NEWTONIAN PHYSICS II

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ABSTRACT

Lecture notes on what the title says and what the subject headings say.

Subject headings: linear force — simple harmonic oscillator — simple harmonic motion — circular motion — uniform circular motion — curve banking — non-uniform circular motion — simple pendulum — non-inertial frames — inertial forces — drag forces — terminal velocity

1. INTRODUCTION

More Newtonian physics, new force laws, still no energy.

2. THE LINEAR FORCE AND THE SIMPLE HARMONIC OSCILLATOR

The linear force is an extremely important force. In this lecture, we only consider it in one dimension. The formula for linear force on an object in one dimension is

$$F = -kx \ , \tag{1}$$

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where k is force constant which is always greater than zero, x is the object position in the one dimension, and $x = 0$ gives the equilibrium point (i.e., the point where the force is zero).

One doesn't need to have the origin as the equilibrium point as in equation (1). One write the linear force formula

$$F = -k(x - x_{\rm eq}) \;, \tag{2}$$

where now $x = x_{\rm eq}$ gives the equilibrium point.

What is the object position for the linear force?

Well it should be the position in the object that makes the linear force law true. This position is **NOT** the center of mass in general.

But we usually implicitly assume in intro physics problems that the object is rigid and constrained not to rotate in anyway during one-dimensional motion. In this case any position fixed relative to the object can serve as the object position. The location of equilibrium point relative to an outside coordinate system depends on the object position choice. For example, if one shifts the object position by Δx, the equilibrium point shifts by Δx. Given our assumption of rigid, constrained object, the obviously best choice for object position is usually the center of mass of the object. This makes the application of 2nd law using the linear force simpler. On the other hand, if the object is small relative to displacements from equilibrium, it becomes particle-like and just some representative point in the object can serve as the object position and also an approximate center of mass.

Usually we will use center of mass for the object position for the linear force and assume that this is valid.

Where does that linear force actually act on the object?

Well this depends on the particular case.

For the problems we deal with, it usually doesn't matter. The linear force is an external force and we are usually just calculating the center-of-mass motion. Recall for center-of-mass where a force is applied to a system is irrelevant in a direct sense. In an indirect sense is can be very relevant since internal motion can dictate what external forces actually apply to an object. Given our assumption of rigid, constrained object, where one-dimensional forces are applied to the object is irrelevant.

Why is the linear force so important?

For the moment, let's go beyond one dimension and consider general systems.

Imagine any system with all parts in equilibrium (e.g., a building). This means the net force on any part is zero.

Equilibria can be **UNSTABLE** or **STABLE**. Unstable and stable are somewhat relative terms note.

A ruler balanced on a finger is **UNSTABLE**. A small perturbation causes an irreversible change.

On the other hand, if an equilibrium of a system is **STABLE**, then any small perturbation of any part from its equilibrium position does not cause the system to undergo irreversible change (e.g., a building to collapse). Rather the system will return to equilibrium after perhaps some oscillations.

For stable equilibrium, there must a **RESTORING FORCE** bring any part back to the equilibrium and also, if the system is not idealized, some damping force to reduce any oscillations or velocity caused by the initial perturbation to go to zero.

The restoring force on a part is zero at equilibrium. The restoring force grows as the part is displaced from equilibrium. Now nature usually has no discontinuities at least on small enough scales, and so one would expect the restoring force to be a smooth function of displacement. For small enough displacement, the restoring force should be linear in displacement. So the restoring force should be a linear force for small enough displacements. Thus, nearly always the linear force is the force that counters small displacements from equilibrium and thus stabilizes the equilibrium. The linear force therefore has almost universal application for stable equilibrium systems.

The linear force for one dimension given by equation (1) is, in fact, a force of the consistent with general restoring forces we've described. If the $x = 0$, there is zero force. If the $x < 0$, there is a force in the positive direction. If the $x > 0$, there is a force in the negative direction. It acts to try to restore the system to the equilibrium position. Damping forces that we will not consider until much later in intro physics will damp out oscillations and velocities caused by the linear force.

We are not going to consider general systems explicitly. We will just consider the linear force as an exact force law and analyze the behavior it gives for an object subject to it. That gives us lots of insight into general systems without stretching our brains.

The linear force is a force of many names. It is called:

1. The linear force by yours truly since it is linear in the spatial coordinate.

2. The linear restoring force since it is linear in the spatial coordinate and tries push an object back to the equilibrium (i.e., it tries to restore the object to the equilibrium of the force) just as was we've described above.

3. The spring or ideal spring force since springs are key examples of systems which exert the linear force: approximately for real springs and exactly for ideal springs. The $x = 0$ location is the equilibrium position of the spring. At the microscopic level, the spring force is an electromagnetic force arising from chemical bonds. Springs exhibiting the

linear force, are very useful devices for measuring forces (e.g., fish scales, bath room scales) and in shock absorbing systems like in car suspensions (Wikipedia: Suspension (Vehicle)). A **DEMONSTRATION** might be good here of any spring system yours truly can find in the demo room.

4. The Hooke's law force since equation (1) is called Hooke's law. Robert Hooke (1635–1703) first published the law in 1676 as a Latin anagram which means that he published a puzzle for folks. In 1678, he unanagramed as *Ut tensio, sic vis* which means "As the extension, so the force." (Wikipedia: Hooke's law).

5. The simple harmonic oscillator force since a simple harmonic oscillator (SHO) is a system in which this force is the only force causing acceleration (or doing work in the language of lecture *ENERGY*). There can be constraint forces in such systems, but they do no work.

What is the motion of an object subject to the linear force alone?

Let's apply Newton's 2nd law:

$$F_{x,\mathrm{net}} = -kx = ma \tag{3}$$

which using $a = d^2x/dt^2$, we can rewrite as

$$m\frac{d^2x}{dt^2} = -kx \ . \tag{4}$$

Equation (4) is often called the SHO equation since it is the equation of motion for a SHO.

Recall the expression "equation of motion" is used in various ways. Yours truly uses it to mean a particular application of the 2nd law.

Equation (4) is a **DIFFERENTIAL EQUATION** (DE) which is an equation whose solution is not a value, but a function and which involves derivatives of that function. The function solution for equation (4) is $x(t)$: i.e., x as function of the independent variable t.

It may be your first DE—at least the first explicit, non-trivial DE. In fact, $F = ma$ is really a DE—and you never guessed that did you—because what we typically want is velocity and position as a function of time and a is derivative of these quantities. When a is a constant (which has usually been the case hitherto) solving the DE is just integration (i.e., antidifferentiation). When a is not a constant as in equation (4), getting a solution is trickier.

The order of the DE is the highest order derivative of the function in the DE: in the present case, the DE is 2nd order since the highest order derivative is the second derivative. It is a linear DE since the function and its derivatives occur only linearly in the DE. If a function or any of its derivatives occur non-linearly, a DE is non-linear. Linear DEs are usually much more easy to solve or approximate than non-linear DEs. It is an ordinary DE (an ODE) because there is only one independent variable time t. Partial DEs (PDEs) have multiple independent variables.

DEs are everywhere in physics. Much of physical law is, in fact, formulated in terms of DEs. They are also everywhere in all other mathematical sciences too.

Methods of solving DEs are beyond our scope—other than simple integration solutions (i.e., antidifferentiation solutions) which we've already done—e.g., in finding the constant-acceleration kinematic equations.

But we can just write down the general solution of equation (4) and verify it.

The general solution is

$$x = A\cos(\omega t) + B\sin(\omega t) \ . \tag{5}$$

The ω is called the angular frequency and it is a constant. Same symbol as for angular velocity, but the two quantities are distinct in most cases. The angular frequency ω is set by the physical parameters of the system as we'll see just below. The A and B are constants of integration to be determined by initial conditions of the SHO oscillator system. We'll show how they are set below in § 2.2.

Find the formulae for v and a. You have 1 minute working in groups or individually. Go.

To summarize the motion variables for the general solution, we write

$$x = A\cos(\omega t) + B\sin(\omega t) \ , \tag{6}$$

$$v = -A\omega\sin(\omega t) + B\omega\cos(\omega t) \ , \tag{7}$$

$$a = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) \ , \tag{8}$$

where ω is called the angular frequency and A and B are constants of integration.

Is it possible verify that for the purported general solution satisfies the SHO DE $ma = -(k/m)x$ and to determine what ω is in terms of the parameters of the system.

You have 1 minute working in groups or individually to do these things. Go.

We find that

$$a = -\omega^2 x \tag{9}$$

which is, in fact, equation (4) when we choose—do so choose—that

$$\omega = \sqrt{\frac{k}{m}} \ . \tag{10}$$

We have explicitly verified that equation (6) is the solution of equation (4) for our choice of ω. Often one sees k replaced by ω as a parameter of the SHO equation (6): this makes either equation (9) or $ma = -m\omega^2 x$the fiducial form of the SHO equation. Note that

$$k = m\omega^2 \ . \tag{11}$$

We note that the constants of integration A and B are not set by the DE itself. Any values of A and B will satisfy the DE. The constants are determined by the initial conditions. We will discuss the imposing of initial conditions below in § 2.2.

The cosine and sine functions are oscillatory and periodic. So the motion of the object is a exactly repeating oscillation.

Actually, the motion is sinusoidal since a linear combination of a cosine and sine is also a sinusoid. This is easy to show. Consider $C\cos(\omega t+\phi)$. A trig identity gives

$$C\cos(\omega t+\phi) = C\cos\phi\cos(\omega t) - C\sin\phi\sin(\omega t) \ . \tag{12}$$

Given $A = C\cos\phi$ and $B = -C\sin\phi$, one finds

$$\phi = \tan^{-1}\left(\frac{-B}{A}\right) + n\pi \qquad \text{and} \qquad C = \frac{A}{\cos\phi} \ , \tag{13}$$

$n = 0$ if the ordered pair $(A, -B)$ is in the 1st or 4th quadrants and $n = 1$ if it is in the 2nd or 3rd quadrants.

Thus, one can always reduce the linear combination of cosine and sine functions to cosine function with a phase constant ϕ or a sine function with a phase constant $\phi = \pi/2$ since by another trig identity

$$\sin\left(\omega t + \frac{\pi}{2}\right) = \cos(\omega t) \ . \tag{14}$$

So a linear combination of a cosine and sine is a sinusoid as aforesaid. Note that the angular frequency of the linear combination sinusoid is the same as for the cosine and sine functions in the linear combination.

Why is the SHO called the SHO?

It's a very simple system and the motion is very harmonious. The motion is called simple harmonic motion. Actually simple harmonic motions turn up all over the place in musical applications. For example, a taut string vibrating in standing waves often exhibits simple harmonic motion at each point along the string axis in a direction perpendicular to the string axis (e.g., Halliday et al. 2001, p. 388).

Now let's first investigate the periodicity of the SHO solution and then consider imposing the initial conditions.

2.1. Angular Frequency ω, Period P, Frequency f

In this section, we investigate the relationships between angular frequency ω, period P, and frequency f.

But first, why is ω called the angular frequency?

Well yours truly can only guess, but the arguments of cosine and sine are usually thought of as being angles. In the context of uniform circular motion, the angular frequency and the angular velocity are the same thing as we'll show just below. In the context of the simple harmonic oscillator, the arguments are **NOT** angles in general. There is a case where they are angles. That is the case of uniform circular motion viewed in projection along an axis. We will see in this in § 4.1. But there are also simple harmonic motion cases where there is angular motion, but the angular velocity and angular frequency are **NOT** the same quantity: e.g., the simple pendulum (see § 6.1). This can give rise to a **CLASH** of symbols.

Now for the relationships.

For any sinusoidal function of time, let P be the period of the oscillation. Also for uniform circular motion, let P be the period of the motion.

Now sinusoidal function repeats its behavior (i.e., completes a cycle) every time its argument increases by 2π and uniform circular motion repeats its behavior every time the angular position increases by 2π We see that P must satisfy

$$2\pi = \omega P \tag{15}$$

where ω is angular frequency for sinusoidal motion and angular velocity for uniform circular motion. We now find

$$P = \frac{2\pi}{\omega} \ . \tag{16}$$

Frequency f is the number of cycles (i.e., repeats of behavior) per unit time. Say we had N cycles which, of course, took time NP. The frequency is given by

$$f = \frac{N}{NP} = \frac{1}{P} = \frac{\omega}{2\pi} \ . \tag{17}$$

The result $f = 1/P$ is general for any periodic motion, not just sinusoidal motion and uniform circular motion.

Since $f = \omega/(2\pi)$ holds for uniform circular motion, one just says ω is also the angular frequency for uniform circular motion as well as for sinusoidal motion.

To summarize our results for sinusoidal motion and uniform circular motion, we have

$$\omega = 2\pi f = \frac{2\pi}{P} \ , \qquad P = \frac{1}{f} = \frac{2\pi}{\omega} \ , \qquad f = \frac{1}{P} = \frac{\omega}{2\pi} \ . \tag{18}$$

Now in the case of the SHO

$$\omega = \sqrt{\frac{k}{m}} \ . \tag{19}$$

Thus, for the SHO we have

$$f = \frac{1}{P} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \ , \qquad P = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \ , \qquad \omega = 2\pi f = \frac{2\pi}{P} = \sqrt{\frac{k}{m}} \ . \tag{20}$$

We can draw a couple of physical observations.

The greater the mass of the object, the longer the period—the object is more sluggish.

The greater the force constant, the shorter the period—the object is livelier.

2.2. Imposing the Initial Conditions

Newton's 2nd law applied to a simple harmonic oscillator is physical law. It says what is always true about the SHO.

But any particular SHO behavior also depends on the initial conditions of the system which are set by—well by all of past history in real cases.

The initial conditions are imposed on the solution equation (6) by the choice of the constants of integration A and B.

We have two unknowns A and B, and so need two equations for a solution. Say we know initial position x_0 and initial velocity v_0 of the SHO at time $t = 0$.

Then from equations (6) and (7) we have

$$x_0 = A \ , \qquad v_0 = B\omega \tag{21}$$

which give the solutions

$$A = x_0 \ , \qquad B = \frac{v_0}{\omega} \ . \tag{22}$$

For example, say $x_0 = 0$ and $v_0 = 0$, then A and B are both zero and the oscillator stays at rest at the equilibrium position.

Now if $x_0 > 0$ and $v_0 = 0$, then $B = 0$ and the oscillation starts from rest and $A = x_0$ is the maximum absolute value of the displacement from equilibrium which is called the amplitude.

In general, physical law tells you what is eternal about a system and initial conditions and boundary conditions tell you what is peculiar or individualistic about the system.

Boundary conditions are conditions at the boundary of a system that dictate some of its features. Initial conditions can be considered boundary conditions in time, but usually one thinks of boundary conditions as be spatial conditions.

2.3. The Linear Force and Gravity

We can complicate our system a bit by adding gravity to the linear force with the one dimension of motion now chosen to be the vertical.

The 2nd law in this case gives

$$ma = -ky - mg \ , \tag{23}$$

where we have used y as the coordinate since the motion is in the vertical direction and we have taken y positive upward, and so the gravitational force is downward.

Now $y = 0$ is the intrinsic equilibrium point of the linear force: i.e., the equilibrium point with gravity turned off.

Now note

$$ma = -ky - mg = -k\left(y + \frac{mg}{k}\right) = -k(y - y_{\rm eq}) \ , \tag{24}$$

where define

$$y_{\rm eq} = -\frac{mg}{k} \ . \tag{25}$$

We see that we have the SHO problem all over again. The equilibrium position is displaced from zero to $y_{\rm eq}$.

One can now define a new coordinate

$$y' = y - y_{\rm eq} \ . \tag{26}$$

Note from the chain rule that

$$\frac{d^2y'}{dt^2} = \frac{d}{dt}\left(\frac{dy'}{dy}\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2} \ . \tag{27}$$

The equation of motion for y' is now seen to be

$$m\frac{d^2y'}{dt^2} = -ky' \ . \tag{28}$$

The solution is already known for y'. It's just

$$y' = A\cos(\omega t) + B\sin(\omega t) \ . \tag{29}$$

So adding constant gravity to the SHO simply shift the equilibrium point and the SHO solution by $y_{\rm eq}$.

2.4. Springs in Parallel: Optional

Say that you had a set of ideal, massless springs attached to an object of mass m and to some rigid wall or rigid walls if you allow springs on both sides of the object. The springs are all in parallel and the only motion allowed is in the one dimension that the springs are in. So the system is one dimensional. A spring i has force constant k_i and the equilibrium position is x_i for the object center-of-mass position for spring i. The net spring force on the object is

$$F = -\sum_i k_i(x - x_i) \ , \tag{30}$$

where x is the object center of mass position.

The object can be viewed as being attached to an equivalent net spring. To prove this note

$$F = -\sum_i k_i(x - x_i) = -\left(\sum_i k_i x - \sum_i k_i x_i\right) = -k\left(x - x_{\rm eq}\right) \ , \tag{31}$$

where we define the equivalent force constant by

$$k = \sum_i k_i \tag{32}$$

and the equivalent equilibrium position by

$$x_{\rm eq} = \frac{\sum_i k_i x_i}{k} \ . \tag{33}$$

Since we find a force formula of the same form as for a single spring (with k being the net spring constant and $x_{\rm eq}$ being equilibrium position for the net system), we have proven the equivalent net spring replacement.

2.5. Springs in Series: Optional

Say that you had a set of ideal, massless springs attached in series. One end of the series attaches to to an object of mass m and the other to some rigid wall. The springs are along one dimension and the only motion allowed is in the one dimension that the springs are in. So the system is one dimensional. A spring i has force constant k_i and the extension from equilibrium length is Δx_i which can be positive or negative.

Recall that standard treatment of massless systems is that they must have zero net force no matter what their acceleration. This is the correct limiting behavior for systems whose mass goes to zero. The acceleration of such systems is determined by larger systems with mass that they are a part of.

Therefore, the forces acting on the ends of each (massless) spring must be equal and opposite in order for the net force on each spring to be zero. By the 3rd law, the force that by any object in the series on another must also be equal and opposite to force the other exerts on that object. In fact, the end forces exerted by all the springs (including those they exert on the wall and object) must be equal in magnitude also by the 3rd law.

Consider the force the springs exert at their positive ends:

$$F = -k_i \Delta x_i \ , \tag{34}$$

where the force is unadorned by a subscript since it is the same at all positive ends. At the negative ends, the force has the same magnitude, but opposite direction.

We see that force exerted by the whole system of springs on the object is just F actually.

Now note that

$$\frac{F}{k_i} = -\Delta x_i \ . \tag{35}$$

Summing over all springs, we get

$$\sum_i \frac{F}{k_i} = -\Delta x \ , \tag{36}$$

where we define the total extension of the spring system by

$$\Delta x = \sum_i \Delta x_i \ , \tag{37}$$

We now see that

$$F\left(\sum_i \frac{1}{k_i}\right) = -\Delta x \ , \tag{38}$$

Now we see that the whole spring system can be viewed as an equivalent net spring with spring constant obeying the linear force law

$$F = -k\Delta x \ , \tag{39}$$

where k is an equivalent force constant given by

$$\frac{1}{k} = \sum_i \frac{1}{k_i} \ . \tag{40}$$

We note that the equivalent force constant k satisfies the inequality

$$\frac{1}{k} \geq \max\left(\frac{1}{k_i}\right) \tag{41}$$

which implies the inequality

$$k \leq \min\left(k_i\right) \ , \tag{42}$$

where equalities only holds for the case that there is one spring in the series.

3. CIRCULAR MOTION

It's usually a good idea to exploit the symmetry of circular motion by using polar coordinates.

Recall from the lecture *KINEMATICS IN TWO DIMENSIONS* that the general polar-coordinate kinematic formulae for displacement, velocity, acceleration are, respectively,

$$\vec{r} = r\hat{r} \ , \tag{43}$$

$$\vec{v} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \ , \tag{44}$$

$$\vec{a} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta} \ , \tag{45}$$

where

$$\omega = \frac{d\theta}{dt} \qquad \text{and} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{46}$$

are, respectively, the angular velocity and angular acceleration (e.g., French 1971, p. 556–557).

A major complication in polar coordinates is that the unit vectors $\hat{r}$ and $\hat{\theta}$ are dependent on the angular coordinate θ. One has

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \qquad \text{and} \qquad \hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y} \ , \tag{47}$$

where $\hat{x}$ and $\hat{y}$ are the constant unit vectors of 2-dimensional Cartesian coordinates.

If you are given r and θ as a function of time, then the whole motion is known.

But, of course, that is not always the case.

What you are given all the time—in classical physics—is Newton's 3 laws and in particular the 2nd law $F = ma$. In polar coordinates, the 2nd law can be written

$$\vec{F}_{\text{net}} = F_r\hat{r} + F_\theta\hat{\theta} = m\left[\left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}\right] = m\vec{a} \ , \tag{48}$$

where $\vec{F}_r$ is the net force in the radial direction and $\vec{F}_\theta$ is the net force in the angular direction.

Note that equation (48) is referenced to inertial frames. Newton's 2nd law in all forms is reference to inertial—unless you introduce non-inertial forces—which we do in § 7.

General polar coordinate systems can be tough.

But we now specialize to circular motion where r is constant.

So specialize the general kinematic formulae to the case of r constant. You have 30 seconds. Go.

The kinematic formulae specialize to

$$\vec{r} = r\hat{r} , \quad (49)$$

$$\vec{v} = r\omega\hat{\theta} , \quad (50)$$

$$\vec{a} = -r\omega^2\hat{r} + r\alpha\theta . \quad (51)$$

The angular components $r\omega$ and $r\alpha$ are, respectively, the tangential velocity and tangential acceleration. The term tangential is because these components are tangent to the circle of motion. Since in circular motion there is no radial velocity, we will usually just use v for tangential velocity, but one can add a subscript like "tan" for clarity. Note

$$v = r\omega \qquad \text{and} \qquad \omega = \frac{v}{r} . \quad (52)$$

For tangential acceleration, we will use a_{tan} when needed: Note

$$a_{\text{tan}} = r\alpha \qquad \text{and} \qquad \alpha = \frac{a_{\text{tan}}}{r} . \quad (53)$$

The radial component of acceleration is always negative for circular motion and hence is called the centripetal acceleration (center-directed acceleration) as we did in the lecture *MULTI-DIMENSIONAL KINEMATICS*. The centripetal acceleration is given by

$$a_{\text{cen}} = -r\omega^2 = -\frac{v^2}{r} . \quad (54)$$

Often the negative sign is dropped if one just means the magnitude or if one counts inward as positive.

For circular motion, the 2nd law specializes to

$$\vec{F}_{\rm net} = F_r\hat{r} + F_\theta\hat{\theta} = m\left(-r\omega^2\hat{r} + r\alpha\hat{\theta}\right) = m\left(-\frac{v^2}{r}\hat{r} + r\alpha\hat{\theta}\right) = m\vec{a} \ . \tag{55}$$

The radial component of the 2nd law gives the centripetal force formula

$$F_{\rm cen} = -mr\omega^2 = -m\frac{v^2}{r} \ , \tag{56}$$

where $F_{\rm cen} = F_r$ and where one drops the negative sign if one only wants magnitudes or if one considers inward as positive.

The centripetal force means center-pointing force. Note that the centripetal force is always radially inward. It must be to cause a radially inward acceleration which is kinematically necessary for circular motion. If the motion is not circular, then outward radial forces and accelerations can occur.

Absolutely, positively, the centripetal force is **NOT** a magical force that turns on when you go into circular motion. It's the radial component of $\vec{F}_{\rm net}$ of the 2nd law in the case circular motion.

The way to view the centripetal force is as a **REQUIREMENT** for circular motion. There must be some actual physical force to supply the centripetal force. For planets or satellites in orbit, the centripetal force is gravity. For object swung around on a string, it's string tension.

Let's now consider uniform circular motion (i.e., constant speed circular motion).

4. UNIFORM CIRCULAR MOTION

In the case of uniform circular motion, r and ω are both constants. For simplicity, we usually assume $\theta = 0$ at time $t = 0$. Thus,

$$\theta = \omega t \ . \tag{57}$$

The polar-coordinate kinematic formulae for displacement, velocity, and acceleration can easily be specialized.

Do so. You have 30 seconds. Go.

Well one gets

$$\vec{r} = r\hat{r} \ , \tag{58}$$

$$\vec{v} = r\omega\hat{\theta} \ , \tag{59}$$

$$\vec{a} = -r\omega^2\hat{r} = -\frac{v^2}{r} \ . \tag{60}$$

The 2nd law specializes to

$$\vec{F}_{\rm net} = F_r\hat{r} = -mr\omega^2\hat{r} = -m\frac{v^2}{r}\hat{r} = m\vec{a} \ , \tag{61}$$

or more simply to

$$F_r = -mr\omega^2 = -m\frac{v^2}{r} \ , \tag{62}$$

where one drops the negative sign if one only wants magnitudes or if one considers inward as positive. As in § 3, the radial component of the net force is called the centripetal force $F_{\rm cen}$ and one writes

$$F_{\rm cen} = -mr\omega^2 = -m\frac{v^2}{r} \ . \tag{63}$$

In uniform circular motion, there is only the centripetal acceleration and centripetal force. There is no tangential acceleration and no tangential force.

Repetition is good. So we'll just repeat some remarks from § 3 here.

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4.1. Uniform Circular Motion and Simple Harmonic Motion

There is an interesting point to be made about uniform circular motion in general. It is closely related to simple harmonic motion.

To see this first consider the displacement in uniform circular motion

$$\vec{r} = r\hat{r} = r(\cos\theta\hat{x} + \sin\theta\hat{y}) = r\left[\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}\right] \tag{64}$$

where we have used equation (47) for the polar coordinate unit vector $\hat{r}$.

Say we take the projection along the x-axis using the dot product. We get

$$x = \vec{r}\cdot\hat{x} = r\cos(\omega t) \tag{65}$$

which is just simple harmonic motion with angular frequency ω equal to angular velocity.

The projection along any other axis through the center of the circle would also yield simple harmonic motion with angular frequency ω. Now say for the axis in the direction of the arbitrary unit vector $\hat{u}$. We get

$$x = \vec{r}\cdot\hat{u} = r[u_x\cos(\omega t) + u_y\sin(\omega t)] \ , \tag{66}$$

where

$$u_x = \hat{x}\cdot \qquad \text{and} \qquad u_y = \hat{u}\cdot\hat{x} \ . \tag{67}$$

So we get a linear combination of cosine and sine functions. But such a linear combination is actually a sinusoid with the angular frequency ω as we showed in § 2.

The conclusion is that the projection along any axis through the center of the circle of a uniform circular motion yields simple harmonic motion with angular frequency ω. So a simple harmonic motion can be generated from a uniform circular motion.

The angular velocity of the uniform circular motion (which is also its angular frequency: see § 2.1) becomes the angular frequency of the generated simple harmonic motion.

The fact that a SHO motion arises as a projection of uniform circular motion and has the same angular frequency is a special case. In most simple harmonic oscillator cases, there is no angular motion. And in some cases where there is an angular motion, the angular velocity and angular frequency are not the same quantity (see § 6.1). An example is the case of the simple pendulum (see § 6.1).

4.2. Self-Rotational Motion of System in Uniform Motion

This section is usually just left as reading. Yours truly won't lecture on it usually.

Do we have to worry about the self-rotational motion of an object in uniform circular motion: i.e., its rotation about its center of mass?

Well not much.

If we idealize the system as point mass, not at all.

If it is negligibly small, not at all it.

If the object is self-rotating at a constant rate and there are no torques about its center of mass, then it will keep self-rotating at a constant rate by conservation of angular momentum.

The concepts of torque and angular momentum are introduced in the lecture *ROTATIONAL DYNAMICS*.

In simple demonstrations, we usually think of the object as self-rotating with a constant angular velocity equal to the angular velocity of the uniform circular motion.

This is the case of swing a bob with a sling.

But in general the uniform circular motion angular velocity and the self-rotation angular velocity do **NOT** have to be the same.

For example, they are not the same in general for astrophysical bodies in orbits. The Earth orbits the Sun in a year, but its self-rotation is daily.

There is a common special case where orbiting bodies have the same orbital revolution rate and self-rotation rate. This case is often called co-rotation.

Co-rotation occurs when an orbiting body becomes tidally locked to it's orbital companion. In such cases, the orbiting body always turns the same face to the orbital companion. Tidal effects effect tidal locking and stabilize it against perturbations that try to delock it. We won't go into tidal locking in more detail here. For more information, see Wikipedia: Tidal locking.

Tidally locking is common for small bodies orbiting much larger ones. In the solar system, most moons are tidally locked to their parent planet, and thus exhibit co-rotation.

The Earth's moon—the Moon—for example is tidally locked to the Earth. We always see the same face. We didn't know what the other side looked like before the Soviet probe Luna 3 took the first images of the far side in 1959—which is why nearly everything on the far side is named for something or someone Russian.

The tidal locking isn't perfect. Small effects cause small oscillations from perfect tidal locking and allow us to see about 59 % of the Moon surface from the Earth.

4.3. Example: Sling and Bob on a Frictionless, Level Surface

In this case, we swing a sling with bob (that we can approximate as a point) on a frictionless level surface. The sling rope is an ideal rope. The bob mass is m. The bob executes uniform circular motion about a center with speed v. The radius of the circle is r.

In the vertical direction, there is no motion and the normal force of the surface balances gravity.

In the horizontal direction, the tension force provides the centripetal force. Therefore, we have

$$T = m\frac{v^2}{r} \ , \tag{68}$$

where we've taken inward as positive for simplicity.

Now say that there is maximum tension $T_{\rm max}$ before the rope breaks.

What is the maximum speed before the rope breaks? You have 1 minute working in groups or individually. Go

Solving for speed v gives

$$v = \sqrt{\frac{Tr}{m}} \ . \tag{69}$$

Since v increases strictly with T from $T = 0$, the maximum tension gives the maximum speed before the rope breaks. Thus,

$$v_{\rm max} = \sqrt{\frac{T_{\rm max} r}{m}} \ . \tag{70}$$

For some numbers, say $m = 4.0\,{\rm kg}$, $r = 1.0\,{\rm m}$, and $T_{\rm max} = 576.0\,{\rm N}$. We get

$$v_{\rm max} = \sqrt{\frac{T_{\rm max} r}{m}} = \sqrt{\frac{576 \times 1}{4}} = 12\,{\rm m/s} \ . \tag{71}$$

The fact that the answer works out to be an exact number means that we've cooked the input numbers.

4.4. Example: The Conical Pendulum

In this case, we create a conical pendulum by swinging a sling and bob (that we can approximate as a point). The sling rope is an ideal rope. The bob mass is m. The bob traces out a horizontal circle in the air and the rope a cone. The bob executes uniform circular motion about a center with speed v. The radius of the circle is r. Note this is the radius of the circle, and so the radius is in the horizontal direction. The length of the rope is ℓ.

In this case, the tension force must supply the centripetal force and the balance gravity.

Since there are no forces on a massless rope between the endpoints, the rope points in the direction of the tension force that it exerts on the bob.

Let θ be the angle of the rope from the vertical. It must hang down. The angle θ must be less than 90°.

Yours truly will now the demonstrate the conical pendulum with my sling—I made this myself—not the parts, of course—anyone can make parts—the total product. Students in the front row are permitted to cower.

We note the demonstration is less than ideal. First, the rope is not ideal. Second, there is air resistance and friction at the pivot point. These forces tend to decelerate the sling. I need to exert a bit of tangential force to make the pivot move in order to keep the sling from decelerating. To anticipate the lecture *ENERGY*, the extra pivot motion compensates for the loss of energy to waste heat.

As you probably note, the faster I set the speed v, the close θ gets to 90°.

Let's analyze the dynamics.

Now for the course mantra:

"Newton's 2nd law is always true and it's always true component by component."

We first note that the conical pendulum is a three-dimensional motion actually. But we can analyze it in two dimensions in a plane that contains the axis of rotation. This plane rotates with the bob and all forces point in this plane. We are exploiting the axial symmetry of the system and using polar coordinates for the plane the bob circles in. The only coordinates we need are the horizontal radial coordinate and the vertical y coordinate.

Apply the 2nd law to the radial and vertical directions. You have 1 minute working in groups or individually. Go.

Since it's a given that the bob is in uniform circular motion, the horizontal component of the tension force horizontal component is the centripetal force. We have

$$T\sin\theta = m\frac{v^2}{r} = \frac{mv^2}{\ell\sin\theta} \ . \tag{72}$$

We've taken inward as positive for simplicity.

In the vertical direction, the vertical tension component cancels gravity and we find

$$T\cos\theta = mg \ . \tag{73}$$

So we have two equations:

$$T \sin\theta = \frac{mv^2}{\ell \sin\theta} , \tag{74}$$

$$T \cos\theta = mg . \tag{75}$$

Those equations represent physical law for the given system.

They contain 5 variables (not counting g): i.e., T, θ, m, v, and ℓ.

The equations only allow us to solve for 2 variables. All the other 3 variables must be determined by the initial conditions of the system. They cannot be determined by physical law. They can be set independently. One can choose them to have any values you like. Those choices are the initial conditions, in fact.

This is usually the way it is in physics. Physical law gives you some relationships, but you also need initial conditions or boundary conditions. Physical law is what is always true. The initial or boundary conditions are what are peculiar to the system at hand.

In principle, any 3 variables can be chosen as initial conditions or knowns. But in simple hand-powered demonstrations with simple equipment m and ℓ are likely to be fixed knowns. So let's say they are knowns. That leaves three unknowns θ, T, and v. If we set any one of them, then the other two can be solved from our equations (74) and (75).

Let's consider in turn the cases of θ, T, and v chosen to be the third known.

4.4.1. Set θ and Get Solutions for T and v

OK, θ is our known along with m and ℓ.

Find solutions for T and v. You have 1 minutes working in groups or individually. Go.

If we set θ, then

$$T = \frac{mg}{\cos\theta} \qquad \text{and} \qquad v = \sin\theta\sqrt{\frac{\ell g}{\cos\theta}} \ , \tag{76}$$

where the latter equation was obtained by dividing equation (74) by equation (75) and rearranging.

Note for θ going to 90°, the tension and velocity would have to go to infinity. It's not possible in the ideal case to swing with the string exactly horizontal. But it's quite easy to get very close to being horizontal.

Let's do some numbers. Say $\ell = 0.40\,\text{m}$, $m = 0.050\,\text{kg}$, and $\theta = 45°$. We find

$$T = \frac{mg}{\cos\theta} = 0.69\,\text{N} \qquad \text{and} \qquad v = \sin\theta\sqrt{\frac{\ell g}{\cos\theta}} = 1.7\,\text{m/s} \ . \tag{77}$$

The results for T and v seem reasonable for what can be obtained from the simple demonstration sling.

4.4.2. Set T and Get Solutions for θ and v

OK, T is our known along with m and ℓ.

Find solutions for θ and v. You have 1 minutes working in groups or individually. Go.

We find θ and v from, respectively,

$$\theta = \cos^{-1}\left(\frac{mg}{T}\right) \qquad \text{and} \qquad v = \sin\theta\sqrt{\frac{\ell T}{m}} \ , \tag{78}$$

Tension actually can be tuned to a value by using, for example, a fish scale inserted into the rope.

Instead of using a fish scale, one could tie a second mass to the other end of the rope and let it hang down a small frictionless pivot point circle. The force of gravity on the second mass must be the tension in the rope.

But in this case, one has m and T as fixed knowns. One can try to set v or θ by controlling the swinging. Then θ or v and ℓ will be determined by physical law.

4.4.3. Set v and Get Solutions for θ and T

Now we set v and for simplicity we take it to be positive. Dividing equation (74) by equation (75) and multiplying by $\sin\theta$ gives

$$\frac{\sin^2\theta}{\cos\theta} = \frac{v^2}{\ell g} = f \ , \tag{79}$$

where we define $f = v^2/(\ell g)$ for simplicity. The last equation is actually a quartic for $\sin\theta$ that can be solved by solving a quadratic for $\sin^2\theta$. Behold:

$$\begin{aligned}
\frac{\sin^2\theta}{\cos\theta} &= f \\
\sin^2\theta &= f\cos\theta \\
\sin^4\theta &= f^2\cos^2\theta \\
\sin^4\theta &= f^2(1-\sin^2\theta) \\
0 &= \sin^4\theta + f^2\sin^2\theta - f^2
\end{aligned} \tag{80}$$

with solution

$$\sin\theta = \sqrt{\frac{-f^2+\sqrt{f^4+4f^2}}{2}} \tag{81}$$

which gives

$$\theta = \sin^{-1}\left(\sqrt{\frac{-f^2+\sqrt{f^4+4f^2}}{2}}\right) \ , \tag{82}$$

where we have taken the positive case solution for the quadratic since the other solution gives an imaginary number and the positive solution for square root since θ is confined to the interval $[0^\circ, 90^\circ]$. Knowing $\sin\theta$, we can find T from

$$T = \frac{mv^2}{\ell\sin^2\theta} \ . \tag{83}$$

Actually equation (81) suggests a little investigation. We know that $\theta \to 90^\circ$ or $\sin\theta \to 1$ for $v \to \infty$, but that isn't immediately obvious. We can make is so using a Taylor's expansion about small $1/f$:

$$\begin{aligned}
\sin\theta &= \sqrt{\frac{-f^2 + \sqrt{f^4 + 4f^2}}{2}} \\
&= f\sqrt{\frac{-1 + \sqrt{1 + 4/f^2}}{2}} \\
&\approx f\sqrt{\frac{-1 + 1 + (1/2)(4/f^2) + (1/2)(1/2)(-1/2)(16/f^4)}{2}} \\
&\approx f\sqrt{\frac{1}{f^2} - \frac{1}{f^4}} \\
&\approx \sqrt{1 - \frac{1}{f^2}}\ , \qquad (84)
\end{aligned}$$

where we have expanded to 4th order in small $1/f$. We can now see clearly that as $v \to \infty$ (which implies that $f \to \infty$) $\theta \to 90^\circ$.

5. UNBANKED AND BANKED CURVES

On unbanked and banked curves, the motion is generally not uniform circular motion. But the quantitative aspects we look at only need the concepts of uniform circular motion.

We will largely be speaking of cars on curves to be concrete in our descriptions, but the analysis is actually more general than cars. It can apply to planes, bobsleds, and other vehicles making curves.

5.1. A Car on an Unbanked Curve

In this subsection, we just qualitatively consider a car on an unbanked curve.

Almost all road curves are banked—and for good reason.

But there are some unbanked curves around and they are rather treacherous. Of course, ordinary road corners are unbanked, but note that you take them slow.

Now on a curve, a car is on a circular path at least at each point if the radius of curvature varies which is usually, but not always, the case.

What provides the centripetal force on the car for an unbanked curve?

Is the normal force? Gravity?

No they are both vertical forces. They have no component in the horizontal direction where we need a centripetal force.

What is it?

Friction.

Which friction is it? Static or kinetic?

Well maybe a bit of both and another friction—that of just sinking into the road. In our previous, discussion of friction implicitly considered just ideal rigid surfaces. But real roads do compress a bit under car weight.

In any case, friction is a reactive force that tries to prevent motion between surfaces and only turns on to try to prevent that. It's like the normal force which only turns on if you press inward on a surface.

In the car-on-a-curve, you angle the car wheels and ideally static friction will be push exactly **OPPOSITE** to the car's current direction.

This static friction can be resolved into two orthogonal components. One component of frictions acts opposite the wheel direction and the other acts perpendicular to that direction and toward a center of curvature. The direction to the center of curvature is actually defined the component perpendicular to the wheel.

In reality, there might be a bit of sliding in turns.

If the size of the friction force is enough, then the car is slowed a bit by the first component of friction and pushed around the curve by the other maybe with a little small scale sliding.

The higher the car speed, the greater the friction force must be to change the car's velocity or—to anticipate the concepts of the lecture *MOMENTUM*—to change the car's momentum (which is mass times velocity).

If the friction force is insufficient, then the car will noticeably slide.

Another factor is that the car is hard to control when turning to fast. Yours truly once had a very bad control problem exiting an interstate at too high a speed—if you notice your exit late, don't try for it—there's always another one.

Whenever one is actually driving, one makes all kinds of minor adjustments to wheel angle and car speed to effect a smooth turn. Lots of experience tells us how to do this without much thought—or any calculations.

5.2. Banked Curves

Say that only gravity and the normal force are present to provide a centripetal force on curved path which is banked. The path is rigid.

Let's analyze the forces and see what relationships we get.

Remember the class mantra:

"Newton's 2nd law is always true and it's always true component by component."

In this case, I chose the x direction to be the horizontal direction with the x axis toward the curve center. The y is vertical with up positive.

We assume a vehicle is making the turn—this is the key condition.

In the x direction, we have

$$F_{\rm cen} = m\frac{v^2}{r} = F_{\rm nor}\sin\theta \ , \tag{85}$$

where $F_{\rm nor}$ is the normal force (with implicit direction normal and outward from the surface) and θ is the angle of the normal force from the vertical and also, by geometry, the angle of the incline or banking from the horizontal. Here the x component of the normal force is providing the centripetal force.

In the y direction, we have no motion. The path is rigid, and so vehicle can't sink into the path and in any perturbation to lift off the path, gravity pulls the vehicle back. So we must have balanced forces. So we have

$$F_{\rm nor}\cos\theta = mg \ , \tag{86}$$

where $F_{\rm nor}\cos\theta$ upward component of the normal force and mg is the force of gravity.

Physics has given us two equations. From these we can solve for two unknowns. The most interesting variables to take to be unknowns are the normal force $F_{\rm nor}$ and the angle of the incline θ (which is also gives the direction of the normal force, of course.)

The knowns we take to be r, v, and g.

First, solve for the normal force $F_{\rm nor}$.

You have 1 minute working individually or in groups. Go.

The normal force magnitude is obtained by adding the squares of equations (85) and (86).

We get

$$F_{\rm nor} = m\sqrt{\left(\frac{v^2}{r}\right)^2 + g^2} \ . \tag{87}$$

Now solve for the incline angle θ.

You have 1 minute working individually or in groups. Go.

To get the incline angle, we divide equation (85) by equation (86) and obtain

$$\tan\theta = \frac{v^2}{rg} \tag{88}$$

and

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right) \ . \tag{89}$$

Note that the argument of the inverse tangent function is dimensionless as it has to be since the tangent function value is dimensionless. To be explicit, the dimension of the argument is

$$\left[\frac{v^2}{rg}\right] = \frac{{\rm L}^2/{\rm T}^2}{{\rm L}\times{\rm L}/{\rm T}^2} = 1 \tag{90}$$

which means dimensionless.

Equation (89) is the banking angle formula. For a given speed and radius, the formula gives the angle one should bank a road at in order for a car to go around the curve with only the normal force and gravity combining to give the acceleration.

Note the formula is independent of mass. This is because the normal force is linearly dependent on mass, and so mass cancels out.

A car moving at the right speed would go around the curve no matter what it's mass.

The normal force provides all the force needed for the motion. A car at the right speed would make a properly banked curve ideally even a road were ideal frictionless ice.

Note also that if v^2/r goes to infinity, θ goes to 90° and the car plane would perpendicular to the ground.

The banking angle formula equation (89) always seems a little mysterious. How does the normal force know to be just strong enough? But the answer is that you have assumed that you are making the curve. Therefore the normal force must be strong enough by the given condition. It's actually a constraint force. We can evaluate it from its assumed effect on the system and not from an intrinsic formula.

Note that we didn't actually have to assume the contact force at angle θ was the normal force. We just had to assume there was such a contact force. But if we adjust the banking of the road such that the road surface is perpendicular to the direction needed for the contact force, then the contact force must be the normal force. The friction force can exert no forces normal to the road, and so cannot supply any component of the contact force.

If you did a banking analysis with the friction force turned on and solved for normal force and friction force, then you would find that the friction force goes to zero for the banking angle given by the banking angle formula. This analysis is done in Appendix A.

Why do civil engineers—who are very polite engineers—bank curves using the banking angle formula. Somewhat obviously so that the cars don't need to rely on friction to make the curve. The normal force is very strong. A car would have to plow into the road to cause it to fail. By fail we mean not be strong enough to give the acceleration around the curve. Failure of the normal force doesn't much happen, except maybe on mud roads. The friction force can fail even on dry roads. Water and ice and snow reduce friction substantially.

Another obvious reason is that turning is much easier on a properly banked road. If you go at the specified speed, there is little need to turn the wheels at all and there is no tendency to loss of control.

Yet another reason it is easier on the road structure, the car structure, and the car contents including the passengers. The above analysis actually applies to every bit of the car and contents. If you go at the specified speed, anticompressional forces do all the acceleration. Antishear forces (of which friction is only one) in the car and the road are never called on. Antishear forces are any force that resists layers sliding over one another including those bonded together inside solids. Calling on such forces repeatedly leads to rapid wear. It's also more comfortable for passengers to be just compressed into their seats making a curve and have no tendency to slide sideways.

Of course, if you drive at the wrong speed around a curve, antishear forces do get called on with all their discomfort, tendency to wear, and chance of failure. So you should always drive at the specified speed.

Too fast and you tend to slide outward. Too slow and you tend to slide inward. It's better to err on the too slow side, of course, because failure is probably less catastrophic. In very icy conditions what are you doing on the road anyway?

Now civil engineers can't use just one banking angle. They must go from straight, unbanked road to maximally banked road continuously. And they can't ask drivers to change speed more than once to the curve speed. So they must vary the radius of curvature to allow a continuous change from unbanked (at infinite radius of curvature) to maximally banked (at smallest radius of curvature) to unbanked again (at infinite radius of curvature).

Now all of the analysis above is a bit idealized. There are probably many complicating effects that real engineers must include in their designs. One thing—which is probably minor in this context—is that cars are not really point masses. Forces are needed not just to make the car center of mass go round curves, but also rotate the car.

5.3. Example: Bobsledding

Making a curve while bobsledding is a special case.

Let's assume the curve is on level ground (except for the banking itself) to avoid the complications about curving and going downhill at the same time.

The speed isn't fixed and one is trying to go as fast as possible—or so one assumes—one knows nothing about bobsledding—maybe the slowest horse wins.

Also the surface has low friction. Ideally none at all.

With no friction and unfixed speed how is the banking angle chosen?

Well it's chosen by allowing a continuously varying banking angle. The angle grows steeper moving outward from the center of curvature.

A bobsled simply slides up the wall until the normal force is enough to accelerate the bobsled around the curve.

In this context, the wall providing the centripetal force seems perfectly natural—we're all been big bobsled fans right.

Recall the banking angle formula

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right) . \tag{91}$$

At the 1994 Winter Olympic in Lillehammer, Norway, the bobsled track at a curve with radius of curvature 24.0 m. Let's say that your bobsled went around the curve at 34.0 m/s. The mass of the bobsled with crew is the maximum allowed: 630 kg.

Taking radially inward as positive, what is your centripetal acceleration a_{cen}, centripetal force F_{cen}, and angle from the horizontal that the bobsled runs at?

You have 1 minute. Go.

Behold:

$$a_{\rm cen} = \frac{v^2}{r} = 48.2\,{\rm m/s^2} \ , \tag{92}$$

$$F_{\rm cen} = m\frac{v^2}{r} = 30300\,{\rm N} \ , \tag{93}$$

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right) = 78.5^\circ \ . \tag{94}$$

6. NON-UNIFORM CIRCULAR MOTION

Non-uniform circular motion is, of course, a vaster field than uniform circular motion.

So we'll just look at one instructive example: the simple pendulum.

6.1. The Simple Pendulum

Say we have pendulum with a point bob of mass m and an arm length of radius r. The arm is massless, infinitely thin, and reaches from the pivot point to the point bob. The arm can be rigid or not, but in either case we assume it stays straight for oscillations. The pivot point is frictionless and there is no air drag. The only forces on the bob are the tension force along the arm and gravity.

Note the arm can only exert a tension force on the bob even if it is rigid. The pivot exerts no angular force on the rigid arm, and so the arm can't transmit an angular force to the bob. Since the arm is massless, it is accelerated in angular motion by zero force from the bob, and so by the 3rd law exerts zero force on the bob. The arm is just constrained to follow the bob's angular motion. One the other hand, the bob is constrained by the arm's tension force to only move on a circular path of radius r.

I will demonstrate the simple pendulum with my sling—you are getting sleepy—but that was happening before the demonstration.

We measure angle θ from the vertical with positive in the counterclockwise direction.

Consider the bob at a general angle θ and analyze the forces using the 2nd law. Remember the class mantra:

"Newton's 2nd law is always true and it's always true component by component."

For this system, we will use a different Cartesian coordinate for every instant in time. At any instant the x axis is tangent to the pendulum circular path with the positive x direction in the positive θ direction. The y axis is aligned with the arm with positive direction in the direction of the pivot.

We assume there is no acceleration of the bob in y direction at any point.

Thus from the 2nd law, we find the tension in the arm to be?

You have 30 seconds. Go.

Behold:

$$T = mg\cos\theta \ , \tag{95}$$

As long as the tension does not go negative, an ideal rope will stay straight. So for $|\theta| < 90°$, the arm will stay straight even if it is an ideal rope instead of a rigid arm. Of course, a non-ideal rope might buckle at smaller angles depending on non-ideal complications. A rigid rod can exert a negative tension (i.e., an anticompression force).

Now what equation of motion does the 2nd law give for instantaneous x direction?

You have 30 seconds. Go.

Behold:

$$m\frac{d^2x}{dt^2} = -mg\sin\theta \ . \tag{96}$$

Now say the path length s along the circle is measured positive counterclockwise from the vertical position for the arm. At the location of the bob at any instant s path and the x axis are aligned. Therefore $ds = dx$. It follows that

$$\frac{d^2s}{dt^2} = \frac{d^2x}{dt^2} \ . \tag{97}$$

Our equation of motion can now be written

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \ . \tag{98}$$

Now

$$s = r\theta \tag{99}$$

for θ measured in radians. Since r is a constant, we find

$$\frac{d^2\theta}{dt^2} = -\frac{g}{r}\sin\theta \ , \tag{100}$$

where we have canceled out m. Gravity is the only force that acts to accelerate the bob in the path length direction, and so it's not surprising that mass cancels out of the equation of motion.

This innocent, little differential equation (DE) equation (100) has no exact solution. It's in fact a non-linear DE. That transcendental sine function destroys the simplicity.

But we can restore simplicity and make progress using the small angle approximation for $\sin\theta$.

How many people have seen Taylor's series? Show of hands.

Some. The others will see it soon. Just accept Taylor's series for now. It's a way of writing a function (which is differentiable to all orders) as a power series which in general is infinite. The series is relative to some point and the power series only converges (i.e., only correctly represents the function) for some region of convergence around that point. I'll leave convergence and regions of convergence to your calculus class. Writing the function as a Taylor's series for a point is called expanding the function in the Taylor's series about a point. The closer you are to the expansion point, the more accurate truncated versions of the series are provided you are in the region of convergence.

If we Taylor expand $\sin\theta$ around θ equals zero, we have to 3rd order in small θ

$$\sin\theta \approx \theta - \frac{1}{6}\theta^3 \ , \tag{101}$$

where θ is in radians, of course. We see in the limit that θ goes to zero, that the linear term approximates $\sin\theta$ exactly. In fact, the linear approximation

$$\sin\theta \approx \theta \tag{102}$$

is 2nd order accurate in small θ. Table 1 illustrates the accuracy of linear or small angle approximation.

Table 1. Accuracy of the Small Angle Approximation for $\sin\theta$

θ (degrees)	θ	$\sin\theta$	Relative Error $\frac{\theta - \sin\theta}{\sin\theta}$
0	0	0	0
10	0.17453	0.17365	0.0050951
20	0.34907	0.34202	0.0206003
30	0.52360	0.50000	0.0471976
40	0.69813	0.64279	0.0861001
50	0.87266	0.76604	0.1391828
60	1.04720	0.86603	0.2091996
70	1.22173	0.93969	0.3001384
80	1.39626	0.98481	0.4178030
90	1.57080	1.00000	0.5707963

Note. — The calculations were done with a double precision fortran-95 program.

One can see from the relative error column in Table 1 that the small angle approximation is quite good for angles less than or equal to 30° and is not wildly wrong even at 90°.

Making the small angle approximation $\sin\theta$ replaced by θ, the exact simple pendulum DE equation (100) changes into the approximate simple pendulum DE

$$\frac{d^2\theta}{dt^2} = -\frac{g}{r}\theta \ . \tag{103}$$

The approximate simple pendulum DE is, in fact, the simple harmonic oscillator DE all over again as Yogi Berra would have said. See § 2.

So what must the formula for the angular frequency be in this case?

You have 10 seconds. Write it down.

Well

$$\omega_{\rm osc} = \sqrt{\frac{g}{r}} \ , \tag{104}$$

where subscript "osc" stands for oscillation.

To summarize, the period-related results, the angular frequency $\omega_{\rm osc}$, frequency f, period of the oscillation p are, respectively,

$$\omega_{\rm os} = \sqrt{\frac{g}{r}} \ , \qquad f = \frac{1}{P} = \frac{\omega_{\rm os}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{r}{g}} \ , \qquad P = \frac{1}{f} = \frac{2\pi}{\omega_{\rm os}} = 2\pi\sqrt{\frac{r}{g}} \ , \tag{105}$$

where we have used results from § 2.1.

Note $\omega_{\rm osc}$ is angular frequency, not angular velocity ω. We use the subscript "osc" to indicate the different meaning. The conventional use of ω for both angular frequency and angular velocity is sometimes confusing. Sometimes they are same as in uniform circular motion (see §§ 4 and 2.1). But often they are not. Context must decide. In the present case, the angular frequency of the oscillation $\omega_{\rm osc}$ is not angular velocity and is a constant which the angular velocity is not.

From your knowledge of the SHO solution, you can write down the simple pendulum solution with just changes in variable symbols.

You have 30 seconds. Go.

Just writing down the solution angular position and its 1st and 2nd derivatives, we find

$$\theta = A\cos(\omega_{\rm osc}t) + B\sin(\omega_{\rm osc}t) \ , \tag{106}$$

$$\omega = -A\omega_{\rm osc}\sin(\omega_{\rm osc}t) + B\omega_{\rm osc}\cos(\omega_{\rm osc}t) \ , \tag{107}$$

$$\alpha = -A\omega_{\rm osc}^2\cos(\omega_{\rm osc}t) - B\omega_{\rm osc}^2\sin(\omega_{\rm osc}t) \ , \tag{108}$$

where A and B are constants of integration determined by the initial conditions.

Remember the simple pendulum relies on making the small angle approximation for sine. Thus, our solution is only approximate. But it is actually quite accurate for oscillations of small amplitude—say amplitude less than 30° or so. And it approaches exactness as the amplitude goes to zero.

Note that the period formula

$$P = 2\pi\sqrt{\frac{r}{g}} \tag{109}$$

does not depend on mass or amplitude, but only on arm length r and g.

The period is linear in the square root of r and inverse linear in the square root of g.

The simple pendulum can thus be used for very accurate measurements of g.

The independence from amplitude can be demonstrate with my sling—or if available the classroom bowling ball—you may have wondered why the bowling ball was here—now you know.

Since there is no dependence on amplitude to good approximation, pendulums can make good oscillator systems for clocks. In a real clock, the motion of a pendulum will damp out without repeated extra forces exerted to cancel the resistive forces. Because of the resistive forces and the extra forces, it is hard to keep the amplitude exactly constant. But since the period does not depend on the period in the limit that the simple pendulum is exactly correct, we know that small amplitude variation has little effect on the regularity of the pendulum. There are means to make real pendulum clocks keep time very accurately.

We can anticipate the lecture *ENERGY* for a moment. It turns out that the mechanical energy of the simple pendulum is independent to good approximation of the period for fixed r and g and depends only on the amplitude. Or inverting, period for fixed r and g is independent to good approximation of mechanical energy. In real pendulums, resistive forces cause loss of mechanical energy to waste heat. Energy injections are needed to keep the periodic motion going. So small variations in mechanical energy will occur in real cases. But since period is independent of mechanical energy to good approximation, the real pendulums can be very regular. To repeat ourselves, there are means to make real pendulum clocks keep time very accurately.

Actually, Galileo (1564–1642) noted that pendulum periods were nearly independent of amplitude. According to the story, he first noticed this watching watching the oscillations of the bronze chandelier of the cathedral of Pisa using his pulse to measure the periods. Galileo did have the idea of using pendulums in clocks, but never built one. Later in 17th century Christian Huygens (1629–1695) invented the pendulum clock. His design was a more advanced than our discussion, of course.

Simple pendulums are idealizations even without the small angle approximation since they have a massless, infinitely thin arm and a point bob.

What if the arm and bob of a simple pendulum are replaced by a finite rigid object that is allowed to freely oscillate about a point. In this case, the object is called the physical pendulum.

The analysis of the physical pendulum with the small angle approximation made again is almost the same as for the simple pendulum. However, one has to make use of the concepts of rotational dynamics. We analyze the physical pendulum in the lecture *ROTATIONAL DYNAMICS*.

But may you wonder what is the cause of the distinction between the simple pendulum and physical pendulum.

Shouldn't the external forces determine the center-of-mass motion for the physical pendulum. Well yes. The external forces are gravitational force and the force at the pivot point. Gravity is easy to determine: it's mg, where m the pendulum mass. But the force exerted at the pivot point by whatever is holding the pendulum there is not easily directly determinable. It is for the simple pendulum, of course, because it is infinitely thin and aligns with the arm. But for physical pendulum, the bob and arm are one rigid object of finite size and mass and we can't determine the pivot force's size and direction easily. The pivot force is a constraint force: we know its effect directly rather than what it is. To find the pivot force you need to go to rotational dynamics. As a matter, of fact, in rotational dynamics, you do solve for the motion without knowing what that pivot force is.

The physical pendulum is easy to treat after one has the formalism of rotational dynamics. One needs the torque and rotational inertia quantities in particular. The physical pendulum is an optional example in the lecture *ROTATIONAL DYNAMICS*.

In the rotational dynamics treatment, the bob and arm become one system and the only external force one needs is to deal with explicitly is gravity.

But now you ask why isn't our simple pendulum formula just right for a finite bob, but still with a massless, infinitely thin arm? Isn't there just the tension force along the ideal arm and gravity and don't they determine the center-of-mass motion of the bob. Well that would be so if the bob somehow didn't rotate around its center of mass. This can be approximately arranged actually by having gap in the bob so that the arm can be attached to near the center of mass of the bob with another free pivot point. But this is a tricky setup. More usually the bob's orientation with respect to the arm is fixed. This means that bob's rotation about it's center of mass is the same as the center of mass's rotation about the pivot. But a varying rotation of the bob about it's center of mass depends extra forces that have to be supplied by the arm. If the arm is an ideal rope, it would have to bend a little to provide that extra force. A ideal rigid arm, can provide forces to cause the bob to rotate without bending. If the bob is very small in size scale compared to the arm, these extra forces are small. In the limit of a point bob, we have the simple pendulum situation.

7. NON-INERTIAL FRAMES AND INERTIAL FORCES

Newton's 3 laws of motion can be generalized to deal with non-inertial frames by introducing the concept of inertial forces.

The use of inertial forces in non-inertial frames is a standard procedure and is often immensely useful. It gives a straightforward way to deal with motion in non-inertial frames making use of laws (i.e., Newton's 3 laws and force laws) and techniques that we know well from inertial frame cases. It the economical approach. In particular, it gives a straightforward way to deal with frames that we approximate as inertial for most purposes, but actually are non-inertial. The Earth is a good example. It's rotating on axis and revolving around the Sun. So strictly speaking it is non-inertial. For most purposes, approximating the Earth as inertial is adequate. But not for all like long-range gunnery and weather.

As we discussed in the lecture *NEWTONIAN DYNAMICS I*, virtually all frames in the universe attached to bodies that have mass are non-inertial. In theory, only frames participating in the mean expansion of the universe and those frames not accelerating with respect to the aforesaid frames are exactly inertial. We can find determine our own local inertial frame as also discussed in the lecture *NEWTONIAN DYNAMICS I*. This just illustrates that real inertial frames are not a myth—and not like the Cheshire Cat vanishing leaving only a theoretical smile.

In dealing with non-inertial frames and inertial forces, there is the problem of vocalizing the words "non-inertial" and "inertial" and the problem of not saying one when means the other and the confusion that that inertial forces occur in non-inertial frames. But tradition dictates the klutzy jargon and we're are stuck with it.

We will not deal with non-inertial frames in total generality. That would be arduous, obscure, and futile.

First, we will discuss non-inertial frames with a single constant acceleration.

Then, we discuss rotating frames, but without hard part.

So on with the show.

7.1. Non-Inertial Frames with a Single Constant Acceleration

Say you are in an non-inertial frame which has a single constant acceleration $\vec{a}_{\rm in}$ relative to inertial frames.

Remember, if $\vec{a}_{\rm in}$ is the acceleration relative to one inertial frame, it is the acceleration relative to all inertial frames since they are **NOT** accelerating relative to each other.

Our frame may actually be attached to solid body (e.g., the proverbial elevator or rocket) or it may just be a defined frame of the mind.

Consider a general system with acceleration $\vec{a}$ relative to inertial frames.

Relative to the non-inertial frame, the system has acceleration (i.e., center-of-mass acceleration)

$$\vec{a}\,' = \vec{a} - \vec{a}_{\rm in} \ , \tag{110}$$

and thus

$$\vec{a} = \vec{a}\,' + \vec{a}_{\rm in} \ . \tag{111}$$

We apply the 2nd law to the system:

$$\vec{F}_{\rm net} = m\vec{a} \ . \tag{112}$$

Nothing forbids us from rewriting this formula thusly:

$$\begin{aligned} \vec{F}_{\rm net} &= m(\vec{a}\,' + \vec{a}_{\rm in}) \\ \vec{F}_{\rm net} - m\vec{a}_{\rm in} &= m\vec{a}\,' \\ \vec{F}_{\rm net} + \vec{F}_{\rm in} &= m\vec{a}\,' \\ \vec{F}'_{\rm net} &= m\vec{a}\,' \ , \end{aligned} \tag{113}$$

where we have defined an inertial force by

$$\vec{F}_{\rm in} = -m\vec{a}_{\rm in} \tag{114}$$

and the net force in the non-inertial frame by

$$\vec{F}'_{\rm net} = \vec{F}_{\rm net} + \vec{F}_{\rm in} \ . \tag{115}$$

The inertial force is not a real force as we define things. It is a quantity that acts like a force in a non-inertial frame. Inertial forces are sometimes called fictitious forces—but that is only marginally easier to vocalize, and so seems to yours truly as a pointless addition to the jargon.

In the derivation above, we arrived at the equation

$$\vec{F}'_{\rm net} = m\vec{a}\,' \ . \tag{116}$$

This equation is Newton's 2nd law for the non-inertial frames. It is a generalization of Newton's 2nd law. Newton's 2nd law for the non-inertial frames holds for general non-inertial frames too with the proper definitions of non-inertial forces.

What kind of force is the inertial force?

Well let's look at the inertial force law again.

$$\vec{F}_{\rm in} = -m\vec{a}_{\rm in} \tag{117}$$

Now our system can be anything from a large complex system to a classical point particle.

So equation (117) applies to the classical point particles that mythically make up our general system.

So on every classical point particle, there is a force per unit mass that is a constant $-\vec{a}_{\rm in}$.

The inertial force is thus a field force with a constant field of $-\vec{a}_{\rm in}$. It requires no body on body contact. It reaches out and acts directly on every particle of system. The inertial force itself is linearly dependent of mass.

All inertial forces, in fact, are field forces that act on every particle of a system. They are also all linearly dependent on mass too. We will not go into a general proof of this.

The inertial force we have found is, in fact, like gravity near the Earth's surface:

$$\vec{F}_{\rm grav} = mg(-\hat{y}) \ . \tag{118}$$

So any object or particle in the non-inertial frame is affected by the inertial force as if it there were a Earth-surface-like gravitational field of $-\vec{a}_{\rm in}$.

Exactly like.

No experiment can tell the difference.

This gravity-like property of inertial forces was a starting point for Einstein's general relativity—but we don't have quite enough time to go down that road—it's a bad road with tensors and differential geometry.

But how can an inertial force which from view of an inertial frame is nothing at all act like a force?

Well looking some special cases will bring some elucidation.

7.1.1. Free Fall in the Non-Inertial Frame

If the non-inertial force is the only external force on the system (and thus the net force on the system) in the non-inertial frame, then the 2nd law for non-inertial frames specializes to

$$\vec{F}'_{\rm net} = \vec{F}_{\rm in} = -m\vec{a}_{\rm in} = m\vec{a}\,' \tag{119}$$

which leads to

$$\vec{a}\,' = -\vec{a}_{\rm in} \ . \tag{120}$$

So in the non-inertial frame, the system acceleration (i.e., the center-of-mass acceleration) is just $-\vec{a}_{\rm in}$.

But what of the subsystems of the system right down to the scale of the mythical classical particles that make up the system.

Well if no other forces act on them except the inertial force itself, they all have acceleration $-\vec{a}_{\rm in}$.

In general, there will be internal forces acting on the subsystems that cause accelerations added vectorially onto the acceleration $-\vec{a}_{\rm in}$.

The situation is just like free fall under gravity near the Earth's surface. In fact, we generalize the term free fall to the case when only an inertial is the net force on the system.

In the gravity free fall, no internal forces act in a direct sense to oppose the gravity force which pulls on the system particle by particle.

The same is true in inertial-force free fall.

In both cases, the system is weightless.

What is the system acceleration relative to inertial frames?

Well

$$\vec{a} = \vec{a}\,' + \vec{a}_{\rm in} = -\vec{a}_{\rm in} + \vec{a}_{\rm in} = 0 \ . \tag{121}$$

So in inertial frames, the object isn't accelerated.

No real forces act on it, and so it shouldn't be.

7.1.2. Equilibrium in the Non-Inertial Frame

If a system is equilibrium in the non-inertial frame $\vec{F}'_{\rm net} = 0$ and $\vec{a}\,' = 0$.

This means that

$$\vec{F}'_{\rm net} = \vec{F}_{\rm net} - m\vec{a}_{\rm in} = 0 \tag{122}$$

or

$$\vec{F}_{\rm net} = m\vec{a}_{\rm in} \ , \tag{123}$$

and so

$$\vec{a} = \vec{a}_{\text{in}} \ . \tag{124}$$

There must be a net real force to cause acceleration of the system equal to the non-inertial frame's acceleration.

What can cause that net force?

Well any real force.

But for many systems in non-inertial frames, it must be normal, tension, and/or pressure forces.

And this is not just for systems as a whole, but for every subsystem down to the scale of the mythical classical point particles.

The subsystems are often accelerated by internal forces. In some cases, there are real field forces that cause the acceleration in whole or in part.

These internal forces on the subsystems are exactly like those needed to support sytems internally against gravity.

So typically in solid object, the internal forces are stress forces forces caused by strains. Stress is force per unit area. Strain is a deformation of rigid body which often results in stress.

In a human body, the stresses and strains in resisting gravity or an inertial force are detected as heaviness. You relieve the stress and strain a bit by lying down.

From an inertial frame perspective, the internal forces are causing the subsystems down to the scale of classical point particles to accelerate with the non-inertial frame.

Of course, one can have cases of non-free-fall acceleration in non-inertial frames. The inertial force is just one of the forces added to give the net force in the inertial frame.

So to accelerate with our non-inertial frame, the objects and you need all the external and internal forces that would be needed to support against gravity.

In such an non-inertial frame, you would feel just like you were in a gravitational field of $-\vec{a}_{\rm in}$.

In a car accelerating forward, you experience an inertial force to the rear that that is just like a gravity force to the rear. In an elevator accelerating upward, you experience an inertial force that is just like an additional gravity force. In an elevator accelerating downward, the inertial force would counteract gravity to some extend and you'd feel lighter.

But wait you say.

What if you just let go and didn't try to accelerate with the frame? All those external and internal forces wouldn't be needed to keep you accelerating with the frame and you'd just be accelerated by $-\vec{a}_{\rm in}$ as we found in § 7.1.1.

You'd feel weightless.

But this also true of gravity.

If you just let gravity alone act on you, you are in free fall and feel weightless.

Gravity is pulling on you particle by particle, and you need no external or internal forces to counteract it.

The feeling of weightlessness is really the feeling of not trying to counteract gravity and/or an inertial forces.

7.1.3. Example: Life in an Elevator

Following in the footsteps of Einstein, let's consider life in an elevator—or alligator as I would have called it when I was a tyke.

The elevator has acceleration $\vec{a}_{\rm in} = a_{\rm in}\hat{y}$.

Then the 2nd law for an object in the elevator frame is

$$\vec{F}'_{\rm net} = \vec{F}_{\rm other} - mg\hat{y} - ma_{\rm in}\hat{y} = m\vec{a}\,' \ , \tag{125}$$

where $\vec{F}_{\rm other}$ is all forces aside from gravity and the inertial force.

For the elevator frame, one can define an effective gravity field y component by

$$-g_{\rm eff} = -g - a_{\rm in} \ . \tag{126}$$

The effective weight of any object of mass m in the elevator is then

$$mg_{\rm eff} = m(g + a_{\rm in}) \ . \tag{127}$$

This is just what a spring scale would measure. The mg part is the upward force the scale would have to supply to counter gravity and the $ma_{\rm in}$ part is the upward force the scale would have to supply to accelerate the mass to acceleration of the elevator frame.

Of course, $a_{\rm in}$ can be greater or smaller than zero.

In the former case, you would feel heavier than normal and in the latter, lighter than normal.

Note the direction of acceleration is not that of the velocity. The elevator can be going up or down and in either direction the elevator acceleration can be upward or downward.

What if the elevator were in free fall?

Then $a_{\rm in} = -g$, and objects in the elevator non-inertial frame are weightless.

Of course, all that weightlessness ends when the big normal force kicks in at the bottom of the shaft.

Just for some number fun say $a_{\rm el} = \pm 2\,{\rm m/s^2}$ which is a respectable elevator acceleration. You have an object of mass 4 kg. It's real weight is

$$mg \approx 4 \times 10 = 40\,{\rm N}\ . \tag{128}$$

But effectively in the elevator frame, it's weight is

$$mg_{\rm eff} = mg\left(1 + \frac{a_{\rm el}}{g}\right) \approx 40 \times \left(1 \pm \frac{2}{10}\right) = \begin{cases} 48\,{\rm N} & \text{for the upper case;} \\ 32\,{\rm N} & \text{for the lower case.} \end{cases} \tag{129}$$

7.2. Rotating Frames

Rotating frames are trickier non-inertial frames than frames non-inertial frames accelerating with a single constant acceleration.

Every point in a rotating frame is accelerating with its own non-constant acceleration. So there is, in fact, not a single non-inertial frame with a constant acceleration relative to an inertial frame, but a continuum of non-inertial frames each with a non-constant acceleration relative to an inertial frame. The situation is tricky.

To be concrete consider a simple frame rotating with constant angular velocity ω relative to an inertial frame around an axis which we label the z axis. Note ω is the frame angular velocity, not the angular velocity of anything in the frame necessarily.

The acceleration for any point relative to an inertial is

$$\vec a = -\omega^2 r \hat r \tag{130}$$

which is just the centripetal acceleration. Note r is the cylindrical radial coordinate: i.e., the radius measured perpendicularly from the z axis.

We see that the acceleration depends on r and on the θ coordinate through $\hat{r}$ which recall is given by

$$\hat{r} = \cos(\omega t) + \sin(\omega t) \tag{131}$$

for the initial position being aligned with the standard x axis.

So every point in frame has a varying acceleration in an inertial frame.

In the rotating frame, of course, fixed points have no acceleration. But one has to invent inertial forces.

There are two inertial forces, in fact, that are needed: the centrifugal force and the Coriolis force.

The centrifugal force is easy to understand and to obtain the formula for.

Say there is a system at rest in the rotating frame.

In the inertial frame, there is a centripetal force acting on the system to make it rotate.

That force is still there in the rotating frame, of course, since it is a real force caused by some interaction the system with another system or a field. That force is

$$\vec{F}_{\rm cp} = -m\omega^2 r\hat{r} \tag{132}$$

But in the rotating frame, the system is motionless. To have Newton's 2nd law apply in the rotating frame, there must be an inertial force exactly canceling the centripetal force. This force is the **CENTRIFUGAL FORCE**

$$\vec{F}_{\rm cf} = m\omega^2 r\hat{r} \ . \tag{133}$$

It's a radially outward point force. It's present whenever you take the rotating frame as the frame of reference. It's the same if the system in the rotating frame is moving or not and the same whether or not any other forces act on the system including those that make up an centripetal force.

In fact, if you turned off centripetal force for a system initially at rest in the rotating frame, the system would accelerate and start to move.

Note that the centrifugal force is a field force and is linearly dependent on mass as all inertial forces are. It acts on all particles directly of a system directly.

The centrifugal force is the force that tries to throw you off carnival centrifuges. The wall normal force of the centrifuge keeps you from moving in the rotating frame. But that wall force is just a contact force on your body. Internal forces have to cancel, the centrifugal force acting directly on all the particles that make you up.

The motion of a system in the rotating frame is complicated by the other inertial force in the rotating frame, the Coriolis force. The Coriolis is capitalized since it is derived from Gaspard-Gustave de Coriolis (1792–1843) who derived its formula in 1835.

The Coriolis force is depends on the velocity relative to the rotating frame and is zero for zero velocity in the rotating frame. This is why the Coriolis force did not come into the analysis leading to the centrifugal force. But once a system starts moving the Coriolis force turns on

The derivation of the Coriolis force formula is well beyond our scope (e.g., Symon 1971, p. 273–279). The derivation is rather tricky and not terribly obvious for yours truly. For the record, the Coriolis force formula is

$$\vec{F}_{\rm co} = -2m\omega(\hat{z} \times \vec{v}\,') \ , \tag{134}$$

where $\hat{z}$ is the unit vector in the positive z direction, $\vec{v}\,'$ is the velocity relative to the rotating frame, and the times symbol $\times$ stands for cross product. One point to note is that the Coriolis force, unlike the centrifugal force, is independent of radius.

We will take up the cross product in detail in the lecture *ROTATIONAL DYNAMICS*. Here we will say the cross product is another way to multiply vectors and the result is a vector (technically a pseudovector, but that's a fine point beyond our scope.)

The direction of the a cross product is perpendicular to the the two factor vectors. It's sense is given by a right-hand rule: put the factor vector tails together and sweep the fingers of your right hand from the first factor vector to the second factor vector and your thumb points roughly in the direction of the cross product vector.

For the Coriolis force, the cross product means that the force is linearly proportional to the velocity $\vec{v}\,'$ magnitude and points perpendicular to velocity $\vec{v}\,'$. So the Coriolis force always tries to drive a system off course.

The Coriolis force formula is tricky and its effects are quantitatively tricky. But qualitatively, the Coriolis force is well known.

Play catch on a rotating merry-go-round and the Coriolis force causes the ball to veer of a straight path in the horizontal direction. The centrifugal force is also acting in this situation. From the ground frame, the only forces acting on the thrown ball are gravity and air drag, of course.

The Coriolis force due to the Earth's rotating frame has a very big effect on weather. Consider the northern hemisphere. Winds going north tend to veer to the east due the Coriolis force acting on them. From an inertial frame, perspective all that is happening is a northward-moving air mass is moving to latitudes with lower eastward Earth velocity than where they started from, and some of that velocity tends not to be canceled, and so the air mass veers east. Similarly, winds going south tend veer west due to the Coriolis force.

Since winds tend to converge at lower pressure zones, the Coriolis force causing counterclockwise inward spiral motion of the winds on the zones as seen from above. These inward swirls are called cyclones. The most extreme cyclones are hurricanes or typhoons. Since winds tend to diverge from high pressure zones, the Coriolis force tends to cause them to swirl clockwise as seen from above. These outward swirls are called anticyclones.

Yes, the Coriolis force will cause water tanks to drain through a small opening with counterclockwise vortex as seen from above. But this effect is swamped by all kinds of other effects including initial conditions unless those other effects are reduced to a minute level not found in everyday life draining events.

In the southern hemisphere, the Coriolis force effects due the Earth's rotating are all the mirror image of those in the northern hemisphere.

8. DRAG FORCES IN FLUIDS

Kinetic friction is a resistive force that acts between macroscopically smooth surfaces slides over each other. In the simple approximation of intro physics , the kinetic friction is independent of the relative velocity of the surfaces (Amontons's 3rd law (Wikipedia: Guillaume Amontons)).

The situation is different for fluids acting as a resistive medium for the motion of solid objects. For fluids, the resistive force is called fluid resistance or **DRAG** (Wikipedia: Drag (physics)).

In complete generality, **DRAG** is complex.

But a few things can be said in general.

The drag force is always opposite to the direction of motion and its zero if the object is at rest in the medium.

The drag force in general depends on the object velocity, shape, orientation, texture, and rigidity.

The drag force depends on the nature of the fluid, in particular its density.

9. AN APPROXIMATE DRAG FORCE LAW

An approximate law for the drag force on an object in a fluid that works well in many cases is:

$$D(v) = D_1 v + D_2 v^2 \ , \tag{135}$$

where v is the object speed, $D_1 v$ is the linear drag, $D_1 \geq 0$ is the linear drag coefficient, $D_2 v^2$ is the quadratic drag, $D_2 \geq 0$ is the quadratic drag coefficient, and the direction of the force is opposite the object's velocity (e.g., French 1971, p. 153).

The coefficients D_1 and D_2 are set by the properties of the object and fluid.

The linear drag dominates at low speed and the quadratic drag at high speed. This can been seen from the ratio of the quadratic to the linear drag forces

$$\frac{D_2 v^2}{D_1 v} = \frac{D_2 v}{D_1} \ : \tag{136}$$

the ratio is much less than 1 for small enough v and much greater than 1 for large enough v. The characteristic transition speed $v_{\rm trans}$ is when the ratio is 1 which implies

$$v_{\rm trans} = \frac{D_1}{D_2} \ . \tag{137}$$

For simple analysis, one only needs to the linear drag for low speeds and the quadratic drag for high speeds.

Quite frequently, one drag force is completely dominant.

The linear drag force is easy to treat analytically. In everyday contexts it applies to small slowly moving objects like dust particles in air (e.g., Serway & Jewett 2008, p. 148). It is principally due to fluid viscosity which is sort of fluid friction (e.g., French 1971, p. 153).

For ordinary human-sized macroscopic objects that are **NOT** streamlined, the quadratic drag usually completely dominates (e.g., Halliday et al. 2001, p. 104). The quadratic drag is actually easy to understand in terms of the transfer of fluid momentum to the object in turbulent flow—but we probably havn't discussed momentum yet (e.g., French 1971, p. 153). A crude proof for quadratic drag is given in Appendix E.

Let's consider the linear and quadratic drag separately and analyze their effect on objects sinking under the force of gravity.

9.1. Linear Drag

The linear drag force is

$$D_{\text{lin}} = D_1 v \ , \tag{138}$$

where the force is opposite to an object's velocity.

To emphasize the main feature of linear drag, the magnitude of the linear drag force increases linearly with the object speed.

To get an understanding of effect of linear drag, we will analyze the case of an object moving under the forces of gravity and the linear drag starting from **REST** at time zero—rest at time zero is our **INITIAL CONDITION**.

We take down as the positive y direction and apply $F = ma$ to get

$$\sum F_i = mg - m_{\text{dis}} g - D_1 v = ma \ , \tag{139}$$

where we have dropped vector notation for this 1-dimensional case, mg is the gravitation force, $m_{\rm dis}g$ is the buoyancy force, $m_{\rm dis}$ is mass of the fluid displaced by the object, and $D_1 v$ is the drag force.

The **BUOYANCY FORCE** which is a subject of a much later lecture: the lecture **FLUIDS**. In words, there is net pressure force equal to the weight of the fluid displaced by the object. Let's we write

$$mg - m_{\rm dis}g = mg_{\rm eff} \ , \tag{140}$$

where we define

$$g_{\rm eff} = \left(1 - \frac{m_{\rm dis}}{m}\right) g = \left(1 - \frac{\rho_{\rm dis}}{\rho}\right) g \ , \tag{141}$$

where $\rho_{\rm fluid}$ is the fluid density and ρ is the object density. We can eliminate the buoyancy force by using $g_{\rm eff}$ (which is object specific). We'll just drop the buoyancy force since it can be dealt with trivially and we arn't interest in it here. In any case, for objects with $\rho_{\rm dis}/\rho << 1$, it is negligible.

Dropping the buoyancy force and rearranging the equation of motion, we find

$$ma + D_1 v = mg \qquad \text{or} \qquad \frac{dv}{dt} + \frac{D_1}{m} v = g \qquad \text{or} \qquad \frac{dv}{dt} + b_1 v = g \ , \tag{142}$$

where we define $b_1 = D_1/m$.

Equation (142) is a 1st order linear ordinary differential equation (i.e., a DE): 1st order

Fig. 1.— Schematic plot of the linear drag force magnitude as a function of object speed.

because the highest derivative order to appear is 1st order linear because the function v and its derivatives only occur linearly in the equation, and ordinary because there is only one independent variable: i.e., t.

Equation (142) is also an **INHOMOGENEOUS DE**—inhomogeneous does not mean not pasteurized—it means that the function and its derivative do not occur in every term.

The inhomogeneous term is a constant g in our case. Inhomogeneous terms (which may be functions) are sometimes called sources. The solution of a DE with a source is often quite different than with the source set to zero.

Before we give the exact solution to DE equation (142), let's see what we can learn by simple means from the equation (142). First, at time zero, $v = 0$ which is our initial condition. Thus, for early times one has

$$\frac{dv}{dt} \approx g \qquad \text{and this solves to} \qquad v \approx gt \ . \tag{143}$$

So initially, we expect free-fall-like acceleration and velocity increases. But as velocity increases, the drag force term $b_1 v$ increases which reduces the accelerating force. Eventually, gravity and drag force must cancel out and make acceleration zero. Then one has a final or terminal velocity (which more properly is terminal speed) satisfying

$$b_1 v_{\text{term}} = g \qquad \text{or} \qquad v_{\text{term}} = \frac{g}{b_1} = \frac{mg}{D_1} \ . \tag{144}$$

This terminal velocity is stable—which means perturbations from it are damped out. If there is perturbation to lower velocity than v_{term}, then $D_1 v$ is less than mg and the object accelerates to higher velocity again. If there is a perturbation to higher velocity than v_{term}, then $D_1 v$ is greater than mg and the object has a negative acceleration that slows it down again.

The exact solution of equation (142) for out initial conditions can be found by elementary means—but those are beyond the scope of this class—but when you take a DE course you'll have them coming out of your ears.

We give the solution procedure in Appendix B for your edification.

Here we just write down the solution:

$$v = v_{\rm term}\left(1 - e^{-t/\tau}\right) \ , \tag{145}$$

where we define the e-folding time (or time constant)

$$\tau = \frac{1}{b_1} = \frac{m}{D_1} = \frac{v_{\rm term}}{g} \tag{146}$$

This solution can be confirmed by substitution back into equation (142). We leave that as an exercise for the student. Remarkably τ is independent of g.

A plot of the solution is presented below in Figure 2.

We can obtain the limiting cases of the solution:

$$v = \begin{cases} v_{\rm term}\left(1 - e^{-t/\tau}\right) & \text{in general;} \\ v_{\rm term}\left[1 - \left(1 - \frac{t}{\tau}\right)\right] = v_{\rm term}\left(\frac{t}{\tau}\right) = gt & \text{for small } t; \\ v_{\rm term} & \text{for small } t \to \infty. \end{cases} \tag{147}$$

We recover the early and late time results form the exact solution that we anticipated.

If we solve the linear approximation to the solution for the time when the terminal velocity is reached, we find

$$t = \frac{v_{\rm term}}{g} = \tau \ . \tag{148}$$

Now linear approximation must have failed by the time, it gives $v_{\rm term}$. Thus, τ is an upper limit on the time for which the linear solution is approximately valid. It is usual to take τ as the characteristic time for the transition between early times (when the linear solution is approximately valid) to late times (when the linear solution is not valid).

We note that for finite time the object formally never reaches the terminal velocity: it only reaches terminal velocity at time infinity. But practically, it does. There are always little perturbations in the fluid that knock velocity up and down. There is also always measurement uncertainty.

Once the object velocity is closer to the terminal velocity than the size of the perturbations or the uncertainty, then for most practical purposes the object has reached terminal velocity.

In fact, it usually isn't too long to practically reach terminal velocity—a few e-folding times. We note that

$$e^{-t/\tau} = \left[10^{\log(e)}\right]^{-t/\tau} = 10^{-0.43429\ldots\times(t/\tau)} \ . \tag{149}$$

Thus for $t/\tau = 10$, the relative difference from terminal velocity is $\sim 10^{-4}$. And for $t/\tau = 100$, the relative difference from terminal velocity is $\sim 10^{-43}$. Now 10^{-43} is minute relative difference for almost any quantity—usually far too minute to be measured. So by 100 e-folding times, the object has almost certainly been at terminal velocity for practical purposes for a long time.

9.1.1. An Example of the Linear Drag Solution

Say a small sphere is released in oil from rest and allowed reach terminal velocity. Say that sphere's mass is $m = 2\,\mathrm{g} = 0.02\,\mathrm{kg}$ and the measured terminal velocity—which took just forever to find—is $5.00\,\mathrm{cm/s} = 0.05\,\mathrm{m/s}$.

What is the e-folding time τ? Well

$$\tau = \frac{v_{\rm term}}{g} \approx \frac{0.05}{10} = 0.005\,\mathrm{s} \ . \tag{150}$$

More exactly one gets $0.0051\,\mathrm{s}$.

Now what is the time when the velocity is 90 % of terminal velocity? Well we must invert the solution to find time as a function of v: behold

$$\begin{aligned}
v &= v_{\rm term}\left(1-e^{-t/\tau}\right) \\
\frac{v}{v_{\rm term}} &= 1-e^{-t/\tau} \\
e^{-t/\tau} &= 1-\frac{v}{v_{\rm term}} \\
\frac{-t}{\tau} &= \ln\left(1-\frac{v}{v_{\rm term}}\right) \\
t &= -\tau\ln\left(1-\frac{v}{v_{\rm term}}\right)
\end{aligned} \tag{151}$$

Solving for t when $v/v_{\rm term} = 0.90$ gives

$$\begin{aligned}
t &= -\tau\ln\left(1-\frac{v}{v_{\rm term}}\right) \approx -0.005\times\ln(1-0.9) = -0.005\times\ln(0.1) \\
&= 0.005\times\ln(10)\approx 0.01\,{\rm s}\ .
\end{aligned} \tag{152}$$

More exactly one gets 0.0117 s.

That wasn't so long. But one still has to wait till the end of time to formally reach terminal velocity.

9.2. Quadratic Drag

The quadratic drag force is

$$D_{\rm quad} = D_2 v^2\ , \tag{153}$$

where the force is opposite to an object's velocity.

To emphasize the main feature of quadratic drag, the magnitude of the linear drag force increases linearly with the object speed. See Figure 3 for a plot.

To get an understanding of effect of quadratic drag, we will analyze the case of an

Fig. 2.— A sketch of the solution of the linear drag equation with the object starting from rest at time zero.

Fig. 3.— Schematic plot of the quadratic drag force magnitude as a function of object speed.

object moving under the forces of gravity and the linear drag starting from **REST** at time zero—rest at time zero is our **INITIAL CONDITION**.

We take down as the positive y direction and apply $F = ma$ to get

$$\sum F_i = mg - m_{\text{dis}}g - D_2v^2 = ma \ , \tag{154}$$

where we have dropped vector notation for this 1-dimensional case, mg is the gravitation force, $m_{\text{dis}}g$ is the buoyancy force, m_{dis} is mass of the fluid displaced by the object, and D_2v^2 is the drag force.

We will neglect the buoyancy force for the same reasons as in § 9.1.

Without the buoyancy force and rearranging the equation of motion, we

$$ma + D_2v^2 = mg \qquad \text{or} \qquad \frac{dv}{dt} + \frac{D_2}{m}v^2 = g \qquad \text{or} \qquad \frac{dv}{dt} + b_2v^2 = g \ , \tag{155}$$

where we define $b_2 = D_2/m$.

This innocent little DE has no exact analytic solution.

It's non-linear because derivatives of v appear non-linearly: in this case it is the zeroth order derivative of v—but that still counts.

Non-linear DEs are in general much harder to solve than linear DEs—but we'll let your differential equation course go into all that. In appendix D, yours truly develops an approximate analytic solution.

But let's see what we can learn by simple means. the equation (155). First, at time zero, $v = 0$ which is our initial condition. Thus, for early times one has

$$\frac{dv}{dt} \approx g \qquad \text{and this solves to} \qquad v \approx gt \tag{156}$$

So initially, we expect free-fall-like acceleration and velocity increases. But as velocity increases, the drag force b_2v increases which reduces the accelerating force. Eventually, gravity and drag force must cancel out and make acceleration zero. Then one has a final or terminal velocity (which more properly is terminal speed) satisfying

$$b_2 v_{\rm term}^2 = g \qquad \text{or} \qquad v_{\rm term} = \sqrt{\frac{g}{b_2}} = \sqrt{\frac{mg}{D_2}} \ . \tag{157}$$

A characteristic time for the transition between the linear phase of velocity growth and the phase of asymptotic approach to the terminal velocity is given by assuming the linear phase continues until $v_{\rm term}$ is reached. This characteristic time is

$$t_{\rm ch} = \frac{v_{\rm term}}{g} = \sqrt{\frac{1}{gb_2}} = \sqrt{\frac{m}{gD_2}} \ . \tag{158}$$

By this time the linear phase must be over. The larger the terminal velocity, the longer the time it takes to approach it.

This terminal velocity is stable—which means perturbations from it are damped out. If there is perturbation to lower velocity than $v_{\rm term}$, then $D_1 v$ is less than mg and the object accelerates to higher velocity again. If there is a perturbation to higher velocity than $v_{\rm term}$, then $D_1 v$ is greater than mg and the object has a negative acceleration that slows it down again.

In a very rough qualitative sense, the quadratic drag solution is similar to the linear quadratic drag solution.

There is actually a useful approximate expression for D_2:

$$D_2 = \frac{1}{2} C \rho_{\rm fluid} A \ , \tag{159}$$

where C is dimensionless drag coefficient that is of order 1 (with actual typical values ranging from about 0.4 to 1), $\rho_{\rm fluid}$ is the density of the fluid, A is the effective cross sectional area of the object perpendicular to the direction of motion. A crude proof is given for equation (159) in Appendix E.

We can now write the terminal velocity as

$$v_{\rm term} = \sqrt{\frac{mg}{(1/2)C\rho_{\rm fluid}A}} \tag{160}$$

$$= \sqrt{\frac{\rho\ell g}{(1/2)C\rho_{\rm fluid}}} \tag{161}$$

$$= 127.145\,{\rm m/s} \times \sqrt{\left(\frac{\rho}{998\,{\rm kg/m^3}}\right)\left(\frac{\ell}{1\,{\rm m}}\right)\left(\frac{1.21\,{\rm kg/m^3}}{\rho_{\rm fluid}}\right)\left(\frac{1}{C}\right)}\ , \tag{162}$$

where ρ is the object's mean density, and $\ell = V/A$ is a characteristic length scale of the object (which has volume V), and the last version is in terms of fiducial values. The fiducial values are the $g = 9.8\,{\rm m/s^2}$, of course, 1 m for ℓ, the density of water $1000\,{\rm kg/m^3}$ (at 20° and 1 atm pressure (Wikipedia: Water))) for ρ and the density of air $1.21\,{\rm kg/m^3}$ (at 20° and 1 atm pressure (e.g., Halliday et al. 2001, p. 323)) for $\rho_{\rm fluid}$.

We have assumed that the buoyancy force is negligible in writing equation (162). If it was not negligible g would have to be replaced by $g_{\rm eff} = (1 - \rho_{\rm fluid}/\rho)g$. The buoyancy force is negligible for the calculations we report below and we have neglected it.

For some representative objects, Table 2 gives terminal velocities calculated from equation (162) using some rough values and $C = 1$, and the corresponding measured terminal velocities and their 95 % distances (the fall distance from rest to distance to reach 95 % of the terminal velocity).

Table 2. Terminal Velocities for Typical Objects

Object	Calculated Terminal Velocity (m/s)	Measured Terminal Velocity (m/s)	Measured 95 % Distance (m)
Shot put shot	114	145	2500
human	127	60	430
baseball ball	24	42	210
tennis ball	17	20	115
raindrop (radius = 1.5 mm	6	7	6
cat tucked	64	27	...
cat spread	64	19	...

Note. — The measured values are from Halliday et al. (2001, p. 105).

The calculated values Table 2 are not so bad considering that very crude estimates were used for ℓ and since $C = 1$ was used for all objects.

One can see that a human doesn't accelerate indefinitely going down.

So there is no problem skydiving without a parachute: aim a haystack or a soft snowdrift: it has been done: check out

http://www.greenharbor.com/fffolder/carkeet.html .

Then there are cats.

Cats have tendency to jump out of high windows—they're just so smart. This is called the high-rise (Wikipedia: High-rise syndrome).

But their survival rate is pretty good—and it increases with fall height.

What happens is a falling cat can orient itself right side up in the air and spread out if it has enough time. Cats do this automatically—millions of years of tree-climbing and tree-off-falling evolution is at work here. In their spread-out form, they have a rather low terminal velocity because of a big cross sectional area for their mass. Of course, they must slow down to that low terminal velocity from pre-spread-out higher one—a fall of more than 6 stories seems to be enough. As the ground approaches, they tuck in again for landing (Wikipedia: High-rise syndrome; Halliday et al. e.g., 2001, p. 104–105).

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A. Banked Turns with the Normal Force and Friction

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B. Solution for a Sinking Object with Linear Drag

Equation (142) (S 9.1), the differential equation (DE) for a sinking object with linear drag,

$$\frac{dv}{dt} + b_1 v = g \tag{B1}$$

can be solved using the old integrating factor trick. The idea is to multiply both sides of the DE by a factor f which makes both sides immediately integrable—and then we just integrate.

Neat you say—but what is f?

Well you have to solve a DE for f, but its a homogeneous DE—and that makes everything so much better. Homogeneous means that all the terms contain the function to be solved for or its derivatives. Multiply equation (B1) by f to get

$$f\frac{dv}{dt} + f b_1 v = f g \ . \tag{B2}$$

We demand—yes demand—that both sides be integrable. This means that

$$(fv)' = f\frac{dv}{dt} + f b_1 v \tag{B3}$$

which implies that

$$f'v + fv' = fv' + fb_1 v \qquad \text{or} \qquad f' = fb_1 \ . \tag{B4}$$

This last equation can be written as

$$\frac{1}{f}\frac{df}{dt} = b_1 \ . \tag{B5}$$

We can now integrate both sides over time and solve for the integrating factor:

$$\begin{aligned} \int \frac{1}{f}\frac{df}{dt}\,dt &= \int b_1\,dt \\ \ln(f) &= b_1 t + \text{constant} \\ f &= e^{b_1 t}\ , \end{aligned} \tag{B6}$$

where "constant" is a constant of integration and we have set $e^{\text{constant}} = 1$ since the scale of the integrating factor is irrelevant—if not 1, it just cancels out in the final solution.

Well now we have

$$\begin{aligned} f\frac{dv}{dt} + f b_1 v &= f g \\ (fv)' &= fg \\ e^{b_1 t} v|_0^t &= \int_0^t e^{b_1 t'} g\,dt' \\ e^{b_1 t} v|_0^t &= \frac{g}{b_1}(e^{b_1 t} - 1) \\ v &= \frac{g}{b_1}(1 - e^{-b_1 t}) \\ v &= v_{\text{term}}(1 - e^{-t/\tau}) \end{aligned} \tag{B7}$$

which is the solution with the initial condition $v(0) = 0$ and where we have used the definitions

$$v_{\text{term}} = \frac{g}{b_1} = \frac{mg}{D_1} \qquad \text{and} \qquad \tau = \frac{1}{b_1} = \frac{m}{D_1} = \frac{v_{\text{term}}}{g}\ . \tag{B8}$$

The quantity v_{term} is the terminal velocity and τ is the e-folding time.

I find it remarkable that τ is independent of g.

C. Solution for a Sinking Object with General Power-Law Drag

The differential equation for sinking with quadratic drag

$$\frac{dv}{dt} + b_2 v^2 = g \tag{C1}$$

as no exact analytic solution.

It's a non-linear DE as one can see and those are trouble.

One can get an approximate solution with some trickery.

But let's be more general and get an approximate solution for

$$\frac{dv}{dt} + bv^p = g \ , \tag{C2}$$

where $p > 0$.

D. Solution for an Object Subjected to Only a Power-Law Drag

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E. Proof for the Quadratic Drag Force

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