GRAVITY

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ABSTRACT

Lecture notes on what the title says and what the keywords say.

Subject headings: gravity — Isaac Newton — apple — Newton's gravitation law — gravitational constant — inverse-square law — gravitational field — gravitational field near the Earth's surface — Gauss's law for gravity — Kepler's laws — gravitational potential energy — escape velocity — black holes

1. INTRODUCTION

Isaac Newton (1643–1727) didn't discover gravity in the sense of discovering that things fall down.

That had always been known—of course.

And since the discovery that the Earth was round by the ancient Greeks in the 5th century BCE (perhaps by Parmenides of Elea in the early 5th century BCE), it was known (to those who knew the Earth was round) that the gravity force pointed toward the Earth's center at least to high accuracy. But this was just for objects close to the Earth. There was no understanding that this force applied to celestial objects.

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What Newton discovered is that the gravity force is the same force that determines the motions of the solar system. This was not at all obvious before. On Earth things fall down under gravity and celestial bodies seem to move in ellipses about the Sun or other bodies. Actually not much before Newton's time, most folks in the tradition descended from the ancient Greeks believed the Sun, Moon, and planets orbited the Earth in circles or compounded circular motions. Nicolaus Copernicus (1473–1543) postulated the Sun was the center of the motion for the solar system and Johannes Kepler (1571–1630) had established that the planet orbits were ellipses with the Sun at one focus. By Newton's time many had accepted the Copernican-Keplerian view without reservations—there were still some holdouts though.

Newton's discovery consisted in finding a general law for gravity (i.e., the Newton's gravitation law) and showing that the special cases (motions on Earth and motions in space) followed from this law together with the rest of Newtonian physics.

Newton's highly accurate predictions of celestial motions from Newtonian physics soon convinced many that Newtonian physics was correct to a very high degree of accuracy at least. It was no surprise that Newtonian physics could be extended to explain the motions beyond the solar system: multiple star systems and eventually galaxies.

Of course, in the modern epoch, we know that Newtonian physics is only an approximation. It is superseded by quantum mechanics and relativistic physics. But Newtonian physics is an excellent approximation in many cases and comparatively easy to use, and so it is still widely used and, in fact, is essential. The modern replacement for the Newton's gravity law is Einstein's general relativity. General relativity explains phenomena that Newtonian physics fails to explain. The universal law gravity is a weak gravitational field limit of Einstein's results. But the general relativity is not the last word either. It is not a quantum theory and everyone believes the true theory of gravity must be a quantum theory since quantum mechanics seems even more fundamental than general relativity.

Now what about that apple:

> "when formerly, the notion of gravitation came into his mind. It was occasioned by the fall of an apple, as he sat in contemplative mood. Why should that apple always descend perpendicularly to the ground, thought he to himself. Why should it not go sideways or upwards, but constantly to the earth's centre."—William Stukely (1687–1765): recalled from a conversation with Newton, 1726 April 15.

2. NEWTON'S GRAVITATION LAW

Newton's gravitation law goes by several similar names. Yours truly prefers Newton's gravitation law or, for short, the gravitation law. Wikipedia—the supreme authority—prefers the law of universal gravitation (Wikipedia: Newton's law of universal gravitation) Yours truly just can get used to it.

But "what's in name?"

Now what is Newton's gravitation law?

Consider two classical particles (particle 1 and particle 2) with masses m_1 and m_2.

The gravitation law for the force of 1 on 2 is

$$\vec{F}_{12} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} \ , \tag{1}$$

where $G = 6.67428(67) \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$ is the gravitational constant, r_{12} is the distance between the masses, and $\hat{r}_{12}$ is a unit vector that points from 1 to 2.

As equation (1) shows, the gravitation law is a mass and distance dependent force law.

Since mass has only one sign which is always positive, the gravity force between point particles is always attractive. In fact, as we'll see soon, the gravitational force between any two systems of particles is always attractive.

The dependence on distance is an inverse-square law dependence. The gravitational force is an inverse-square law force. Such forces have remarkable properties. For example, they allow a Gauss's law for the force as we'll see in § 6. These properties are also extremely important since the two key inverse-square law forces of nature—gravity and the Coulomb force—determine much of the structure of our world.

The gravitation law is very similar to Coulomb's law for the Coulomb (or electrostatic) force:

$$\vec{F}_{12} = \frac{kq_1q_2}{r_{12}^2}\hat{r}_{12} \ , \tag{2}$$

where q_1 and q_2 are two point charges, $k = 8.9875517873681764 \times 10^9\,\mathrm{N\,m^2/C^2}$ is Coulomb's constant (which is exact by definition), and the other quantities are the same as for equation (1). Gravity and the Coulomb force are profoundly similar in some respects as one can see comparing equations (1) and (2). They are both inverse-square laws as mentioned above and in Coulomb's law charge plays the role that mass plays in the universal law of gravity. But in other respects, the two laws are profoundly different. Mass has only one sign, positive, and an explicit minus sign. This makes gravity always attractive and gives rise to long range gravitational effects of large accumulations of mass such as planets, stars, galaxies, and the universe as a whole. But charge can be positive and negative, and there is no explicit minus in Coulomb's law. So Coulomb force can be repulsive and attractive. Also positive and negative charge effects can cancel at long range from accumulations of equal amounts of these charges. The universe insofar as we can tell is overall neutral and most large regions are nearly neutral. This means that long range effects of Coulomb force are vastly suppressed relative to those of gravity. Thus, the effects of mass and charge in determining the structure of matter and the universe at large and at small are profoundly different.

Returning to equation (1), we note that

$$\vec{F}_{21} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{21} = -\vec{F}_{12} \ , \tag{3}$$

and so the gravitation law obeys Newton's 3rd law. This is not surprising: Newton would never have formulated the 3rd law if gravitation—his example par excellence of a force law—did not obey the 3rd law.

But what good, you say, is the gravitation law since classical point particles do not actually exist and in any case we almost always have to deal with finite mass systems.

Well classical point particles are ideal limiting systems. You can regard them as quantum mechanical particles assigned only their average classical behavior. This allows the large scale average classical behavior to be calculated from large accumulations of quantum mechanical particles. The gravitational force between systems of particles can be determined from the gravitation law by a general procedure. There are also important special cases, where the gravitation law can be used directly. We investigate the general procedure and the special cases in the following subsections of this section.

2.1. The Gravitational Force for Systems of Particles

Say we have two systems of particles with system 1 particles labeled by index i and system 2 particles labeled by index j. Using universal law of gravitation equation (1), we find the net force of system 1 on system 2 by summing over all the inter-particle forces. We obtain

$$\vec{F}_{12} = \sum_{ij} -\frac{Gm_im_j}{|\vec{r}_j - \vec{r}_i|^2}\frac{(\vec{r}_j - \vec{r}_i)}{|\vec{r}_j - \vec{r}_i|} \ , \tag{4}$$

where m_i is the mass of particle i from system 1, m_j is the mass of particle j from system 2, $\vec{r}_i$ is the position of particle i from system 1, $\vec{r}_j$ is the position of particle j from system 2, and

$$\frac{(\vec{r}_j - \vec{r}_i)}{|\vec{r}_j - \vec{r}_i|} \tag{5}$$

is the unit vector that points from particle i to particle j.

If we go to the continuum limit for the systems, where they have densities $\rho_1(\vec{r})$ and $\rho_2(\vec{r})$. The differential gravitational force between two infinitesimal bits of the two systems follows from the gravitation law:

$$d\vec{F}_{12} = -\frac{G\rho_1(\vec{r}_1)\rho_2(\vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|^2}\frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|}\, dV_1\, dV_2 \ , \tag{6}$$

where $\vec{r}_1$ and $\vec{r}_2$ are, respectively, dummy variables of integration for systems 1 and 2, and dV_1 and dV_2 are, respectively, differential volumes for systems 1 and 2.

The net gravitational force of system 1 on system 2 in the continuum limit is found by integration of equation (6):

$$\vec{F}_{12} = -\int\int \frac{G\rho_1(\vec{r}_1)\rho_2(\vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|^2}\frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|}\, dV_1\, dV_2 \ , \tag{7}$$

where the integrals are over the whole systems.

Using equation (4) with the roles of systems 1 and 2 exchanged, we see that

$$\begin{aligned}
\vec{F}_{21} &= \sum_{ji} -\frac{Gm_j m_i}{|\vec{r}_i - \vec{r}_j|^2}\frac{(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|} \\
&= \sum_{ij} \frac{Gm_i m_j}{|\vec{r}_j - \vec{r}_i|^2}\frac{(\vec{r}_j - \vec{r}_i)}{|\vec{r}_j - \vec{r}_i|} \\
&= -\vec{F}_{12} \ .
\end{aligned} \tag{8}$$

Thus, we find the 3rd law holds for systems of particles too.

Similarly in the continuum limit,

$$\vec{F}_{21} = -\int\int \frac{G\rho_2(\vec{r}_2)\rho_1(\vec{r}_1)}{|\vec{r}_1 - \vec{r}_2|^2}\frac{(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}\, dV_2\, dV_1$$

$$
\begin{aligned}
&= \int\int \frac{G\rho_1(\vec{r}_1)\rho_2(\vec{r}_2)}{|\vec{r}_2-\vec{r}_1|^2}\frac{(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|}\,dV_1\,dV_2 \\
&= -\vec{F}_{21}\ . \qquad (9)
\end{aligned}
$$

Again, we find the 3rd law holds for systems of particles too.

Because the 3rd law holds for gravitation for systems of particles (and two particles too), one often just says "the force between two systems" meaning that the magnitude of the force of one on the other is equal to the magnitude of the force of the other on the one.

The force between the systems is always attractive as we foretold in § 2.

Note absolutely, positively, equations (4) and (7) just give net forces between systems. The part-to-part gravitational forces and internal gravitational forces are all important in general in fully understanding systems. But we won't go on to treat those gravitational forces in general—it's all very hard.

Now evaluating equation (4) or equation (7) in general can be laborious. In most cases, a numerical solution is all one can do. But one can do it, and so the gravitation law is universally useful.

Though one can, we don't actually want to in this course.

Fortunately, three special cases exist where the gravitational force between two systems can be found easily by treating them as point particles. We consider them in turn below.

2.2. The Gravitational Force between Remote Systems

Say you have two systems (1 and 2) with masses m_1 and m_2.

Each system has a characteristic length scale: they are ℓ_1 and ℓ_2. A characteristic length scale ℓ is one that characterizes a system. Measured along any direction the system length is of order ℓ. A single characteristic length is not possible for all systems, but we assume it is for ours.

Both systems have an average position which could be the center of mass, but it does not have to be. The average position just characterizes the position of the system in space. Let r_{12} be the distance between the average positions and $\hat{r}_{12}$ be the unit vector pointing from the center of system 1 to the center of system 2.

If $r_{12} >> \ell_1$ and $r_{12} >> \ell_2$, then the gravitation law

$$\vec{F}_{12} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} \tag{10}$$

holds approximately for the two bodies.

We will not go into the proof of this result. A similar result can be proven for the Coulomb force between systems of electric charge. This proof may be given in the 2nd semester of intro physics.

We will also not go into how accurate the result is. We will say that the larger r_{12} is relative to ℓ_1 and ℓ_2, the more accurate the result. If r_{12} is of order ℓ_1 and/or ℓ_2, the result is usually only crudely valid.

2.3. The Gravitational Force Exerted by a Spherically Symmetric System

The gravitational force exerted a spherically symmetric system on objects outside of the spherically symmetric system is exactly the same as if the spherically symmetric system were replaced by a point particle that was located at the system's center and had the system's mass.

This remarkable result is actually a consequence of the inverse-square law nature of the gravitational force. We prove the result using Gauss's law for gravity in § 6.3.

Now I know what you're thinking.

If a spherically symmetric system exerts a gravitational force as if it were a point particle, are the gravitational forces exerted on it exactly as if it were a point particle?

Yes.

Consider two systems (1 and 2). System 1 is spherically symmetric and system 2 is general, except that it doesn't overlap with system 1. We imagine replacing system 1 by a particle version. We know

$$\vec{F}_{1(\text{particle}),2} = \vec{F}_{12} \ , \tag{11}$$

where $\vec{F}_{1(\text{particle}),2}$ which is the exerted by a particle version of system 1 on system 2. By the 3rd law, we know that

$$\vec{F}_{2,1(\text{particle})} = -\vec{F}_{1(\text{particle}),2} \tag{12}$$

is the force system 2 exerts on the particle version of system 1. Now we see

$$\vec{F}_{21} = -\vec{F}_{12} = -\vec{F}_{1(\text{particle}),2} = \vec{F}_{2,1(\text{particle})} \tag{13}$$

where we've used the 3rd law again for the first equality. That completes the proof.

An obvious corollary of the above results is that non-overlapping spherically symmetric bodies interact gravitationally just like point particles.

That this is so is an immense simplification in celestial mechanics. Many astro-bodies (large moons, planets, stars) are approximately spherically symmetric. That they can be treated to high accuracy as point particles in many calculations, makes their motions in complex systems (e.g., solar systems) tractable. Let's call this the point-particle result for brevity.

Newton, using primitive calculus tools (of his own invention), had to work hard to derive the point-particle result (e.g., French 1971, p. 265). If he had not derived the point-particle result, he would not have been able understand the motions of solar system without assuming the result as an ad hoc hypothesis.

2.4. The Gravitational Force Between a Large Spherically Symmetric System and an External, Small System

Say that you have a large spherically symmetric system like a planet (e.g., the Earth) with radius R.

Then you have a small system external to the large system: i.e., it is at a radial location $r \geq R$ relative to the large system. The small system has a characteristic size $\ell << R$.

From § 2.3, we know that as far its gravitational effects at $r \geq R$, the large system can be replaced by a point particle at its center that has the large system mass. The gravitational effects of the large system are just the same as that of the particle replacement.

Since $\ell << R$ and a particle has zero size, we conclude from § 2.2 that the gravitational force between the particle replacement and the small system is approximately that between point particles. This means that the gravitational force between the large system and the small system is approximately that between point particles: i.e., the force is given by the universal law of gravity equation 1

$$\vec{F}_{12} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} \ . \tag{14}$$

We have stated the result for large spherically symmetric systems and external, small systems as approximate. However, for planets and human size objects the result virtually exact. This is convenient for dealing with gravitational force on external, relatively small bodies (e.g., humans, space probes, etc.) near the Earth and other planets.

3. The Gravitational Force is "Weak"

The gravitation force is "weak".

That is to say it is a weak force in certain senses.

Let's consider a few illustrations of gravity as a weak force.

For a first illustration, say we have two spherical symmetric systems each of mass 1 kg separated by 1 m.

What's the gravitational force between them?

Recall $G = 6.67428(67) \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$.

You have 30 seconds. Go.

Behold:

$$F = \frac{Gm_1 m_2}{r_{12}^2} = 6.67428 \times 10^{-11}\,\mathrm{N} \approx 1.5 \times 10^{-11}\,\mathrm{lb}\ . \tag{15}$$

This calculation shows why we don't see gravity between human-size objects. It's far too weak.

Everyone in this room is attracted to everyone else.

But only weakly—and a good thing too or we'd be a mass of arms and legs in the center of the room.

For a second illustration in which gravity is weak can be see by yours truly holding up this marker.

Yes, the Earth's gravity pulls it down with significant force.

But it takes the whole mass of the Earth to do that.

My hand with its rather complex structure due to the electromagnetic force resists the whole Earth.

For a third illustration, let's compare the gravitational force between a proton and electron to the Coulomb force between them.

Note we have

$$\begin{aligned}
G &= 6.67428(67) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \quad (16)\\
m_p &= 1.672621637(83) \times 10^{-27}\,\mathrm{kg} \quad (17)\\
m_e &= 9.10938215(45) \times 10^{-31}\,\mathrm{kg} \quad (18)\\
k &= 8.9875517873681764 \times 10^{9}\,\mathrm{N\,m^2/C^2} \quad (19)\\
q_e &= -e = -1.602176487(40) \times 10^{-19}\,\mathrm{C} \quad (20)\\
q_p &= e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}\ , \quad (21)
\end{aligned}$$

where e is the elementary charge. Also recall Coulomb's law

$$\vec{F}_{12} = \frac{kq_1q_2}{r_{12}^2}\hat{r}_{12}\ . \quad (22)$$

Calculate ratio of the gravitational to Coulomb force between the proton and electron.

You have 1 minute working in groups or individually. Go.

Behold:

$$\frac{F_{\mathrm{gravity}}}{F_{\mathrm{Coulomb}}} = \frac{Gm_pm_e}{ke^2} = 4.408 \times 10^{-40}\ . \quad (23)$$

Note the ratio is independent of distance since both forces are inverse-square law forces and distance cancels out.

We can see why the electromagnetic force rather than internal gravity determines the structures of atoms, molecules, and solids.

Consider this table. It's held together by the electromagnetic force, not internal gravity which is negligible. But it's all short-range manifestations of the electromagnetic force.

This is true of all solids and liquids. The atoms and molecules are strongly bonded by the electromagnetic force to their near neighbors over microscopic distances.

Of course, the external gravity of planets (e.g., Earth) also has an effect on the structures of macroscopic objects located on those planets. Still many structures are not strongly changed by putting them in free fall: e.g., this chair.

Nevertheless, its true to say that the electromagnetic force largely determines the small scale structure of materials and solids and liquids.

The electromagnetic force does have long-range manifestations: long-range Coulomb forces and magnetic forces. The long-range Coulomb force can only occur when there is net charge. An overall neutral or nearly object cannot have a strong long-range Coulomb force. The effect of the equal amounts of positive and negative charges cancel at long range—but not at short range as just discussed above.

The long-range manifestations of electromagnetic force are often very important. For example, the long-range magnetic force of the Earth's magnetic field.

But it's true to say that gravity which is always a long-range force. Gravity is always long range since there is no cancellation of mass which is gravity's single "charge". So gravity just increases with mass accumulation.

It's often said that gravity determines the large-scale structure of the universe.

This is true, but with qualifications.

Consider astro-bodies ranging in size from small asteroids to the universe as whole.

Small asteroids can have any shape. There structure is largely determined by the electromagnetic force.

The largest asteroids have been pulled into roughly spherical shape by their internal gravity (Wikipedia: Asteroid belt). But they don't collapse to black holes because of their pressure forces which are electromagnetic forces. Pressure forces resist compression, but not shearing. Thus, pressure forces cannot support protuberances which can buckle under internal gravity. Non-pressure manifestations of the electromagnetic force cannot reach the strength of pressure forces.

Planets and the stars are more strongly pulled into nearly spherical shape by strong gravity.

So the structure of solid or liquid objects from large asteroids up the largest stars are determined by a combination of gravity and pressure forces. Rotation also plays a role. The centrifugal force also gives a resistance to gravity. Of course, since it is an inertial force, the centrifugal force could be regarded as a non-force and as the effect of motion on structure rather than the effect of a force on structure.

Gas and dust clouds in space have pressure and centrifugal force and gravity interacting to give them structure too.

Other structures like star clusters, galaxies, galaxy clusters have have structure determined by gravity and the centrifugal force. Pressure plays little or no role. But one must make the qualification that the items that make star clusters, galaxies, and galaxy clusters include gas and dust clouds, stars, planets, and asteroids.

The structure of the universe as whole is strongly determined by gravity, but it is currently thought that a dark energy pressure force plays a role too.

So gravity is very important in determining the large-scale structure of the universe, but it doesn't do it alone.

If one goes to the subatomic realm, the strong and weak nuclear forces play a role too.

In fact, there is a complicated interplay between all the fundamental forces (gravity, electromagnetic force, strong nuclear force, and weak nuclear force) in making our universe.

4. THE GRAVITATIONAL FIELD

Newton, contemporaries, and later physicists were all a bit unhappy with the **ACTION AT A DISTANCE** that gravity seemed to have in the pure, original Newtonian formulation.

They thought there ought to be some mediating thing.

Today, we think of the **GRAVITATIONAL FIELD** as being the mediating thing.

A field in mathematical physics is just a function defined everywhere in space or some specified region of space. In physics, the functions are usually continuous.

One can consider temperature and density as fields in many cases.

They are scalar fields in that temperature and density are characterized by one real number at each point in space and number that is independent of the coordinate system and there is no direction in space assigned to temperature and density.

The velocity of a flowing fluid is an example of a vector field. At every point in the fluid, the fluid has velocity specified by a magnitude and a direction.

I like to picture vector fields as a bunch of little arrows attached to every point in space.

The arrows point in space space, but their extent is in an abstract vector space. For example, a velocity vector points in real space, but its extent is in an abstract velocity space.

GRAVITATIONAL FIELD is a vector field.

4.1. Description of the Gravitational Field

The idea is that a mass distribution creates a **GRAVITATIONAL FIELD**.

The **GRAVITATIONAL FIELD**, then causes the gravitational force on objects located in the **GRAVITATIONAL FIELD**.

This eliminates **ACTION AT A DISTANCE**.

The conventional **GRAVITATIONAL FIELD** symbol is $\vec{g}$—we'll show how $\vec{g}$ connects to the free-fall acceleration $\vec{g}$ in bit.

The force on a point particle m located at $\vec{r}$ is by the formula

$$\vec{F} = m\vec{g}(\vec{r}) \ . \tag{24}$$

The **GRAVITATIONAL FIELD** can be described as the gravitational force per unit mass as equation (24) shows.

The MKS units of $\vec{g}$ must be N/kg = m/s^2 which are also the MKS units of acceleration, but **GRAVITATIONAL FIELD** is **NOT** an acceleration. But it will equal an acceleration if only gravity acts on a point particle located in a **GRAVITATIONAL FIELD**. Applying $F = ma$ to a point particle acted on only the force of the gravitational field, one gets

$$m\vec{a} = m\vec{g}(\vec{r}) \ , \tag{25}$$

or

$$\vec{a} = \vec{g}(\vec{r}) \tag{26}$$

since gravity cancels out.

What a bout for system of particles in a gravitation field. Applying $F = ma$ for a system of particles gives

$$m\vec{a} = \sum m_i \vec{g}(\vec{r}_i) \ , \tag{27}$$

where the particles making up the system are labeled by i, m_i is particle mass, $\vec{r}_i$ is particle position, $g(\vec{r}_i)$ is the gravitational field due to external sources at $\vec{r}_i$. If the gravitational field can be approximated as a constant over the system, then

$$\sum m_i \vec{g}(\vec{r}_i) = m\vec{g} \tag{28}$$

where $\vec{g}$ is the approximate constant gravitational field. Then we can cancel gravity in the $F = ma$ expression for a system to get

$$\vec{a} = \vec{g} \ . \tag{29}$$

Equation (29) often applies and is considered the standard free-fall result.

We can derive the **GRAVITATIONAL FIELD** caused by a point particle of mass m_1 from the gravitation law equation (1)

$$\vec{F}_{12} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} \ . \tag{30}$$

We simply divide both sides by m_2 and suppress the subscripts. We obtain

$$\vec{g}(r) = -\frac{Gm}{r^2}\hat{r} \ , \tag{31}$$

where r is the radial position measured from the point particle.

What happens if $\vec{r}$ goes to zero in equation (31)?

The field diverges to infinity. The same divergence happens to the force between two point particles brought together.

What does the divergence mean? There really isn't a good answer. Maybe there really are no point particles or maybe our law of gravity fails for very small r. The latter point is almost certainly true since people believe that on small enough scales gravity must become quantum mechanical. There are theories that deal with this issue of diverging gravitational field, but none have gained wide acceptance.

From our discussion in § 2, we recognize that equation (31) is also the gravitational field for a spherically symmetric mass distribution outside of that distribution with m being the total mass and the origin being at the distribution center. Such a distribution can be replaced by a point particles of the same mass located at the origin.

As a matter, of fact, equation (31) is valid inside of a spherically symmetric mass distribution as long as m is taken as the mass enclosed by a sphere of radius r. This result is proven in § 6.3.

A schematic picture of the field for a point particle would be a bunch of little arrows pointing toward the origin. One could imagine arrow lengths as proportional to magnitude of the field at each point—but remember the field vectors actually extend in an abstract gravitational field space, not space space. They point in space space, of course.

If one has a distribution of point particles, then the gravitational field at a point $\vec{r}$ is

$$g(\vec{r}\,) = -\sum_i \frac{Gm_i}{|\vec{r}-\vec{r}_i|^2}\frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|} \ , \tag{32}$$

where the sum is over all particles i and the position vectors are defined relative to an arbitrary origin.

If one goes to the continuum limit for a mass distribution, then gravitational field at a point $\vec{r}$ is

$$g(\vec{r}\,) = -\int_V \frac{G\rho(\vec{r}\,'\,)}{|\vec{r}-\vec{r}\,'|^2}\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dV' \ , \tag{33}$$

where the integral is over all space (i.e., all volume V), ρ is the mass density, and the position vectors are defined relative to an arbitrary origin.

4.2. Utility of the Gravitational Field

Is the **GRAVITATIONAL FIELD** actually a **REAL THING** that causes the gravitational force and eliminates **ACTION AT A DISTANCE** really, really?

As we've introduced it, the **GRAVITATIONAL FIELD** doesn't do anything new physically.

It's a mathematical tool for calculating the gravitational force.

It actually is a very useful tool—but is it anything more?

Yes.

In Newton's pure, original theory of gravity, gravity effects propagated instantly through space—or such was the conventional interpretation. This interpretation is consistent with **ACTION AT A DISTANCE**.

If you move a mass here, it's gravitational force on mass over there changed instantly in a continuous fashion as you move the first mass.

But, in fact, as we know from various—but rather tricky-to-go-into—evidences, changes in gravitational force are known to propagate only at the vacuum speed of light.

The finite propagation time for gravitational force changes is a consequence of finite time for the gravitational field changes to propagate through space as the source of the field moves.

So yes the gravitational field is a real thing.

It's beyond our scope to go deep into gravitational field theory. But we can say it is a structure in space and it takes energy to create it which becomes has its own associated field energy. Actually changes in gravitational field energy can be calculated using the concept of potential energy. We take up the subject of gravitational potential energy in § 8. Calculating the field energy directly is beyond our scope and yours truly's knowledge.

The concept of gravitational field gets significantly modified in the theory of general relativity. But that modification is not the last word either. As mentioned in § 1, people believe the fundamental gravity theory must be a quantum theory of gravity. There are ideas about what that theory will be like, but none have gained acceptance.

5. THE NEAR-EARTH-SURFACE GRAVITATIONAL FIELD

Since the Earth is spherically symmetric to high accuracy, we can use the gravitational field formula for a point particle equation (31)

$$\vec{g}(r) = -\frac{Gm}{r^2}\hat{r} \ . \tag{34}$$

to find the Earth's gravitational field outside of the Earth: i.e., for $r > R_{\rm Ea}$, where $R_{\rm Ea}$ is the Earth's radius.

Let's use the mean radius of the Earth for $R_{\rm Ea}$. The mean radius is 6371.0 km according to some prescription (Wikipedia: Earth). The equatorial radius and polar radius are, respectively, 6378.1 km and 6356.8 km (Wikipedia: Earth). The Earth is slightly oblate which accounts for the difference in equatorial and polar radii.

The Earth's mass is $M_{\rm Ea} = 5.9736 \times 10^{24}\,{\rm kg}$ and $G = 6.67428(67) \times 10^{-11}\,{\rm N\,m^2/kg^2}$ recall.

Calculate the magnitude of $\vec{g}$ for the Earth's mean radius distance (i.e., for the Earth's surface).

You have 30 seconds. Go.

Behold:

$$g(R_{\rm Ea}) = 9.82\,{\rm N/kg}\ . \tag{35}$$

One can also write a fiducial value formula for the gravitational field around a spherically symmetric planet:

$$\vec g(r) = -9.82\,{\rm N/kg} \times \left(\frac{M}{M_{\rm Ea}}\right)\left(\frac{R_{\rm Ea}}{R}\right)^2 \hat r\ , \tag{36}$$

using the Earth's surface gravitational field, Earth's mean radius, and Earth's mass as fiducial values.

Not surprisingly the Earth g value we've calculated is nearly the fiducial g value of 9.8 N/kg we have used throughout the the earlier intro physics lectures. Actually, Earth's g varies with latitude, altitude, and local geography as discussed in the lecture *ONE-DIMENSIONAL KINEMATICS*. Moving around the solid and ocean Earth surface, g varies by only about 0.5 %. The variations are quite measurable and such measurements are important in geological and Earth science studies. But the variations are below human sense perception.

By the way, the near-Earth gravitational field magnitude g is sometimes informally called little g and the gravitational constant G is sometimes informally called big G (Wikipedia: Earth's gravity).

Here a question for the class.

How is the mass of the Earth determined?

Note we can measure g and G in the laboratory.

We inverse the equation for Earth's gravitational field and solve for Earth's mass.

What about if one is off the Earth in low Earth orbit? What is g then?

Low Earth orbit is accepted to be between altitudes 160 km and 2000 km (Wikipedia:

Low Earth orbit). Below about 160 km, air drag causes orbital decay to be very rapid.

Making use of equation 36, we find

$$g = \begin{cases} 9.35\,\mathrm{N/kg} & \text{for altitude } 160\,\mathrm{km}; \\ 5.69\,\mathrm{N/kg} & \text{for altitude } 2000\,\mathrm{km}. \end{cases} \tag{37}$$

One can see that in low Earth orbit, gravity does not just turn off.

So why are the astronauts in orbit weightless?

They are in free fall.

They are literally perpetually falling all the time, but they keep missing the Earth because they have angular momentum around the Earth. In free fall, you feel weightless because gravity pulls on you particle by particle and you are not resisting gravity at all.

We will discuss orbits in § 7 on Kepler's laws of planetary motion.

6. GAUSS'S LAW FOR GRAVITY: READING ONLY

Gauss's law for gravity is equivalent to the gravitation law. Equivalent means each can be derived from the other.

Gauss's law follows from the inverse-square law nature of the gravitational force. There are also Gauss's law for electricity and for magnetism since they are inverse-square law forces too although in other respects different from gravity and each other.

One of the reasons, in fact, for looking at Gauss's law now is help understanding the electrical and magnetic Gauss's law later. Another reason is to introduce the concept of solid angle.

At our level Gauss's law is useful for find the gravitational field for systems of very high symmetry: spherical, cylindrical, and planar.

6.1. Solid Angle

Solid angle is the three-dimensional equivalent of angle.

Let's see how solid angle arises.

It is just a basic feature of the three-dimensional Euclidean space—which is the space that describes our local reality to very high accuracy—that surfaces of objects scale up as the second power of a characteristic length.

Consider as sphere.

The most obvious characteristic length is the radius r. The surface area of a sphere is $4\pi r^2$. The ratio of area to radius squared is 4π. This ratio holds for sphere of any size or any scale. We say that 4π is the solid angle subtended by the sphere at the origin.

Now say we take any fraction of the surface area of a sphere $\Delta A_{\rm sph}$. The ratio of this fraction to radius squared is defined as the solid angle subtended by the fraction. The symbol for solid angle is Ω (the capital Greek omega). For our fraction of surface area, one has

$$\Delta\Omega = \frac{\Delta A_{\rm sph}}{r^2} \ . \tag{38}$$

Like the whole sphere, the solid angle of the fraction is independent of the fraction scale. Solid angle is a dimensionless quantity since it is the ratio of two quantities of the same dimensions (i.e., two quantities of dimension length squared). Nevertheless, there is a unit steradian (abbreviation "sr").

A differential amount of solid angle for a spherical surface is

$$d\Omega = \frac{dA_{\rm sph}}{r^2} \ . \tag{39}$$

What is the differential amount of solid angle subtended by an arbitrary differential bit of area dA? Well first let's vectorize dA to $d\vec{A} = dA\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to the dA. Remember a differential bit of surface is so small that it is planar—or in math speak, the shape of a smooth surface taken in the limit of vanishing size it planar.

What is the sense of $\hat{n}$? For a piece of open surface, the sense is arbitrary. For a closed surface, the convention is that $\hat{n}$ points outward.

Now

$$\hat{r} \cdot d\vec{A} = dA_{\rm sph} \ , \tag{40}$$

where $\hat{r}$ is the unit radius vector. The quantity $dA_{\rm sph}$ is a differential bit of spherical surface area centered at the same point as $d\vec{A}$. It is the projection of dA onto a spherical surface. Since $dA_{\rm sph}$ is differentially small it is also planar and the last equation is an exact differential result. The solid angle subtended by $d\vec{A}$ is obviously defined to be

$$d\Omega = \frac{\hat{r} \cdot d\vec{A}}{r^2} \ . \tag{41}$$

Note that $d\Omega$ can be negative if $d\vec{A}$ is at greater than 90° from $\hat{r}$.

The solid angle for a finite surface or a differential bit of surface can be pictured by a continuum of rays emanating from the origin and touching the boundary of the surface at every point on the boundary.

Finding the solid angle subtended by a finite surface can be done analytically for some simple surfaces and numerically otherwise.

For two special cases, it easy to find the total angle subtended by a surface.

Say the surface is closed and encloses the origin. The solid angle subtended is obviously 4π. Note that for convoluted surfaces, negative contributions are always canceled by positive contributions. This can be easily pictured.

Say the surface is closed and does **NOT** enclose the origin. Then the solid angle subtended is obviously 0. Positive and negative contributions cancel exactly. This can be easily pictured too.

To summarize, the two special cases:

$$\oint d\Omega = \oint \frac{\hat{r}\cdot d\vec{A}}{r^2} = \begin{cases} 4\pi & \text{for origin enclosed;} \\ 0 & \text{for origin not enclosed.} \end{cases} \tag{42}$$

6.2. Now Gauss's Law Itself

Consider the gravitational field of a point particle of mass m at the origin:

$$\vec{g} = -\frac{Gm}{r^2}\hat{r} \ . \tag{43}$$

We now take the surface integral of this field over any closed surface:

$$\oint \vec{g}\cdot d\vec{A} = -Gm\oint \frac{\hat{r}\cdot d\vec{A}}{r^2} = \begin{cases} -4\pi Gm & \text{for point particle enclosed;} \\ 0 & \text{for point particle not enclosed,} \end{cases} \tag{44}$$

where we have used equation (42).

Equation (eq-gauss-law-point-particle) is actually independent of the coordinate system origin, just not of the origin for the calculation of the solid angle. To be explicit say that $\vec{r}\,'$ is point particle location for an arbitrary origin. Then the gravitational field is

$$\vec{g}(\vec{r}\,) = -\frac{Gm}{|\vec{r}-\vec{r}\,'|^2}\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \tag{45}$$

and the surface integral of this field over any closed surface:

$$\oint \vec{g}\cdot d\vec{A} = -Gm\oint \frac{[(\vec{r}-\vec{r}\,')/|\vec{r}-\vec{r}\,'|]\cdot d\vec{A}}{|\vec{r}-\vec{r}\,'|^2} = \begin{cases} -4\pi Gm & \text{for point particle enclosed;} \\ 0 & \text{for point particle not enclosed.} \end{cases} \tag{46}$$

Now we had a system of point particles with masses $\{m_i\}$. For each one with the same closed surface we find

$$\oint \vec{g}_i\cdot d\vec{A} = -Gm_i\oint \frac{[(\vec{r}_i-\vec{r}\,'_i)/|\vec{r}_i-\vec{r}\,'_i|]\cdot d\vec{A}}{|\vec{r}_i-\vec{r}\,'_i|^2} = \begin{cases} -4\pi Gm_i & \text{for point particle enclosed;} \\ 0 & \text{for point particle not enclosed,} \end{cases} \tag{47}$$

where g_i is the gravitational field due to m_i alone, $\vec{r}_i$ is a coordinate where the gravitational field is evaluated, and $vecr'_i$ is a coordinate where point particle of mass m_i is located.

If we sum equation (47) over all point particles, we obtain

$$\oint \vec{g} \cdot d\vec{A} = -4\pi G m_{\rm encl} \ , \tag{48}$$

where $\vec{g}$ is the net gravitational field and $m_{\rm encl}$ is the total mass enclosed by the surface. Note that the exact distribution the mass of the system is not used in the formula.

Equation (48) is Gauss's law for gravitation.

It is an integral equation for the gravitational field.

You have to solve it to find the gravitational field.

In three special cases of very high symmetry, Gauss's law can be used directly to solve for the gravitational field.

In other cases, it can be used to find the gravitational field approximately.

Gauss's law can also be converted from an integral equation into a differential equation also called Gauss's law. The differential equation form is more tractable for obtaining certain solutions.

An absolutely key point about Gauss's law is that it exists because gravity is an inverse-square law force. The derivation as one can see from equations (44), (46), and (47) is based on the inverse-square law nature.

So it is no surprise that there are Gauss's laws for other inverse-square law forces. Most importantly, the electromagnetic force is an inverse-square law force. So there are, in fact, Gauss's laws for the electric and magnetic fields. If one says Gauss's law without qualification, many people will assume that you are discussing Gauss's law for the electric field.

Another point is that Gauss's law is a classical gravitational law. It holds in the classical limit and when classical physics fails, one needs general relativity or whatever gravity law supersedes general relativity.

Gauss's law should hold as masses move as long as speeds are much less than the speed of light which allows one to assume that the gravitational fields adjust instantaneously. Since we know that gravitational field changes propagate as the speed of light from general relativity, assuming Gauss's law applies as masses move is consistent with being in the classical limit.

6.3. Applications of Gauss's Law

The three cases of high symmetry that allow Gauss's law to be used to find the gravitational field exactly are spherical symmetry, planar symmetry, and cylindrical symmetry.

The planar and cylindrical cases we will not consider here since they are explicated in the lecture **GAUSS'S LAW** which is on the electric field Gauss's law. Common applications of the planar cylindrical case solutions are for electric field rather than for gravitational fields.

On the other hand, the case of spherical symmetry is a very important case for the gravitational field Gauss's law—it's important for electromagnetic systems too. Knowing the gravitational field for spherically symmetric systems is vital in astrophysics where large dense gravitating astro-bodies, most obviously, planets, stars, and large moons are nearly spherically symmetric as we noted in § 2.

As we noted in § 4, the gravitational field of spherically symmetric system when one is outside of the system is exactly the same as if the system was point particle with all system mass concentrated at the system center. We can now prove this result using Gauss's law.

Consider a spherically symmetric mass distribution. By symmetry the gravitational field must be symmetric. The gravitational field vectors must everywhere point radially inward and they depend only radially position the center of symmetry which we take as the origin. Now consider a general spherical surface of radius r concentric about the origin. There may be nothing at this surface or there may be. We will apply Gauss's law to the surface. In the jargon of Gauss's law application, a surface to which Gauss's law is applied is called a Gaussian surface.

Using symmetry and Gauss's law (eq. (48),

$$\oint \vec{g} \cdot d\vec{A} = -4\pi r^2 g = -4\pi G m_{\rm encl} \tag{49}$$

from which it follows that

$$\vec{g} = -\frac{G m_{\rm encl}}{r^2}\hat{r} \ , \tag{50}$$

where $m_{\rm encl}$ is the mass enclosed by the surface.

Equation (50) is the general formula for the gravitational field of a spherically symmetric system. Note the mass outside of radius r has no effect at all on the gravitational field. This is remarkable feature is a direct consequence of the inverse-square law nature of the gravitational force.

If the radius r is outside of the whole distribution, equation (50) specializes to

$$\vec{g} = -\frac{G m}{r^2}\hat{r} \ , \tag{51}$$

where m is the total mass of the system. Equation (51) is just the result we stated in § 4.1: the gravitational field of spherically symmetric system when one is outside of the system is exactly the same as if the system was a point particle with all system mass concentrated at the system center. So the forces a spherically symmetric system exerts on objects outside of the system is exactly as if the system were a point particle with all the system mass concentrated at the system center. We first stated this result in § 2.3.

We have now proven the point-particle-like result from Gauss's law with great ease. As we mentioned in § 2.3, Newton using primitive calculus tools (of his own invention) had to work hard to get the result (e.g., French 1971, p. 265).

Note if the mass distribution is, in fact, a point particle, then we can derive from equation (51), the gravitation law. Say there is a first point particle wiht m_1 that causes a gravitational field and a second point particle with mass m_2 located a displacement $\vec{r}_{12}$ away.

$$\vec{F}_{12} = m_2\vec{g} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} \ . \tag{52}$$

Thus, Gauss's law is actually equivalent to the universal law of gravitation as we mentioned above: each can be derived from the other. One could start gravitation theory with Gauss's law rather than the gravitation law if one wished to.

A few other results can be easily derived from equation (50). Say that there is a mass-free spherical hollow in the center of the mass distribution. For a spherical Gaussian surface inside this hollow, the mass enclosed is zero and equation (50) specializes to

$$\vec{g} = 0 \ . \tag{53}$$

Another remarkable result. One would just float around inside a hollow planet.

If one is given the density distribution $\rho(r)$, then one can write

$$\vec{g} = -\frac{Gm_{\text{encl}}}{r^2}\hat{r} == -\frac{G}{r^2}\hat{r}\int_0^r 4\pi r'^2\rho(r')\,dr' \ . \tag{54}$$

If one has sphere of radius R with constant density ρ (i.e., was a uniform sphere), then

$$m_{\text{encl}} = m\left(\frac{r}{R}\right)^3 \ , \tag{55}$$

and one obtains

$$\vec{g} = \begin{cases} -\dfrac{Gm}{R^2}\left(\dfrac{r}{R}\right)\hat{r} & \text{for } r \leq R; \\ -\dfrac{Gm}{r^2}\hat{r} & \text{for } r \geq R. \end{cases} \tag{56}$$

Remarkably the force inside the uniform sphere is linear with radius. If one had a narrow shaft along a sphere diameter, a particle in the shaft could execute simple harmonic motion.

In actual fact, large astro-bodies are not uniform spheres. In order to support themselves against collapse the must have an internal pressure force. This force increases with depth to support the greater overlying mass. Density increases as pressure increases. So density increases with depth.

7. KEPLER'S LAWS OF PLANETARY MOTION

8. GRAVITATIONAL POTENTIAL ENERGY

9. BLACK HOLES

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