like  $1/r^3$ . Higher order terms are quadrupole (decays like  $1/r^4$ ), octopole (decays like  $1/r^5$ ), etc. If the molecule is ionized (i.e., non-

neutral), then a monopole term exists that decays like  $1/r^2$ . This sort of expansion is called multipole expansion (e.g., Griffiths 1999, p. 147–148).

But most molecules in materials are neutral and the dipole term in their electric field is dominant. The strength of the term is the dipole moment which like a pure dipole has dimensions of charge times length.

Polar molecules have permanent dipole moments: e.g., the water molecule  $\text{H}_2\text{O}$ . In an applied electric field, the dipole moments will be subject to the E-field torque that will try to align them with the E-field.

Nonpolar molecules have sufficient charge symmetry that they have no permanent dipole moment: e.g., carbon dioxide  $\text{CO}_2$ . But in an applied field the charge distribution of a nonpolar molecule will be perturbed such that positive charge is shifted somewhat in the E-field direction and negative charge shifted is shifted opposite to the E-field direction. Thus the nonpolar molecule becomes polarized.

The polar properties of molecules have many interesting properties. For example, neutral molecules do feel a force in a non-uniform electric field. Say there is net positive charge in the vicinity of molecules. It will polarize neutral molecules with the negative end nearer the net positive charge and the positive end farther from the net positive charge. If the net positive charge field falls off with distance (which is the usual case: e.g., if the net charge is a point charge), the negative end is more attracted to the net positive charge than the positive end is repelled. The result is a net attraction. A same argument holds for a net

Fig. 18.— Cartoon of the polar molecule  $\text{H}_2\text{O}$ .

negative charge *mutatis mutandis*. A similar argument shows that neutral dipoles can be attracted to one another.

In a homogeneous dielectric, the molecules are nonpolar or if polar have random alignment.

An applied field polarizes non-polar molecules and aligns polar molecules (or changes their polarization). Thus, a net polarization of the dielectric occurs. The net polarization usually decreases with temperature which tends to randomizes the molecular alignments and charge distributions: the degree of alignment often goes approximately as  $1/T$ , where  $T$  is Kelvin or absolute temperature (e.g. Tipler & Mosca 2008, p. 825).

The polarization, of course, tends to reduce the electric field below the level of the applied field since the negative pole is attracted to the negative-pointing direction of the applied field and the positive pole to the positive-pointing direction of the applied field. Thus, the E-field of the polarization points opposite to the applied field.

The effect of the polarization is something like shifting all the negative charge one way and all the positive charge the other way. In the interior of the material, any macroscopic region has charge neutrality and is there no average separation of charge. But on the surface there is a tendency to have an induced surface charge density: the positive induced surface charge density tends to be on the surface from which E-field lines emerge from the dielectric and the negative induced surface charge density tends to be on the surface from which E-field lines enter the dielectric.

Fig. 19.— A cartoon of a polarized dielectric.

It is this induced surface charge that causes the polarization field that added to the applied field gives the actual net field.

In the dielectric, the net field  $\vec{E}$  is, as discussed in § 6,

$$\vec{E} = \frac{\vec{E}_0}{\kappa}, \quad (113)$$

where  $\vec{E}_0$  is the applied field and  $\kappa$  is the dielectric constant or relative permittivity (e.g., Griffiths 1999, p. 180). This result holds only for homogeneous linear dielectrics, but most ordinary materials fall into this category. The development of equation (113) is beyond our scope and my knowledge.

Outside of the dielectric, the polarization field can contribute to the net field too. How it does this in general might be tricky. But for the ideal parallel-plate capacitor it's not so bad as we saw in § 6.1.

## 9. CONCLUSION

Lecture notes generally don't have a conclusion—they just peter out.

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