NEWTONIAN PHYSICS I

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ABSTRACT

Lecture notes on what the title says and what the subject headings say.

Subject headings: dynamics — Newtonian physics — forces — inertial frames — classical point particles — systems of particles — Newton's three laws of motion — center of mass — free body diagrams — gravity near Earth's surface — normal force — tension — friction — static friction — kinetic friction

1. INTRODUCTION

The two key components of first semester intro physics are Newtonian physics and energy.

Energy is included in Newtonian physics, but it is more general than Newtonian physics and is part of all physics.

Most people have some understanding of both components—and that understanding is both in the vague sense in which they turn up in everyday nonscientific live and in the precisely defined scientific sense.

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In this lecture, we go into precisely defined Newtonian physics—at our level of understanding. We leave the introduction of energy to the lecture *ENERGY*. A higher level, Newtonian physics from the start already incorporates energy and a lot more.

Newtonian physics is dynamics as opposed to kinematics. Kinematics is the description of motion. Dynamics is kinematics plus the causation of motion. In Newtonian physics, forces cause motions as we'll see.

Newtonian physics also allows for the formation of structures.

We certainly arn't doing Newtonian physics as Newton formulated it.

Some of the terminology and math methods have changed and some concepts have evolved—Newton didn't use energy for a main example. The beginnings of the concept of energy go back to Galileo (e.g., Caldwell 1994, p. 87), but Newton made no use of beginnings though he may have been aware of them.

Still the Newtonian physics we do in this lecture is not a million light-years away either from Newton's work.

Newton would recognize it if he were here today.

2. THE COMPONENTS OF NEWTONIAN PHYSICS

Newtonian physics has several components.

The Euclidean geometry of space is one of them. Let's just assume that we all know what that is—parallel line theorems, triangles, etc.

Then there is time.

Of course, concepts of time pre-dated Newtonian physics.

But as discussed in the lecture *INTRODUCTION TO INTRODUCTORY PHYSICS*, Newtonian physics made time well defined by showing that certain ideal systems would have exactly periodic motions as a parameter time advanced. That parameter time was consistent with earlier concepts of time, and so caused no shock in people's understanding. The ideal periodic systems are ideal clocks which real clocks used by people approximate to some degree.

Newtonian time flows the same in all places and frames of reference. This means that in Newtonian physics ideal clocks everywhere and everywhen can be synchronized and will stay synchronized. Newton himself wondered if this had to be true, but it was the simplest hypothesis and nothing people knew until end of the 19th century caused any doubt.

Actually, nowadays we believe that Euclidean geometry only approximately describes only part of physical space and we understand that time does flow differently in different frames of reference. Special and general relativity are needed to understand the differences from Newtonian concepts.

But in the realm of much of science and human activity, Newtonian space and time very accurately describe things.

Another component of Newtonian physics is kinematics which is the description of motion. The most prominent quantities in the description of motion are displacement, velocity, and acceleration.

Other Newtonian prominent concepts are force and mass.

Force is the cause of accelerations.

A force can also cancel another force and can cause body deformations.

A force or forces can cause structure: e.g., of solids, liquids, planets, stars, galaxies,

cluster of galaxies, and—getting beyond Newtonian physics again—nuclei, atoms, molecules, and the cosmos as whole it is thought.

Mass is the resistance of body to acceleration under a net force.

Still another component is Newton's three laws of motion which give the exact relationships among force, mass, acceleration. One also needs the concept of center of mass to make use of Newton's three laws for objects larger than point particles. We introduce the concept of center of mass in § 6.1.

Then there is the concept of inertial frames which will introduce in § 4 below.

One also needs force laws that are independent of Newton's three laws of motion.

All the components mentioned above constitute Newtonian physics. Other ones are probably needed for an exhaustive discussion—but we're exhausted enough.

In my opinion, the components cannot be adequately defined separately, except for Euclidean geometry.

For example, velocity and acceleration depend on the time concept. But that depends on ideal periodic systems which are understood in terms of forces, masses, and Newton's three laws of motion. As a cause of acceleration, force requires understanding the concepts of acceleration and mass. Mass requires understanding acceleration and force.

And so on.

Except for Euclidean geometry, there seems to be a circularity of definitions.

Things can only be adequately defined in terms of other things that in turn depend on the first things defined.

But the circularity is not vicious.

One just has to accept Newtonian physics as a package. The components (excepting Euclidean geometry) are inextricably connected.

All the bits fit together and give a very adequate description of the much of the world that we observe and much of the scientific and technological realms of interest.

Of course, as discussed in the lecture *INTRODUCTION TO INTRODUCTORY PHYSICS*, we know Newtonian physics is not exactly true. But is believed to be exactly true in the classical limit (i.e., the limit of relative speeds much less than the vacuum speed of light, weak gravity, and macroscopic size but much less than the cosmological size scale) and "the much of the world that we observe and much of the scientific and technological realms of interest" are close to the classical limit to high accuracy: i.e., they are in the classical realm.

The classical limit is itself is not really a well defined mathematical limit—as you approach it from one extra-classical realm, you stop getting closer to it after awhile and head off into another extra-classical realm. For example increasing in size from the realm of quantum mechanics, things get more and more classical for awhile—a long while—but then you reach cosmological sizes and things get non-classical again. The classical limit is loosely speaking the middle of the realm where classical physics works well.

The classical realm is very big.

Thus, Newtonian physics is eminently useful.

It's also much simpler to understand and use than more fundamental theories.

It's also wonderfully beautiful—just take my word for it.

In much of our discussion, we just speak as if Newtonian physics were exactly true: i.e., that we live in a Newtonian universe.

Always having to remark that it isn't exactly true would be useless and tedious. We

know that it isn't exactly true well enough to know what we mean.

3.   FORCE CONCEPT

As mentioned in § 2, a force is the cause of an acceleration of a system and can also cancel another force.

A force can also cause deformations of a system. This is not really an independent property of force. If accelerations of a system happen relative to other parts of the system, then there will be deformations. Constant velocity deformations can happen too, but an acceleration was needed to create the velocity doing the deforming in the first place. Once the energy concept is introduced, one can also say that forces cause or mediate the transfer or transformation of energy.

In § 5, we'll delve into the exact way a force causes an acceleration.

Here we can remark that force is a **VECTOR QUANTITY**. Force cancellation occurs through vector addition. If the forces sum to zero, there is zero net force.

Forces as the causes of deformations, we touch on as the course proceeds.

In everyday understanding, a force is push or a pull which can be described as a physical relation between bodies that changes or tries to change something. We know a push or pull does all of the things we say forces do in a vague way.

Forces are understood to be relationships between **THINGS**. At our level, the **THINGS** are systems with mass for contact forces and system with mass and fields for field forces. We'll discuss contact and field forces in § 3.1.

In nature, there are only four fundamental forces:

1. gravity.

2. the electromagnetic force.

3. the strong nuclear force.

4. the weak nuclear force.

The two nuclear forces are intrinsically quantum mechanical and cannot be described by Newtonian physics. In intro physics, we only touch on them at times. They only have limited impact on our ordinary surface understanding of everyday life.

The electromagnetic force is an immensely complex force.

It has many complex manifestations: the electrostatic force (i.e., the Coulomb's law force), the magnetic force, the chemical bonding forces, the elastic force, friction forces, tension forces, pressure forces, and on and on.

Actually, the various manifestations can't be clearly separated in most cases. For example, an elastic force is a macroscopic manifestation of chemical bonding forces. So are friction forces.

Also special contexts often give rise to special names for the electromagnetic force in that context: e.g., the normal force which we'll discuss below in § 11.

The complexity of the electromagnetic force makes it hard to understand.

But on the bright side, it makes all of chemistry, and therefore, life possible.

Gravity is arguably much less complex than the electromagnetic force.

Still it's plenty complex in many contexts: e.g., the behavior of galaxies.

In the early part of intro physics, we largely restrict ourselves to the case of gravity

near the Earth's surface which is very simple. In this lecture, we consider gravity near the Earth's surface in § 10. In the lecture *GRAVITY*, we will consider gravity more generally.

In the most fundamental theory of modern physics, the standard model of particle physics, the electromagnetic force and the strong and weak nuclear forces are united as manifestations of a single force. Nevertheless, it remains conventional to say there are 4 fundamental forces.

The faith is that in the theory of everything (TOE) (discussed in the lecture *INTRODUCTION TO INTRODUCTORY PHYSICS*) all the forces will be united. But gravity is really stubborn and has not yet been brought into the fold—there are theories for how to do it, but no one established theory.

It will probably remain conventional to say there are 4 fundamental forces even after the advent of TOE.

3.1. Contact and Field Forces

There is another categorization of forces useful in the macroscopic realm: contact and field forces.

A contact force is a force between systems with mass where the systems have to be touching macroscopically for the force to be exerted.

Most everyday manifestations of the electromagnetic force (e.g., elastic force, tension force, pressure force, normal force) are contact forces.

A field force is a force caused by a vector field.

A field is a quantity defined at every point in some region of space.

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Temperature and pressure are scalar fields. They just have scalar values at each point in space.

A vector field has a magnitude and direction at each point in space.

I have this mental picture of little arrows attached to each point in space. One can only draw a representative subset in a diagram, of course.

Vector fields causing forces are macroscopic electric fields and magnetic fields and gravity. Gravity only manifests itself as a field force.

Here's a demonstration we/you can do of the macroscopic electric field using a comb and bits of paper.

Tear up paper into fingernail or sub-fingernail bits. Comb your hair vigorously. Put the comb near the bits, and we/you will see that the comb attracts the bits. What happens is that the comb becomes negatively charged and your hair positively charged—for reasons we won't go into and yours truly is not too clear on anyway. It is enough to say here of charge that like charges repel and unlike charges attract by the Coulomb force. The comb then attracts the bits of paper. The bits are actually neutral. But the bits become polarized because the individual molecules that make up the paper bits become polarized. Becoming polarized means there is a charge separation into positive and negative regions. The positive charge of the molecules is shifted toward the negative comb and the negative charge away. Distance matters too: the closer the charges the stronger the Coulomb force. The closer positive charge of the molecules is more attracted to the comb than the farther negative charge is repelled. So the bits are attracted to the comb even though they are neutral. Actually, low levels of charging and polarization go on all the time without us noticing too much.

Force-causing fields will be elucidated as intro physics proceeds.

Inertial forces are also field forces—but they aren't really forces—it's tricky—we'll discuss them in the lecture *Newtonian Physics II*.

Often fields emanate from a body and cause a force on another body. But one can just have the field itself without an emanating body. One can a field alone just as an ideal case for studying the effects of the field. Of course, real fields have to arise somehow, but that arising may be a long a convoluted story. For example, electromagnetic radiation—which is a self-propagating electromagnetic field—can cause a pressure force. The electromagnetic radiation arose from source, but that source could have been billions of years ago, billions of light-years away and may not exist in the present of cosmic time.

Field forces are sometimes called body forces since they can act directly whole bodies not just on surfaces. Contact forces can act directly only on a body surface. They can cause other forces internal to the body, and so have effects throughout the body and to other bodies.

Field forces can be very long-range. Contact forces are necessarily short-range.

At the microscopic level, all forces are really field forces. The idea of a contact force only arises at the macroscopic level.

4.   INERTIAL FRAMES

A frame of reference is just a set of coordinates that one uses for measuring motions. The frame may be attached to some physical body: e.g., the Earth, a car, the Sun, the center of the Milky Way. But it may just be an abstract frame one defines in space.

Inertial frames are a special class of frames of reference that are **UNACCELERATED** with respect to each other.

The concept of inertial frame is often skirted in intro physics textbooks because it's tricky.

But it's also basic and essential.

Newton's laws of motion are referenced to inertial frames—or non-inertial frames when inertial forces are introduced, but let's leave this exception to the lecture *Newtonian Physics II*.

Newton's laws are valid in all of space—in a classical-physics universe, of course.

It's just that in using them and calculating with them, one uses inertial frames.

This means that **SPACE** has a physical nature besides just being space in which things can be located relative to each other. This physical nature manifests itself in that Newton's three laws are referenced to inertial frames defined on **SPACE**. We'll see this referencing is done in § 5. Actually, many basic physical laws from all branches of physics are referenced to inertial frames.

Where are the inertial frames?

How do we know one when we see one?

It's all a bit tricky.

Newton himself thought that the fixed stars defined the basic inertial frame. The fixed stars are the nearby stars that seemed unmoving since time immemorial.

All frames unaccelerated with respect to that frame were also inertial. This is still a vital ingredient in the inertial frame concept. It it implies that acceleration is inertial frame invariant since changing from one inertial frame to another can add nothing to an acceleration vector. As we'll show explicitly in § 9, an acceleration is invariant in value on transformation between inertial frames. So an acceleration in one inertial frame, is the same

in all inertial frames: same direction, same magnitude.

Naturally, frames of reference attached to the rotating and revolving planets are not inertial frames.

A system rotating with respect to the basic inertial frame must be an accelerated frame, and so a non-inertial frame.

Well all non-inertial frames are non-inertial, but some are less non-inertial than others.

The smaller the acceleration of a non-inertial frame relative to the set of inertial frames, the better it approximates an inertial frame.

Depending on the case, a non-inertial frame may sufficiently like an inertial frame that one can treat it as such.

For many, but not all purposes, the surface of the Earth can be treated as an inertial frame.

But what is the basic inertial frame nowadays?

We now know that the stars are not fixed.

They revolve around the center of the Milky Way in complex orbits.

The Milky Way itself is in a complex orbit around the center of the local group of galaxies. This center seems to be accelerated as well.

In fact, it seems that virtually all systems with mass are accelerated to one degree or another. If any arn't, it's probably just a fluke.

So where is the elusive basic inertial frame.

Is it like the Cheshire Cat, vanished leaving only its smile, the approximate inertial frames?

There's sort of a shaggy dog story.

In our best modern theory, there isn't really a single basic inertial frame.

The universe is expanding.

This is literally a growth of space accounted for in general relativity.

The analogy often used is the surface of a balloon. The surface a balloon is a unbounded, finite two-dimensional surface. Put dots on the balloon and then blow it up. The two-dimensional space grows and the distances between the dots increases. Of course, real space is three-dimensional—which makes picturing its growth tricky, but mathematically it's intelligible.

In the big bang theory, the universe started off with rapid growth which gravity is slowing down. Since circa 1998, we've had evidence that the growth is accelerating and not decelerating. The cause of the acceleration is named dark energy which is name covering our ignorance. Their are theories for the acceleration, but no established theory.

Gravitationally bound systems like clusters of galaxies don't participate in the expansion and neither do smaller bound systems like us. We're unexpanding dots on the balloon.

You may ask what is the universe expanding into and is there a center of expansion.

In the pure general relativity big bang theory, there is no center. The universe may be infinite and always was. It's an infinity getting bigger which mathematically is intelligible. On the other hand, it could be a finite, unbounded universe which is the three-dimensional surface of four-dimensional hypersphere. Such a space is hard to picture, but mathematically it is intelligible. It you head off in a local straight line in such a space, eventually you come back to where you started from—if the universe didn't grow to fast for you? The space of this pure theory is homogeneous (i.e., the same everywhere) and isotropic (i.e., the same in

all directions) if viewed on a large enough scale. Observations are consistent with the pure theory. But you have to look at a pretty large scale for the homogeneity and isotropy to be noticed since on small scale matter is evidently clumped into galaxies, stars, planets, and us.

The pure general relativity big bang theory may not be correct. The universe may be bounded and finite, but the boundaries are evidently so far off we have no observable evidence for them. Outside of such a bounded universe conditions may be quite different where different laws of physics applies.

Now what does the expansion of the universe have to do with inertial frames.

Well frames of references that participate in the mean expansion of the universe seem to the best candidates for a class of basic inertial frames.

There is an infinite continuum of such frames.

For every point in space there is one.

Virtually, all matter is revolving and rotating with respect to these inertial frames participating in the mean expansion of the universe.

But in principle we can determine inertial frames.

In fact, we have actually identified our local basic inertial frame from the cosmic microwave background radiation (CMB) and distant extragalactic radio sources. The CMB is a radiation field pervading all space. It's mostly in the microwave band—like the stuff in your microwave ovens—space is being nuked. The CMB was left over from the big bang. It is theorized that it should be nearly isotropic in frames that are at rest or only rotating with respect to the continuum of basic inertial frames. In all other frames, the Doppler shifts due to the motion of the frames will cause the CMB to be anisotropic. The Doppler effect causes an increase in frequency of wave phenomena if you are moving toward the source and

a decrease in frequency of wave phenomena if you are moving away the source relative to what one would observe if you were at rest with respect to the source. The Doppler effect depends on relative motion and so occurs if you are at rest and the source moves. The Doppler effect happens for electromagnetic radiation like the CMB and for sound. You commonly notice the sound Doppler effect. An approach siren has a higher pitch (i.e., frequency) than a receding one.

Distant extragalactic radio sources relative to whatever point in space you are considering should in theory have asymptotically vanishing rotation relative to the basic inertial frame at that point. By asymptotically vanishing, one means that in the limit as you look farther out in from the point, the better they are as non-rotating reference.

Now from Earth, the CMB is not isotropic, but shows an overall Doppler shift pattern and the distant extragalactic radio sources are rotating. Therefore, the Earth is moving and accelerated relative to the local basic inertial frame. But our measurements tell us by how much and allow us to identify that local basic inertial frame. We can than use that frame to determine what other motions are relative to the local basic inertial frame. For example, the local group of galaxies is moving at $627 \pm 22 \, \mathrm{km/s}$ relative to the CMB in a particular direction (Wikipedia: Cosmic microwave background radiation).

The CMB and distant extragalactic radio sources are used to establish the International Celestial Reference Frame (Wikipedia: International celestial reference frame).

5.   NEWTON'S THREE LAWS OF MOTION

Newton's three laws of motion, or Newton's laws for short, are amazingly simple to write down and recite.

Many people can just rattle them off—often neglecting the part about inertial frames.

There are, in fact, no standard wordings for them. Every textbook just uses its own wordings as far as I can tell. I just use my own too.

But they are not at all obvious.

No one ever rediscovers those laws for themselves.

Even Newton didn't discover them starting from nothing.

He was at the end of a long discussion about what the basic laws of motion were that started before Aristotle with the Pre-Socratic philosophers of ancient Greece.

Why arn't Newton's laws obvious?

Well many motions are immensely complex. Walking for example. We do it easily enough. But hundreds of millions of year of evolution were needed to develop walking. Making robots walk is a formidable task that is only being solved in recent years.

But even simpler motions like projectile motion are not so simple.

Near the Earth's surface there is always gravity causing an acceleration downward. Then there is air resistance causing a force that always acts opposite to the direction of motion.

In fact, it takes very carefully controlled experiments to observe Newton's laws in simple manifestations to high accuracy.

Historically, such experiments began with Galileo.

Of course, it takes more than experiments. One has to theorize the exact mathematical law to explain what one sees. Also, as Galileo understood, experimental error will always cause deviations from a mathematically exact physical law. You have to imagine ideal cases to observe in your imagination the laws acting exactly. In practice, one can always try approach the ideal result more and more closely.

Idealization has been a tool of physics and all of science ever since Galileo.

One idealization we start with is the idealization of the classical point particle—which we go into just below in § 5.1

5.1.   The Classical Point Particle and Systems of Particles

The classical point particle has no extent in space and no internal structure. It has mass and it can exert forces and have forces exerted on it.

But the classical point particle has this little problem.

It doesn't actually exist.

Now quantum mechanical particles do exist. They are discrete, small entities and so can be classified as particles. Some like molecules, atoms, nuclei, protons, and neutrons have finite extent and are not point-like. Others like electrons and quarks may be point-like. We don't know for sure.

But these quantum mechanical particles obey quantum mechanics and not classical physics.

So how can we make use of the classical point particle?

Well the classical point particle can be regarded as an entity which has the average behavior of quantum mechanical particles making up a macroscopic system of ordinary matter in regard to macroscopic system motion.

A classical point particle with this definition will never give the right behavior of a single quantum mechanical particle. To some degree of approximation it can give quantum mechanical behavior like macroscopic electrical and heat conduction.

But a system of imagined classical point particles will give the correct macroscopic motion of a macroscopic system made up of actual quantum mechanical particles.

One might ask which set of quantum mechanical particles are being modeled as ideal classical point particles. One does not need definite answer. The question doesn't arise in the derivations and no one knew anything much about quantum mechanical particles before the 20th century and didn't have to in order to make use of classical point particles in derivations. So any set one likes to imagine: molecules, atoms, nuclei, protons, neutrons, electrons, quarks, et cetera. One might also ask which point in non-point quantum mechanical particles is to regarded as the location of the point classical particle. One does not need definite answer again since again the question doesn't arise in the derivations. But a reasonable answer is the center of mass of the quantum mechanical particles. We introduce center of mass in § 6.1.

Why is using unreal, ideal hypothetical classical point particles useful?

One posits Newton's three laws of motion for classical point particles and then shows that macroscopic motions follow from that are experimentally correct.

Could one dispense with classical point particles as a starting point for classical physics?

I think so, in principle.

But throughout physics, classical and non-classical, using point particles seems to be the easiest, clearest, most fruitful way of developing fundamental physical laws.

So much so that not starting a fundamental theory from point particles seems artificial and is certainly awkward. This is true today and is historically true. There is a big exception which is modern string theory. In string theory the fundamental entities are not particles, but strings which have extent and internal properties. But we won't go into all that—string theory may be wrong anyway.

So point particles are inescapable nearly in physics in general.

And in classical physics, classical point particles are nearly inescapable as starting points.

Hereafter, we will usually just refer to classical point particles as particles for brevity.

But from Newton's laws for particles to Newton's laws for macroscopic systems of particles is a short step. Such macroscopic systems are just macroscopic objects. But in the jargon of intro physics, they are usually called systems of particles or just systems—get used to it.

Now for Newton's three laws.

The three laws are stated for particles since they are the fundamental entities of classical physics.

5.2.   The 1st Law

A short statement of the 1st law is:

A particle is unaccelerated relative all inertial frames unless the system is acted
on by a net force.

Note that concepts of time, space, force, net force, and inertial frames are all needed understand this definition.

Everything has to be accepted as part of the Newtonian physics package.

"Unaccelerated" means that the particle is in straight-line, uniform (i.e., constant speed) motion with respect to all inertial frames. Note also that we could say unaccelerated with respect an inertial frame. Since all inertial frames are unaccelerated relative to each other, that implies unaccelerated with respect to all inertial frames.

Since the statement of the 1st law is the same for all inertial frames, we say it is inertial frame invariant. But note that in non-inertial frames, one can have accelerations without forces.

Note that there is no fundamental difference between being in uniform straight-line motion or at rest relative to an inertial frame. For any particle in uniform straight-line motion relative to an inertial frame, there is always an inertial frame in which it is at rest: i.e., the frame of the particle itself.

Note also that **NO** net force acts on the particle. There may lots of forces on the particle, but they sum to zero. The case of no forces acting on the particle is special case of no net force acting on the particle.

And another thing.

Since any force can be a net force, the 1st law implies that if a force is zero in an inertial frame it is zero in all inertial frames.

We will say more about the transformations of forces under frame transformations in § 5.3.

5.3.   The 2nd Law or $F = ma$

The 2nd law is best expressed by a formula:

$$\vec{F}_{\mathrm{net}} = m\vec{a} \; , \tag{1}$$

where $\vec{F}_{\mathrm{net}}$ is the net force on a particle, m is the particle mass, and $\vec{a}$ is the particle acceleration relative to all inertial frames. This acceleration remember is invariant on transformations between all inertial frames: i.e., it has the same value in all inertial frames. Note that equation (1) is a vector equation.

To complete the statement of the 2nd law, one must add that it is inertial-frame-invariant. This means that if you make a transformation between inertial frames, the formula of the 2nd law is the same. Since acceleration is inertial-frame-invariant just by itself, the implication is that mass and net force are also inertial-frame-invariant just by themselves. One could imagine that they vary by a common scalar factor. However, an extra classical hypothesis is that mass is intrinsic to the particle and independent of frame whether inertial or not. Thus net force should also be independent of inertial frame. Actually, it must be independent of frame whether inertial or not. If you reference a particle to a non-inertial frame, some of its acceleration is due to the frame acceleration and the rest can be accounted for by the 2nd law referenced to an inertial frame. It adds nothing to the description to say that the net force has a different value in the non-inertial frame.

Since any force can be a net force, all forces are frame-invariant. Moreover, classical force laws (see § 6.4) tells us that classical forces are frame-invariant not just inertial frame invariant. These force laws depend only on the relative positions and velocities of systems and positions of fields of force. Thus, they should not depend on reference frame. Actually, there is a qualification for forces on systems caused by electromagnetic fields since the relative amounts of electric force and magnetic force in the net electromagnetic force are frame dependent, but not the force itself in the classical limit. There may be other complex situations.

One must add that in relativistic physics mass and force become frame dependent (e.g., Lawden 2002, p. 43–44). But in classical physics, they are frame-invariant.

Of course, the components of a force are frame dependent, but its magnitude and direction are frame-invariant and that is what we mean by saying it is frame-invariant.

The significance of the inertial frame invariance of Newton's laws of motion is elucidated in § 9.

To return to elucidating the 2nd law itself.

The net force and acceleration are vectors that point in the same direction. The mass is a scalar quantity that is always positive. Being always positive, mass cannot cause a change in direction between acceleration and force.

Note the **NET FORCE** is the vector sum of all forces acting on the particle. It is not any particular force.

You may wonder where $\vec{F}_{\text{net}}$ actually is? For a particle, one can think of it as actually right at the particle. I like to picture the tail of $\vec{F}_{\text{net}}$ as resting on the particle with the vector arrow pointing in the direction of $\vec{F}_{\text{net}}$. But this is just a mental picture for convenience. In particular, like all vectors, except displacement, $\vec{F}_{\text{net}}$ has no extent in space space. Its extent is in an abstract space. It does have a direction in real space.

One often refers to the 2nd law as "$F = ma$" without the vector signs or any indication of F being the net force. This is OK as long as one knows one is simply using $F = ma$ as expression that means the 2nd law.

The 2nd law is understood as the net force causing the acceleration.

One can see from the formula for the 2nd law that the magnitude of the acceleration is directly proportional to the magnitude of the net force and the direction of the acceleration is the direction of the net force.

Now any force can cause an acceleration if it acts alone on the body. This is why we say that forces cause accelerations. But in the formula it is the **NET FORCE** that causes the acceleration as aforesaid.

The mass is the resistance of the particle to the acceleration. It is an intrinsic property of the particle and, as mentioned above, it is always positive. One can see how the resistance

to acceleration arises by writing the formula as

$$\vec{a} = \frac{\vec{F}_{\text{net}}}{m} \; . \tag{2}$$

The acceleration is inversely proportional to the mass. One can determine masses in principle by using a fixed force and measuring accelerations. In actual practice, masses are often determined using the equivalence of inertial and gravitational mass which we discuss in § 10.

One sometimes hears that mass is the quantity of matter. This is a sometimes useful definition. For example, if you had a system made all of one kind of atom. Each atom has the same mass very nearly, and therefore the number of atoms is proportional to the mass—at least to high accuracy: there are qualifications, we won't go into here.

But quantity-of-matter definition is not the fundamental one. Of course, one could just say that quantity of matter by definition is the resistance to acceleration. But then quantity of matter becomes a redundant name.

Now for the course mantra:

"Newton's 2nd law is always true and it's always true component by component."

In any intro physics problem involving forces and motions, recite this mantra to yourself—at least until you have totally internalized it—as our friends in the social sciences would say.

The "component by component" part of the mantra is, of course, because the 2nd law is a vector law. It is valid for all components. Thus in Cartesian components, we have

$$F_{x,\text{net}} = ma_{\text{x}} \; , \qquad F_{y,\text{net}} = ma_{\text{y}} \; , \qquad F_{z,\text{net}} = ma_{\text{z}} \; . \tag{3}$$

Note that if $\vec{F}_{\text{net}} = 0$, then $\vec{a} = 0$ assuming we are not dealing with a zero-mass system. This means that the particle is unaccelerated in all inertial frames. The 1st law is now seen as a special case of the 2nd law.

So in actual fact, one really only has two laws of motion in Newtonian physics: the 2nd law and the 3rd law. But Newton said he had three laws and three laws it remains by his convention. So for historical and pedagogical reasons this is a valid convention.

Why did Newton keep the first law? Well I don't know. But even for him, it may have been a traditional law since Descartes had earlier considered the motion of a particle in the absence of a net force.

What if a particle has zero mass?

Well Newtonian physics does not deal with zero-mass particles or zero-mass systems of particles in one sense. But one can deal them as the limiting case of particles or systems with small mass that are part of larger systems. We'll discuss zero-masses systems further in § 6.7.

5.4. The 3rd Law

In short form, the 3rd law (sometimes called the action-reaction law or law of reaction) is:

For every force there is an equal and opposite force.

This form though a good shorthand is not adequate as the following question illustrates.

Question: If for every force there is an equal and opposite force why don't all forces cancel out pairwise and and result in no acceleration at all?

a) They do cancel out pairwise and there is no acceleration at all and physics is all a crock.

b) The forces **ARN'T** on the same particle, and so they don't cancel out of the net force .

c) The forces **DO** have to be on the same particle, and so they don't cancel out of the net force.

Yes, it's (b).

"Pairwise" is a common physics jargon meaning "in pairs".

A better statement of the 3rd law is:

For every force exerted by a first particle on a second particle there is a force equal in magnitude but opposite in direction exerted by the second particle on the first. Expressed by a formula, the 3rd law is

$$\vec{F}_{21} = -\vec{F}_{12} \;, \tag{4}$$

where $\vec{F}_{12}$ is the force the first particle exerts on the second and $\vec{F}_{21}$ is the force the second particle exerts on the first. The forces have the same fundamental nature: i.e., the reaction force to an electromagnetic force is an electromagnetic force and the reaction force to a gravity force is a gravity force. Since forces are frame-invariant as discussed in § 5.3, the 3rd law is frame-invariant.

For brevity, the forces of the 3rd law are often referred to as the "force between the particles" or when we get to systems of particles, the "force between systems" or the "force between the objects". The interpretation of this expression is that particle/system 1 exerts force $\vec{F}_{12}$ on particle/system 2 and particle/system 2 exerts force $\vec{F}_{21} = -\vec{F}_{12}$ on particle/system 1. But since saying this is all the time is tedious, one usually just says the "force between the particles/systems" and knows what one means when one says it.

A secret kept in most intro physics textbooks is that the 3rd law is not always valid even in classical physics (e.g., Goldstein et al. 2002, p. 8). Newton himself didn't know this—but then he didn't know all of classical physics as we now define it.

The exceptions (of which only ones I know involve the magnetic force) almost never turn up in intro physics—and so we don't need to worry about them. We'll just assume the 3rd law is valid usually without mentioning the qualification that it isn't always valid.

In fact, there are replacements for the 3rd law that account for the exceptions (e.g., Goldstein et al. 2002, p. 8), and so the situation is not so scandalous as it first seems. We won't go into those replacements of which I know less than I should.

The nature of reaction forces takes a bit more explication.

With gravity, everything is clear at least in classical physics. The reaction force to a gravitational force is a gravitational force.

On the other hand, the situation with the electromagnetic force is rather complex. For one thing, the electromagnetic force comes in many different manifestations. For another thing, at the macroscopic level, the force and reaction force can be different manifestations: e.g., the elastic force of a solid can be the reaction force to air pressure force. For third thing, the sum of the electric and magnetic force on a particle is inertial frame invariant, but the electric and magnetic forces are individually not inertial frame invariant. For a fourth thing, the 3rd law is not obeyed in all cases by the magnetic force manifestation of the electromagnetic force as we discussed above.

Fortunately, in simple macroscopic cases, the force and reaction forces can usually be identified without difficulty.

What of the those two other forces: the strong nuclear and weak nuclear forces? They are outside of the classical realm, and so intro physics we don't need to worry about them.

There is probably some conservation law that stands in place of the 3rd law that applies to them and identifies reaction forces—but yours truly is actually ignorant on this point.

6. NEWTON'S THREE LAWS OF MOTION FOR SYSTEMS OF PARTICLES

In this section, we generalize Newton's three laws of motion for classical point particles in order to treat systems of particles.

It's actually pretty easy.

What is a system of particles?

A system of particles is anything with mass that we define to be a system of particles. It could be a single solid object. It could be a sample of fluid. It could be a bunch of objects or a bunch of classical point particles. It could be a single classical point particle too.

We can treat such a system as a single thing and apply the generalized versions Newton's three laws to it.

But we need to introduce the concept of center of mass first.

6.1. Center of Mass

What is center of mass?

The center of mass of system is its mass-weighted mean position. The definition for a system of particles is

$$\vec{r} = \frac{\sum_i m_i \vec{r}_i}{m} \; , \tag{5}$$

where $\vec{r}$ is the center of mass, each particle is identified by index i, m_i is the mass of particle

i, m is the mass of the whole system, and $\vec{r}_i$ is the position of particle i.

Frequently in dealing with macroscopic systems, one imagines smearing the classical point particles out into a continuum of matter. This is an excellent macroscopic approximation in almost all cases. The exceptions are when one is treating individual quantum mechanical particles (e.g., electrons) in classical approximation. One treats those as classical point particles.

For a continuum of matter, the center of mass definition goes over to an integral for a continuous system:

$$\vec{r}_{\text{cm}} = \frac{\int \rho(\vec{r})\vec{r}\,dV}{m} \;, \tag{6}$$

where $\vec{r}_{\text{cm}}$ is the center of mass adorned here with a subscript "CM" for clarity, $\vec{r}$ is the displacement variable, $\rho(\vec{r})$ is system density, dV is the differential of volume, and the integral is over the whole volume. An integral recall is just a way of summing a continuum of stuff: in this case the differential masses $\rho(\vec{r})\,dV$.

In our formal developments, we continue to use the discrete particle formalism for simplicity. Formal developments with integrals are completely analogous, and so can be omitted without loss of content. One often has to do integrals to evaluate the centers of mass of actual objects.

Now the center of mass is an actual point in space. It has a single definite position given by the center of mass definition. The center of mass can in fact be regarded as a classical point particle. But does it obey Newton's three laws of motion? Yes, and we prove this below in § 6.2.

Since the center of mass is a single definite point in space, there are single definite center-of-mass velocity and acceleration.

The formulae for velocity and acceleration are obtained by differentiation of equation (5).

We assume that all the particles have constant mass here. In lecture, **SYSTEMS OF PARTICLES AND MOMENTUM**, we will consider mass-varying systems briefly.

Starting from equation (5), write down the center-of-mass velocity and acceleration formulae. You have 30 seconds. Go.

To make a complete set, the formulae for center of mass, its velocity, its acceleration are, respectively:

$$\vec{r} \;=\; \frac{\sum_i m_i \vec{r}_i}{m} \;, \tag{7}$$

$$\vec{v} \;=\; \frac{\sum_i m_i \vec{v}_i}{m} \;, \tag{8}$$

$$\vec{a} \;=\; \frac{\sum_i m_i \vec{a}_i}{m} \;, \tag{9}$$

where $\vec{v}_i$ is the velocity of particle i and $\vec{a}_i$ is the acceleration of particle i.

Usually in the future developments when we refer to object or system position, velocity, or acceleration, it will be understood that we mean the object's center-of-mass position, velocity, or acceleration. The context usually makes it clear what is meant. We do this because, in fact, we usually deal with systems and not single particles and it is tedious to always repeat the adjective "center-of-mass".

For clarity, we sometimes do say center-of-mass position, et cetera.

Similarly, to avoid tedium we usually do not—as we've already been doing—adorn center-of-mass quantities with subscripts indicating that they are center-of-mass quantities.

An important point is that the center of mass for a set of subsystems treated as particles located at the subsystem centers of mass is the center of mass of the system of particles. The proof is simple:

$$\vec{r}\,' = \frac{\sum_i m_i \vec{r}_i}{m} = \frac{\sum_i m_i (\sum_j m_{ij} \vec{r}_{ij}/m_i)}{m} = \frac{\sum_{ij} m_{ij} \vec{r}_{ij}}{m} = \vec{r} \;, \tag{10}$$

where $\vec{r}\,'$ is the center of mass for a set of subsystems treated as particles in their own right, m_i is the mass of subsystem i, $\vec{r}_i$ is the center of mass of subsystem i, m_{ij} is the mass of the particle j of subsystem i, $\vec{r}_{ij}$ is the position of the particle j of subsystem i, and $\vec{r}$ is the center of mass of the system of particles. So

$$\vec{r}\,' = \vec{r} \tag{11}$$

as the proof shows.

The above proof shows that as long as you have the centers of mass of some set of subsystems of a system, you can always calculate the system center of mass. This is an immensely useful result since frequently systems are built up of subsystems whose centers of mass are known.

Now for the generalization of Newton's laws of motion for systems of particles.

6.2. The Generalization of the Laws of Motion for Systems of Particles

Say you have a system of particles labeled by index i.

Newton's 2nd law applied to particle i is

$$\vec{F}_{i,\text{net}} = m_i \vec{a}_i \; , \tag{12}$$

where $\vec{F}_{i,\text{net}}$ is the net force on particle i, m_i is the mass of particle i, and $\vec{a}_i$ is the center-of-mass acceleration of particle i.

Since nothing forbids us, we now sum the 2nd law applications over all particles i to get

$$\vec{F}_{\text{net}} = \sum_i \vec{F}_{i,\text{net}} = \sum_i m_i \vec{a}_i = m \frac{\sum_i m_i \vec{a}_i}{m} = m\vec{a} \; , \tag{13}$$

where $\vec{F}_{\text{net}}$ is the net force on the system and m is the system mass, and where we have used the center-of-mass acceleration equation (9) from § 6.1.

The fact that the center-of-mass acceleration formula turns up naturally in our development is the whole reason for defining center of mass.

You might feel queasy for a moment by the fact that the forces in the summation act on particles at different points in space and that the accelerations are at different points in space. But there is nothing mathematically wrong with summation. Remember, forces and accelerations actually have, respectively, their extents in abstract force and acceleration spaces.

Now the fact that the net force in equation 13) includes all the internal forces would be a major complication to application of equation 13) in accounting for motion if we had to actually deal with those internal forces. We don't as we will now show.

Nothing also forbids us from decomposing the forces on each particle into external and internal contributions. Thus,

$$\vec{F}_{i,\text{net}} = \vec{F}_{i,\text{net,ext}} + \vec{F}_{i,\text{net,int}} \ . \tag{14}$$

The external contribution $\vec{F}_{i,\text{net,ext}}$ is for forces arising from sources outside the system and the internal contribution $\vec{F}_{i,\text{net,int}}$ is for forces arising from other particles inside the system. Now

$$\vec{F}_{i,\text{net,int}} = \sum_{j, j \neq i} \vec{F}_{ji} \ , \tag{15}$$

where $\vec{F}_{ji}$ is the force of particle j on particle i. And now

$$\vec{F}_{\text{net}} = \sum_i \vec{F}_{i,\text{net}} = \sum_i \vec{F}_{i,\text{net,ext}} + \sum_i \vec{F}_{i,\text{net,int}} = \sum_i \vec{F}_{i,\text{net,ext}} + \sum_{i,j, j \neq i} \vec{F}_{ji} \ . \tag{16}$$

Consider general particles k and ℓ in the system. By the Newton's 3rd law for particles, we must have $\vec{F}_{\ell k} = -\vec{F}_{k\ell}$. Thus all the internal forces cancel out pairwise in equation (16). Note we are assuming the 3rd law is valid for the particles. As we noted in § 5.4, the 3rd

law is **NOT** always valid in classical physics. But we won't consider those cases. They can be dealt with and they are not important for most ordinary macroscopic systems.

So we find

$$\vec{F}_{\text{net}} = \sum_i \vec{F}_{i,\text{net}} = \sum_i \vec{F}_{i,\text{net,ext}} = \vec{F}_{\text{net,ext}} \; , \tag{17}$$

where $\vec{F}_{\text{net,ext}}$ is the net external force on the system.

So the net force on the system is the net external force on the system too.

So we don't have to deal with internal forces in order to treat the center-of-mass motion. Those internal forces do things. They can hold systems (e.g., solids) together and can cause oscillations of the parts of system and internal collisions. But they don't effect the center-of-mass motion. So if we just want to treat the center-of-mass motion, we can neglect the internal forces and the internal motions in a direct sense. In an indirect sense, we can't entirely neglect internal forces and motions as we discuss in § 6.6.

Finally, then

$$\vec{F}_{\text{net}} = m\vec{a} \; . \tag{18}$$

This Newton's 2nd law for a system of particles.

Note that its form is identical to Newton's 2nd law for particle equation (18). The interpretation of the quantities is slightly different: $\vec{F}_{\text{net}}$ is the net force (and net external force) on the system, m is the system mass, and $\vec{a}$ is the center-of-mass acceleration of the system.

Note that the center of mass is particle-like. It's just a point space.

The 1st law for a system of particles is derived by setting $\vec{F}_{\text{net}}$ to zero.

This implies what?

That acceleration $\vec{a} = 0$.

This means what?

It means that the center of mass of the system is in straight-line uniform motion relative to all inertial frames. This is just the same as the 1st law for a particle with the word "system" replacing the word "particle".

What of the 3rd law for a system of particles?

Consider two systems consisting of particles for which the 3rd law holds. The net force of system 1 on system 2

$$\vec{F}_{12} = \sum_{ij} \vec{F}_{ij} \ , \tag{19}$$

where particle i is in system 1, particle j is in system 2, and F_{ij} is the force of particle i on particle j.

Now prove that $\vec{F}_{12} = -\vec{F}_{21}$, where $\vec{F}_{21}$ is the net force of system 2 on system 1. You have 30 seconds. Go

We make use of the 3rd law between the particles to find

$$\vec{F}_{12} = \sum_{ij} \vec{F}_{ij} = -\sum_{ij} \vec{F}_{ji} = -\vec{F}_{21} \ . \tag{20}$$

Thus,

$$\vec{F}_{21} = -\vec{F}_{12} \ . \tag{21}$$

This is Newton's 3rd law for systems of particles.

In form, it is the same as the 3rd law for particles. The interpretation of the quantities is slightly different: $\vec{F}_{12}$ is the net force of system 1 and system 2 and $\vec{F}_{21}$ is the net force of system 2 and system 1.

Recall from § 5.4 that the 3rd law for particles is not in fact always valid even in classical physics and so the 3rd law for systems is not always valid either. But we will not worry

about the exceptions to the 3rd law in either senses. They are negligibly important for most macroscopic systems.

6.3.   Remarks on Newton's Laws for Systems of Particles

A few important remarks need to made about Newton's Laws for systems of particles.

First, recall that accelerations are inertial-frame-invariant and mass and forces are frame-invariant from § 5.3. Thus, Newton's laws for systems of particles are also inertial-frame-invariant. One could also say that we derived them from Newton's laws for particles in a general inertial frame, we must get the same results in any inertial frame. So again Newton's laws for systems of particles are inertial-frame-invariant.

Second, Newton's laws of motion for systems are Newton's laws of motion for classical point particles if the system centers of mass are considered classical point particles with mass equal to the system mass.

So formally there is no great distinction between Newton's laws for classical point particles and systems of particles.

In fact, when one says Newton's three laws of motion, one is almost always talking about Newton's three laws of motion for systems of particles.

In almost all applications, one uses Newton's three laws of motion for systems of particles and not Newton's three laws of motion for particles. After all the classical point particle is an imagined entity that doesn't really exist.

Since it is tedious to say "Newton's three laws of motion for systems of particles" and so one almost never does. One just says "Newton's three laws of motion". Hereafter, we will almost always follow this usage, except when we need to be more explicit for clarity.

Third, what about non-inertial frames? We can't apply Newton's laws of motion to them as stated. Absolutely positively, they works anywhere in classical physics space, but we have to reference the treatment to inertial frames.

But, in fact, we can further generalize Newton's laws of motion for non-inertial frames by introducing non-inertial forces. We do this in the lecture *Newtonian Physics II*.

Fourth, could we not have taken Newton's three laws of motion for systems of particles as axioms? Then the laws of motion for classical point particles would follow as a special case.

Wouldn't this have been better than introducing non-existent classical point particles at the start?

Maybe.

But traditionally, classical point particles are used as a starting point for Newton's three laws of motion. So we've followed tradition.

Also not starting from classical point particles could get us into trouble in dealing with cases where Newton's 3rd law for particles doesn't hold (see § 5.4). Newton's laws for systems of particles won't hold exactly in those cases and we'd have to have other axioms. Starting from classical point particles allows a cleaner treatment of cases where 3rd law fails. Not that we will do that treatment in this course.

Also recall what we discussed in § 5.1. Throughout physics, classical and non-classical, using point particles seems to be the easiest, clearest, most fruitful way of developing fundamental physical laws.

6.4.   Force Laws Needed

Newton's three laws of motion would be of limited use if there was no way to calculate forces independently of the three laws.

For example, you can calculate a net force from a observed acceleration of a system, but then what? Well one could then use that known net force to calculate the acceleration of another system to which the net force would be applied.

For another example, you can calculate some collisional behavior among systems using Newton's laws of motion and their corollary the conservation of momentum when no net forces external to the collision act.

But these examples are limited and don't allow you to predict the whole future or past behavior of a system from initial conditions. Both theoretically and practically this unsatisfactory.

One needs force laws that give you force in terms of other physical variables.

In §§ 10, 11, 12, and 13, we introduce some important force laws that allow the analysis of the motions of many systems.

6.5.   Non-Center-of-Mass Motion

We should emphasize that the 2nd law equation (18) $\vec{F}_{\mathrm{net}} = m\vec{a}$ for system of particles tells us nothing about the internal motions of the systems.

Any general internal motion is allowed subject to the constraint on the center-of-mass motion. For example, the system could be rotating and/or oscillating in some complex way.

It takes a more detailed analysis of the system to understand the internal motions.

In general, we would need to analyze the position of application of external forces on the system and the internal forces to understand all the motions of a system.

In general, such an analysis would be immensely difficult.

So we won't ever try anything immensely difficult.

But we will treat some systems where the internal motions are analyzed.

For example, systems consisting of an ideal rope with attached masses in § 12. We will get some more experience with internal motions in later lectures.

For example, in the lecture *ROTATIONAL DYNAMICS*, we will consider the rotational motion of a rigid body about a single fixed axis.

6.6. Separation of Center-of-Mass Motion and Internal Motions

The 2nd law $\vec{F}_{\text{net}} = m\vec{a}$ for systems of particles actually suggests greater simplicity than is the case.

Even if we don't analyze internal motions, which external forces act on a system will in general depend on the structure of the system and its orientation in space.

For a simple example, throw a ruler at a chair. Whether the ruler hits the chair or not depends on the relative orientation of ruler and chair in general. Thus, whether the chair exerts an external force on the ruler is dependent on the ruler's structure and the time evolution of that structure.

So, in fact, Newton's 2nd law for a system of particles does not really allow a complete separation of the center-of-mass motion and the internal motions.

But in many examples, the internal motions that affect the center-of-mass motions are

easily handled. Usually, the idealized setup of an example just ignores such complications. For example, in simple examples of pushing a block across a frictionless horizontal surface with a constant force, we never specify where the force acts on the block or if the block is rotating. We usually implicitly assume the block is not rotating. In fact, unless we push it in just the right way, it would rotate. Everyday life experience tells us this without deep analysis. That analysis gets into rotational dynamics which we do a bit of in the lecture *ROTATIONAL DYNAMICS*.

6.7.   Zero-Mass Systems

What if a system has zero mass?

Well there actually are no real zero-mass systems in Newtonian physics. So there is no in-principle problem in treating them in Newtonian physics.

However, we often use zero-mass systems as an ideal limit for negligibly small mass systems. The usual rule is that such systems must have zero net force on them and no matter what their acceleration is.

But how do we know what the acceleration of a zero-mass system is? Well we can't if it is just isolated system. But if it is a subsystem of a larger system with non-zero mass and is constrained to accelerate with that larger system, then that determines the zero-mass system's acceleration. The subsystem is constrained by some constraint forces.

This treatment is the correct limiting treatment. Consider a small-mass subsystem of a larger system and constrained to accelerated with it: i.e., its center of mass is constrained to accelerate with the center of mass of the larger system. The larger system has some acceleration determined by the net force acting on it. The net force causing the acceleration of the subsystem is small. In the limit that the subsystem mass goes to zero, the larger

system goes to a finite acceleration that the subsystem is constrained to share. The net force on the subsystem goes to zero as its mass does in obedience to the 2nd law.

Note that the class mantra from § 5.3 holds for zero-mass systems: i.e., we still recite for them

"Newton's 2nd law is always true and it's always true component by component."

An example of zero-mass subsystem is the ideal rope (which we discuss in § 12). It is usually (but not always) defined to have zero mass and this a valid approximation as long as it is part of a large system with mass.

There are systems (e.g., light or electromagnetic radiation) which have no defined treatment in formal Newtonian physics. One can treat them in Newtonian physics with ad hoc hypotheses. "Ad hoc" means for a particular purpose. So ad hoc hypotheses are a special hypotheses which often are only used to fix up a theory to make give the right predictions in special cases. Ad hoc hypotheses can be justified by more general theories. For example, they can be used to treat electromagnetic radiation in Newtonian physics to some accuracy.

6.8. Units of Force

Since Newton's 2nd law for particles equation (1) is exact in classical physics, it gives the exact relation between the units of force and the units of mass and acceleration. The units of force are

$$\text{unit}[F] = \text{unit}[ma] = \text{kg\,m/s}^2 \ , \tag{22}$$

where unit[ ] is my own idiosyncratic unit function. The unit of force is a derived unit: it is defined exactly in terms of the base units of kilogram, meter, and second.

The unit of force is given it's own name special name and symbol. These are, respec-

tively, newton and N. Thus,

$$N = \mathrm{kg\,m/s^2} \ . \tag{23}$$

It turns out that for everyday purposes, the newton is a bit small. It's only about 1/5 or 1/4 of a pound (the US customary unit of force with abbreviation lb). Unfortunately, the newton straddles the line between being 1/5 and 1/4 of a pound and which you use depends on how you round: I round to 1/5. We have the following relationships with the pound:

$$1\,\mathrm{N} \ = \ 0.224808943\ldots\,\mathrm{lb} \approx 0.2\,\mathrm{lb} \ , \tag{24}$$

$$1\,\mathrm{lb} \ = \ 4.4482216152605\,\mathrm{N} \approx 4\,\mathrm{N} \ , \tag{25}$$

where $1\,\mathrm{lb} = 4.4482216152605\,\mathrm{N}$ is exact in the modern definition of the pound (Wikipedia: Pound-force).

Actually, in some contexts, the newton seldom appears explicitly. One just uses MKS units and the unit force is the MKS unit of force without bothering to name it.

But in intro physics, we use the newton all the time.

7.   CALCULATING CENTER OF MASS

In this section, we will just give some examples of calculations of centers of mass.

7.1.   Example: Particles on a Line

Recall the center-of-mass formula

$$\vec{r} = \frac{\sum_i m_i \vec{r}_i}{m} \ . \tag{26}$$

Write down the x component center-of-mass formula. You have 30 seconds. Go.

Now say we have three particles on the x axis: 1 at $x_1 = 1$ with mass $m_1 = 1$, 2 at $x_2 = 2$ with mass $m_2 = 2$, and 3 at $x_3 = 3$ with mass $m_3 = 3$.

What is the x coordinate center of mass? You have 30 seconds. Go.

The x coordinate center of mass is

$$x = \frac{\sum_i m_i x_i}{m} = \frac{1^2 + 2^2 + 3^3}{1 + 2 + 3} = \frac{14}{6} \approx 2.333 \ . \tag{27}$$

What is simple average x position of the particles? You have 30 seconds. Go.

The simple average position of the particles is, of course,

$$x_{\text{ave}} = \frac{\sum_i x_i}{\sum_i 1} = \frac{1 + 2 + 3}{3} = 2 \ . \tag{28}$$

The greater mass of particle 3 drives the center of mass position closer to particle 3 than the simple average position.

Since the particles are points on the x axis, their center-of-mass y and z positions are both zero.

7.2. Example: Systems Symmetric in Three Dimensions

Say you have system that exhibited mirror symmetry about three orthogonal planes that meet a point which we can call the geometric center.

Examples of such systems are a uniform density sphere, cube, or cylinder.

Where would you guess the center of mass had to be?

Right.

At the geometric center.

Anything else would break the symmetry and there is nothing in the systems to break the symmetry.

But one can give a concrete proof too.

Consider an object with the specified mirror symmetry planes.

Let's make use of the three-dimensional Cartesian coordinate system.

Put **GEOMETRIC CENTER** the origin of coordinate system.

Align the x-y, x-z, and y-z planes of the coordinate system with the symmetry planes.

First consider the x-y plane and the z component of the center of mass.

For every mass differential element dm at (x, y, z), there is a mirror differential mass dm element at $(x, y, -z)$.

Thus the sum for the center-of-mass z component consists of pairs of terms $z\,dm + (-z)\,dm = 0$. All these pairs of terms cancel pairwise as we say. Thus, center-of-mass z component is $z_{\mathrm{cm}} = 0$.

Notice if there were any break in the mirror symmetry about the x-y plane we would not have a pairwise cancellation in general, and thus would not in general have $z_{\mathrm{cm}} = 0$. Of course, without mirror symmetry about the x-y plane it is still possible that $z_{\mathrm{cm}} = 0$ by some special arrangement of the matter. But there is no simple way to predict that $z_{\mathrm{cm}} = 0$. One would usually have to do a detailed calculation to find z_{cm} and then one would know if it were zero or not.

The same argument for the x-z, and y-z planes can be given for the x-y plane.

Thus, the origin of our system (which is the geometric center) is the center of mass.

We conclude that a system with 3 mirror symmetry planes has its center of mass at its

geometric center

We also conclude that if any of the mirror symmetries are not present for a system, then the center of mass cannot be found without an at least somewhat detailed calculation or by empirical method. There is in fact a simple empirical method to find the center of mass of a rigid object. We present this method in § 7.3.

Question: Where is the center of mass of a hula hoop?

a) At the center.

b) Nowhere since the center is empty space.

c) At the left end of the hula hoop.

Yes, it's (a). The center of mass does **NOT** have to be in the material object.

If I tossed hula hoop right now, it would probably rotate around the center of mass in some fashion (e.g., about the symmetry axis or perpendicular to it or both) depending on how I threw it. But the center of mass at the geometric center would follow a parabolic trajectory if air resistance is negligible.

7.3.   Example: An Irregular Rigid Object

Consider an irregular rigid object.

How does one find it's center of mass?

Well one could work very hard and do a numerical calculation.

One could guess.

With some experience, the guess might not be so bad.

But there is a simple empirical means to find the center of mass.

Hang a rigid object from a free pivot point (i.e., a pivot point about which the object can swing freely) and let it come to rest.

The center of mass must be below the pivot point—which we won't prove here. Thus, there is a line from the pivot point that passes through the center of mass.

Now hang the object from another free pivot point.

Again center of mass must be below the pivot point. Thus, there is another line from the pivot point that passes through the center of mass.

Where those two lines (taken as fixed to the object) intersect is the center of mass.

The proof of this method we leave to the lecture *ROTATIONAL DYNAMICS*.

7.4.   Example: A Non-Rigid System

For a non-rigid system (e.g., a squishy system, flexible system, fluid system), the position of any part of the system relative to the center of mass is variable in general and depends in a complicated way on the motion and external conditions of the system.

So calculating where the center of mass located relative to all the parts of the system is in general tough.

But take a ragged old sweater for example and toss it. With negligible air resistance, the center of mass follows a parabolic trajectory no matter what the parts are doing. What the parts are doing in detail involves a complex calculation—which we will never do in this class.

Although calculating where the center of mass is difficult for non-rigid systems. Some

systems can actively control where it is. For example, those non-rigid systems human beings learn to control where their center of mass is even if they don't know what it is.

If you are just standing upright, your center of mass is probably somewhere in your lower torso. If you flex your center of mass can move outside of your body. In fact, gravity would torque you over if you didn't keep your center of mass essentially over your feet. So bending over toward the ground, either your bottom goes back or your hands must rest on the ground. A center of mass over a free pivot point is an unstable equilibrium as we'll discuss in the lecture *ROTATIONAL DYNAMICS*. Humans who often approximate an object supported by a free pivot point—especially when standing on one foot—actively adapt to maintain the center of mass over the pivot point or keep our balance as we call it. Animals who stand on four legs avoid a lot of active balancing work by not approximating an object supported by a free pivot point.

In some sports, moving the center of mass out of the body is critical. A main example is the high jump. All a jumper's launch speed will put their centers of mass on a parabolic trajectory. But they can go over a bar that is higher than highest point on the trajectory their center of mass reaches. They curl around the bar putting their center of mass below their bodies. This is an in-flight tactic that is somewhat independent of how much launch speed they can generate. Their center of mass can be up to 20 centimeters below the bar (Wikipedia: Fosbury Flop). The modern optimum jump is the Fosbury Flop where the jumper goes over the bar head first with their backs toward the ground and arched around the bar. The Fosbury Flop wasn't possible before soft foam mats were introduced for landing since you tend to land on your head.

8. FREE BODY DIAGRAMS AND SIMPLE EXAMPLES USING
NEWTON'S THREE LAWS

In this section, we do some simple examples using using Newton's three laws of motion. These examples are all without force laws. We just assume forces. In § 10 and following sections, we introduce some important force laws that allow the analysis of the motions of many systems.

Before we turn to our examples, first we'll introduce a useful diagram for analyzing systems to which Newton's laws are applied—the free body diagram.

8.1. Free Body Diagrams

Free body diagrams are often helpful in analyzing simple force problems.

On a purist's free body diagram, the body is represented by a point at the center of the diagram and the forces that act on the body are drawn with the tails of their vector arrows on the point (e.g., Halliday et al. 2001, p. 77). The point is usually thought of as the body's center of mass. No axes or components of the forces are drawn.

Do **NOT** draw the forces the body exerts. That tends to create a mess of seemingly canceling forces.

Those of us who are less than pure often draw the object schematically and add axes. We still often draw the point to represent the center of mass. Sometimes we draw forces with their tails on the center of mass and sometimes we draw them with their tails on the points on the body where the forces act. However impure, these additions are often mentally helpful.

Examples help to see the how free body diagrams are useful in analyzing force problems.

8.2.   Example: Hockey Puck on Ice

We have a hockey puck on a horizontal plane of frictionless ice.

In the vertical direction, there is no net force and no motion.

Fig. 1.— A free body diagram for a hockey puck acted on by two forces in the horizontal plane.

Thus, the problem is entirely in the two-dimensional horizontal plane of ice. We setup a set of the usual Cartesian x-y axes on the plane.

The puck has mass $0.30\,\mathrm{kg}$ and two forces act on it.

Force 1 has magnitude $5.0\,\mathrm{N}$ and points at $-20°$ from the x axis. Remember we conventionally measure angle counterclockwise positive from the x axis.

Force 2 has magnitude $8.0\,\mathrm{N}$ and points at $60°$ from the x axis.

Figure 1 is an impure free body diagram for the system.

What is the acceleration of the puck in magnitude-direction format?

You will need to use trig to vector components of the forces.

Then remember the class mantra: $F = ma$ is always true and it is always true component by componet.

You have 2 minutes working individually or in groups or individually. Go.

To find the acceleration, we need to find the components of the acceleration.

To find the components of the acceleration, we need to find the components of the forces and sum them to find the net force components.

Behold:

$$
\begin{aligned}
F_x &= F_1 \cos\theta_1 + F_2 \cos\theta_2 = 8.698 \\
a_x &= \frac{F_x}{m} = 28.99 \\
F_y &= F_1 \sin\theta_1 + F_2 \sin\theta_2 = 5.218 \\
a_y &= \frac{F_y}{m} = 17.39 \\
a &= \sqrt{a_x^2 + a_y^2} = 34\,\mathrm{m/s^2} \\
\theta &= \tan^{-1}\left(\frac{a_y}{a_x}\right) = 31° \, ,
\end{aligned}
\tag{29}
$$

where we have used only MKS units, and so don't need to keep track of them explicitly and where we only round off to significant figures in the last two expressions.

9.   NEWTON'S THREE LAWS OF MOTION AND INERTIAL FRAMES

Newton's three laws are inertial-frame invariant as discussed in §§ 5.3. 5.4, and 6.3. This means that they have the same form in any inertial frame and can be applied in the same way in any inertial frame.

As well as having the same form, the values of the quantities that enter the three laws are inertial-frame invariant: force, mass, and acceleration. In fact, force and mass are frame-invariant whether the frame is inertial or not. We presented the arguments for these statements in § 5.3. 5.4, and 6.3.

Note that the components of the force and acceleration vectors do vary with frames in general, but they themselves don't: i.e., their magnitudes and directions don't vary. If one changes the orientation of the axes, the components of force and acceleration must change.

In this section, we will consider what the inertial frame invariance of Newton's laws implies.

Fig. 2.— Displacement in inertial frames 1 and 2.

First, consider displacement, velocity, and acceleration. They are all frame dependent in general. Say we had frames 1 and 2 where $\Delta\vec{r}_{12}$ is the displacement vector from the origin of frame 1 to the origin of frame 2. Euclidean geometry tells us that the displacement of particle in frame 1 $\vec{r}_1$ is related to that in frame 2 by

$$\vec{r}_1 = \vec{r}_2 + \Delta\vec{r}_{12} \ . \tag{30}$$

The situation is illustrated in Figure 2.

Differentiating displacement with respect to time, we find that the transformations between the frames for velocity, and acceleration. To summarize, all the transformations are

$$\vec{r}_1 \ = \ \vec{r}_2 + \Delta\vec{r}_{12} \ , \tag{31}$$

$$\vec{v}_1 \ = \ \vec{v}_2 + \Delta\vec{v}_{12} \ , \tag{32}$$

$$\vec{a}_1 \ = \ \vec{a}_2 + \Delta\vec{a}_{12} \ . \tag{33}$$

These relationships are actually Galilean transformations of the quantities.

It turns out that the Galilean transformations are only valid in the classical limit where relative velocities are small compared to the vacuum light speed and gravity fields relatively weak. The correct transformations for relativistic speeds are the Lorentz transformations. For strong gravity, one needs general relativistic transformations. The key difference from Lorentz and general relativistic transformations is that for the Galilean transformations one assumes that time flow and length are reference frame invariant. We did this implicitly, in the derivation of transformations above.

As mentioned in § 2, Newton himself wondered if the time flow had to be frame invariant, but it was the simplest hypothesis that it was and nothing people knew until end of the 19th century caused any doubt. In § 9.1 we discuss the Galilean transformations further.

Note some folks restrict the meaning of Galilean transformations to the case where there is no acceleration between the frames.

If we restrict ourselves to inertial frames which are unaccelerated with respect to each other, equations (33) reduce to

$$\vec{r}_1 \;=\; \vec{r}_2 + \Delta\vec{r}_{12} \; , \tag{34}$$

$$\vec{v}_1 \;=\; \vec{v}_2 + \Delta\vec{v}_{12} \; , \tag{35}$$

$$\vec{a}_1 \;=\; \vec{a}_2 \; . \tag{36}$$

So acceleration is inertial-frame invariant.

Given the above assumptions about inertial-frame invariance of the force, mass, and acceleration, it follows that Newton's three laws of motion are inertial frame invariant just as we said above.

Actually, the inertial-frame invariance of Newton's laws is part of their statements. So really the laws imply the force, mass, and acceleration are inertial-frame invariant.

What does inertial-frame invariance mean?

Well many things, but **TWO** important consequences come to mind first of all.

FIRST, for a given particle (which could be the center of mass of a system), frame 1 and frame 2 observers would calculate the same acceleration and calculate velocity and position that differ only by some calculable additional terms.

To see this say a frame 2 observer (i.e., an observer who uses frame 2 as the frame of reference for calculations) calculates motions for the particle using known forces and the 2nd law, and finds $\vec{a}_2$, $\vec{v}_2$, and $\vec{r}_2$. Now a frame 1 observer (i.e., an observer who uses frame 1 as the frame of reference for calculations) calculates motions for the particle using known same forces and the 2nd law, and finds $\vec{a}_1$, $\vec{v}_1$, and $\vec{r}_1$.

Well $\vec{a}_1 = \vec{a}_2$ according to the 2nd law—which is indeed true.

What are the relationships between $\vec{v}_2$, and $\vec{r}_2$. and $\vec{v}_1$, and $\vec{r}_1$?

Integrating $\Delta \vec{a}_{12} = 0$, we find

$$\Delta v_{12} = \Delta v_{12,0}$$

and

$$\Delta r_{12} = \Delta v_{12,0} t + \Delta \vec{r}_{12,0} , \qquad (37)$$

where the subscript zero indicates time zero values. Thus, the relationships are

$$\vec{r}_1 = \vec{r}_2 + \Delta v_{12,0} t + \Delta \vec{r}_{12,0} ,$$
$$\vec{v}_1 = \vec{v}_2 + \Delta v_{12,0} ,$$
$$\vec{a}_1 = \vec{a}_2 . \qquad (38)$$

These predictions, of course, are verified since Newton's 2nd law is well verified.

Frequently, the origins of the two frames related by equations (38) are chosen to be at the same point and time zero. This means that $\Delta \vec{r}_{12,0} = 0$.

The **SECOND** consequence of the inertial-frame independence of the 2nd law is that identical particle trajectories relative to different inertial frames are obtained for identical conditions relative to those frames. The identical conditions include identical initial position and velocity relative to those frames and identical applied forces.

This sounds a bit abstract, but your whole life experience confirms it.

For example, play catch on the ground and play catch in a unaccelerated train and everything behaves the same way. Both frames are inertial to high accuracy.

You'd be shocked if it didn't.

On the other hand, throw a ball on a rotating playground merry-go-round (which is a non-inertial frame) and the ball follows a curved path relative to the merry-go-round frame

when viewing the path projected onto the horizontal plane. Throwing in an identical manner in an inertial frame and the ball follows a straight-line path when viewing the path projected onto the horizontal plane. All the conditions relative to the frames can be chosen to be the same, but you get different outcomes. The merry-go-round frame is not an inertial frame, and so there is no violation of Newton's 2nd law. Note that relative to the ground (which is an inertial frame to high accuracy), the ball thrown on the merry-go-round does follow a straight line path when viewing the path projected onto the horizontal plane.

The above example leads to the general conclusion that in non-inertial frames there can be accelerations without net forces. We take up the subject of accelerations in non-inertial frames in the lecture *Newtonian Physics II*.

9.1. Inertial Frames and Physical Law: Optional

It is an axiom of physics, Newtonian and modern, that physical laws—except perhaps those applying to the universe as a whole—should have the same formulae in all inertial frames. In physics jargon terms, the formulae should be invariant under transformations between inertial frames.

The inertial-frame-invariance axiom means that the identical setups in inertial frames lead to identical results in those inertial frames.

Newton's 2nd law obeys this axiom as we discussed in § 9 and so does Newtonian physics in general. But this obedience is based on the assumption of the Galilean transformations for quantities in going from one frame to another. The Galilean transformations assume that forces are frame invariant and the kinematic variables (i.e., displacement, velocity, and acceleration) differ between frames only by additive terms which are the relative displacement, velocity, and acceleration between the frames. The Galilean transformations also assume

that time flows the same in all frames of reference and that length is a frame invariant quantity.

Both the inertial-frame-invariance axiom and assumption of the Galilean transformations seemed perfectly natural from the time of Newton to circa 1905.

But in the 19th century there arose a paradox.

Classical electromagnetism formulated in the 1860s by James Clerk Maxwell (1831–1879) was **NOT** invariant under Galilean transformations between inertial frames. It was thought that this meant there was some error in classical electromagnetism or that it was only valid in the rest frame of a medium called the luminiferous ether. The luminiferous ether was the medium of electromagnetic (light) wave propagation. Since ordinary mechanical waves propagate in media, it seemed reasonable that light waves should too. But the luminiferous ether seemed to have no properties other than satisfying the demand that there be a medium for light wave propagation. You couldn't touch it or see it or anything. That lack of other properties seemed odd.

Now classical electromagnetism turned out to be so accurate and the luminiferous ether so hard to find that questions about the status of both things were raised.

The situation was clarified in 1905 when Albert Einstein (1879–1955) introduced special relativity.

In special relativity, the transformations between inertial frames are **NOT** the Galilean transformations, but the Lorentz transformations. We will not present the Lorentz transformations here. But the Lorentz transformations lead to the results that time flows differently in different frames and that length is a frame dependent quantity. Both these relativistic effects are very strange in our everyday life understanding of things. In fact, special relativistic effects in general are small for everyday speeds and only become significant when

relative speeds approaching the vacuum speed of light are important. This is why we never notice them in everyday life and why science didn't begin to to notice them before the 1880s. The Lorentz transformations reduce to the Galilean transformations in the classical limit of relative velocities much less than the vacuum light speed.

Under the Lorentz transformations, classical electromagnetism obeys the inertial-frame-invariance axiom.

But Newtonian physics doesn't—except in the classical limit of the Lorentz transformations (i.e., when they reduce to the Galilean transformations).

So Newtonian physics was **NOT** exactly right. It is only the low velocity limit of special relativistic physics. Other limitations on Newtonian physics soon appeared in the 20th century. But as discussed in the lecture *INTRODUCTION TO INTRODUCTORY PHYSICS* and § 2, Newtonian physics remains valid and useful in the classical realm.

10. GRAVITY NEAR THE EARTH'S SURFACE

As discussed in § 6.4, Newton's three laws of motion would be of limited use if there was no way to calculate forces independently of the three laws.

One needs force laws that give you force in terms of other physical variables.

Gravity has one of the best examples of a force law.

The general gravitational force law is

$$\vec{F} = m\vec{g} , \tag{39}$$

where m is the mass of a point particle and $\vec{g}$ is the gravitational field which in general is position and time dependent. For a system of particles, the general gravitational force law

is

$$\vec{F} = \sum_i m_i \vec{g}_i \ , \tag{40}$$

where the sum is over all the particles i, m_i is the mass of particle i, and g_i is the gravitational field at the location of particle i and the time of evaluation. If the gravitational field is uniform over the system, then equation (40) yields

$$\vec{F} = \sum_i m_i \vec{g}_i = \left(\sum_i m_i \right) \vec{g} = m\vec{g} \ , \tag{41}$$

where m is the mass of the system and $\vec{g}$ is the uniform gravitational field.

The gravitational field is caused by all the mass in the universe and depends on its amount and arrangement. In general, the gravitational field is complicated to evaluate, but it is fairly simple for spherically symmetric bodies. The Earth is approximately a spherically symmetric body. We'll investigate the general law for the gravitational field in the lecture *GRAVITY*.

Here we will only give the special case of the law for gravity near the Earth's surface:

$$\vec{F} = -mg\hat{y} \ , \tag{42}$$

where m is the system mass, g is the magnitude of the gravitational field approximated as uniform, and $\hat{y}$ is a unit vector point pointing upward from the surface. The gravitational field near the Earth's surface is due to the Earth's mass.

The gravitational field near the Earth's surface is uniform to high accuracy as discussed in the lecture *ONE-DIMENSIONAL KINEMATICS*. Its magnitude varies only by $0.5\,\%$ depending on latitude and altitude in the near-Earth surface environment. The direction to high accuracy is toward the center of the Earth. Locally, the direction of the gravitational field is usually taken to define the downward direction or unit vector $-\hat{y}$. So the gravitational force points down always in the near-Earth surface environment by definition.

In this course, we usually just take g to have fiducial value $9.8\,\mathrm{N/kg}$, where we note that $\mathrm{N/kg} = \mathrm{m/s^2}$.

The quantity g is also our old friend the acceleration due to gravity when a body is in free fall: i.e., free fall in the sense that gravity is the only external force acting on the body.

We'll prove this right now by analyzing the motion of a body under gravity only.

We only need to consider the vertical direction since there are no forces or motion in the other directions.

The free body diagram is trivial: see Figure 3.

The net force in the y direction is $-mg$. Thus, 2nd law or $F = ma$ for the y direction becomes

$$-mg = ma \tag{43}$$

which we solve immediately to get the y direction acceleration

$$a = -g \; . \tag{44}$$

Mass has canceled out of the result. Recall the acceleration is the center-of-mass acceleration for the object. We can say nothing about the internal motions of the object from our simple analysis.

Fig. 3.— A free body diagram for a system acted only by gravity near the Earth's surface.

The mass independence of the acceleration due to gravity acting alone is the Galileo result. No matter what the mass, all objects near the Earth's surface accelerate downward with the same constant acceleration of magnitude g to good approximation when air resistance can be neglected.

This remarkable result originates in the fact that gravity is a force proportional to mass as equations (39) and (41) show. This leads to the cancellation of mass in the formula for acceleration due to gravity.

The appearance of mass in the gravitational force law is independent of its property as the resistance to acceleration.

Mass in a sense is like the charge of the gravitational force just as positive and negative charges are the charges of the electric force. There is a major difference though. Mass only comes in one charge flavor and all mass attracts other mass. There are two flavors of electric charge and like charges repel and unlike charges attract.

In Newtonian physics, the exact equivalence of mass in $F = ma$ (which is called inertial mass) and mass in the gravitational force law equation (39) (which is called gravitational mass) is a coincidence. In general relativity, the equivalence of inertial mass and gravitational mass is an ingredient in the axioms of the theory (e.g., Lawden 2002, p. 128–129).

But people have wondered if the equivalence is actually exact. Say inertial mass m_{in} and the gravitational mass m_{gr} were different. Then $F = ma$ would lead to a free-fall acceleration formula

$$a = -\frac{m_{gr}}{m_{\mathrm{in}}}g \ . \tag{45}$$

If the gravitational mass depended on the material of the object, then the acceleration due to gravity would vary with material. Many careful experiments have found no such variation. For the time being, the equivalence of inertial mass and gravitational mass seems

well founded.

10.1.   Example: Free-Fall Time

Recall free fall is the when an object is accelerated by gravity alone in one meaning of the expression free fall.

Since the acceleration due to gravity acting alone is g near the Earth's surface, free-fall time t for a given displacement y downward near the Earth's surface is the same for all objects starting from rest.

What is the free-fall time t given y and g? This is a kinematics problem. You have 1 minute working individually or in groups. Go.

Well 2nd kinematic equation in this case is

$$y = \frac{1}{2}gt^2 \tag{46}$$

which we solve immediately to get

$$t = \sqrt{\frac{2y}{g}} \; . \tag{47}$$

This, of course, is another result discovered by Galileo.

Of course, the free-fall times of actual objects varies from our formula due to air drag or measurement errors. But in the ideal limit of no air drag and no errors—which can be approached very closely in fact—actual free-fall times approach our result.

11.   NORMAL FORCE

The normal force is the component perpendicular to a surface that contact force of the surface exerts to resist compression. The normal force points outward from the surface.

"Normal" in this context means perpendicular.

One usually only thinks of normal force for solid surfaces. For fluid surfaces, one thinks of pressure forces and surface tension forces.

The normal force is actually an elastic force. The normal force can in principle be calculated from the elastic properties of the material and the degree of compression the surface undergoes.

But that is too hard for intro physics courses.

In intro physics courses, we only consider the ideal contact force for an ideal solid surface.

An ideal solid surface is perfectly rigid and exerts an outward force to resist all compression. The normal component of this force is normal force. There can be an parallel component of the contact force, but it is some **SURFACE FORCE** that acts parallel to the surface that provides the parallel component in a direct sense. The internal forces that resist compression or stretching keep the surface of the solid from sliding. We won't treat those internal forces in this course.

One surface force that acts parallel to the surface is friction. Friction is the only contact force parallel to the surface that we will consider.

Because the surface of an ideal solid is perfectly rigid, we have **NO** intrinsic formula for the (ideal) normal force.

We know it only turns on to resist compression and is always outward from the surface. It never attracts. One can, of course, invoke attractive sticky forces on a surface, but those are not the normal force.

We immediately know the effect of the ideal normal force on a system. Forces whose effect is known immediately are called constraint forces in physics (e.g., Goldstein et al. 2002,

p. 13).

Constraint forces (like the ideal normal force) for which no intrinsic formula is given can only be determined using the 2nd and 3rd laws.

11.1. Normal Force from the 2nd Law

Consider an object in contact with an ideal surface.

Let's just consider the direction perpendicular to the surface with the outward direction being positive. The system is one-dimensional.

From the 2nd law, we know that

$$F_{\text{net}} = ma \tag{48}$$

for the object where we have dropped the vector symbols since the system is one dimensional.

Let's decompose the net force into the normal force F_{nor} and other forces F_{oth}. Thus,

$$F_{\text{net}} = F_{\text{nor}} + F_{\text{oth}} \ . \tag{49}$$

If we solve for F_{nor} assuming all other quantities are known, we get

$$F_{\text{nor}} = ma - F_{\text{oth}} \ . \tag{50}$$

Everything is consistent if $F_{\text{nor}} \geq 0$ and we have obtained the true value from the 2nd law.

Question: What if $F_{\text{nor}} < 0$ (i.e., the normal force points into the surface)?

a) We've used wrong values for $ma - F_{\text{oth}}$.

b) There must be some unidentified force that attracts to the surface.

Either (a) or (b) can be right.

As an example of determining a normal force from the 2nd law, consider a toaster sitting on a table. The free body diagram is in Figure 4.

There is no acceleration of the toaster and, in the vertical direction, only the normal force and the gravitational force act. The 2nd law in the vertical direction gives.

$$F_{y,\text{net}} = F_{\text{nor}} - mg = ma = 0 \; , \tag{51}$$

where m is the toaster mass.

We find then

$$F_{\text{nor}} = mg \; . \tag{52}$$

The normal force on the toaster in this case is equal in magnitude to the gravitational force and points upward.

Absolutely, positively, the normal force does **NOT** always have the magnitude mg. It is in this case and similar cases: an object sitting at rest on a level surface.

11.2. Normal Force from the 3rd Law

This is a very trivial case.

If one exerts an applied force $\vec{F}_{\text{app}}$ into a surface, by the third law one find the normal force is

$$\vec{F}_{\text{nor}} = -\vec{F}_{\text{app}} \; . \tag{53}$$

What if $\vec{F}_{\text{app}}$ is actually outward from the surface?

Well if you are succeeding in applying an outward force, there must be an inward force being exerted on the applier by the surface, but it's **NOT** the normal force.

11.3.   Object on an Inclined Plane

There is an object on an ideal frictionless, perfectly rigid inclined plane. The plane is at angle θ from the horizontal. The free body diagram is in Figure 5.

We let the x direction be parallel to the plane and point down the incline. Along the incline is the only direction the object can move in as we'll explain just below, and so choosing that direction as the x direction simplifies the analysis immensely.

We let the y direction be normal to the plane and increase outward. The y direction must be perpendicular to the x direction for orthogonal coordinates.

From § 5.3, remember the class mantra:

"Newton's 2nd law is always true and it's always true component by component."

Now we apply the 2nd law to the y and x directions for the object.

First note that there can be no motion in the y direction. The normal force completely resists negative motion in the y direction since the plane is an ideal rigid surface. But the normal force only counters the y component of gravity and cannot give an accelerating force in the y direction. The x component of gravity cannot give any y acceleration definitionally. Now if the plane were not ideal, it could have some roughness which would could allow gravity to give a bit of y direction acceleration. The object sliding along could lift a bit if a roughness directs the object away from the surface. But if the object lifts at all from the surface, the normal force turns off completely and gravity pulls the object back to the surface. So our analysis should be pretty good for inclined planes that are not perfectly

Fig. 4.— A free body diagram of toaster at rest on a table. Only the gravitational force and the normal force act on the toaster.

Fig. 5.— A free body diagram of an object on an inclined plane.

planar too.

Since there is no motion in the y direction, there is no acceleration in the y direction. Thus, the 2nd law gives

$$F_{\text{nor}} + F_{y,\text{grav}} = F_{\text{nor}} - mg \cos \theta = 0 \; , \tag{54}$$

where F_{nor} is the normal force which only has a component in the y direction and $F_{y,\text{grav}} = -mg \cos \theta$ is the component of gravity in the direction.

Thus, from the 2nd law, we find the

$$F_{\text{nor}} = mg \cos \theta \; . \tag{55}$$

This result means that the normal force magnitude equals the magnitude of the gravity component in the y direction. The two forces are in opposite directions, of course.

In the x direction where only x component of gravity acts, we find

$$F_{y,\text{grav}} = mg \sin \theta = ma \; , \tag{56}$$

where a is the acceleration down the slope. We find that

$$a = g \sin \theta \; . \tag{57}$$

Remarkably the acceleration is mass independent. This happens when gravity or a component of gravity is the only force accelerating an object in some direction. The mass cancels out of $F = ma$ since gravity is a homogeneous linear function of mass. There are, however, tricky cases where all traces of forces other than gravity have vanished from the acceleration formula, but where acceleration still depends on mass. For example, there is a mass dependence for acceleration for the double-incline-pulley system and Atwood's machine (see § 12.6).

Without looking at the formulae we've just derived, write them down: i.e., write down the formulae for the normal force, the force of gravity in the x direction, and the acceleration in the x direction. Go.

Behold:

$$F_{\text{nor}} = mg\cos\theta \; , \qquad F_{y,\text{grav}} = mg\sin\theta \; , \qquad a = g\sin\theta \; . \tag{58}$$

Note absolutely, positively, these are **NOT** general formulae. They are just formulae for this particular system. But this particular system occurs over and over again in intro physics.

Now say $\theta = 30°$—which is my is my favorite angle for inclined planes since the sine of $\theta = 30°$ is exactly $1/2$ which is easy to make use of. Then

$$a = g\sin\theta = 9.8 \times \frac{1}{2} = 4.9\,\text{m/s}^2 \; , \tag{59}$$

where we have used our fiducial value $g = 9.8\,\text{m/s}^2$.

Say the object starts from rest and slides for $10\,\text{s}$. How far does it go? You have 1 minute working individually or in groups. Go.

From the appropriate constant-acceleration kinematic equation, we find

$$x = \frac{1}{2}(g\sin\theta)t^2 = 245\,\text{m} \; . \tag{60}$$

12.   TENSION FORCE AND TENSION

Tension force is the force that resists expansion of a system.

Usually one thinks of solids as having tension forces, but liquids also have a tension force called surface tension (Wikipedia: Surface tension).

We are **NOT** going to be general.

We'll just consider tension forces in ropes.

Ropes have been technologically important since prehistory because they can be used to **TRANSMIT** forces over distances and around corners.

The transmission is done by the tension forces in the ropes.

Note we say "in" not "of" although a rope does exert tension forces on external objects—which is one of the important features of a rope. But there are also internal tension forces that have to be considered. We cannot treat ropes as point particles since their shape is important in our analysis.

What is tension force and tension.

Say we draw an imaginary cut line through a rope at some point. The tension force one side of the cut line pulls on the other side and vice versa. These tension forces are equal in magnitude by the 3rd law.

The magnitude of the tension forces at any point along the rope is is called the tension (Wikipedia: Tension (physics)).

Does a rope need to have tension?

No.

It only has tension when applied forces are trying to stretch it along its length.

A rope with non-zero tension is often described as being under tension or taut.

12.1.   Ideal Ropes

We will usually only consider ideal ropes (AKA an ideal cord, string, cable, or chain).

An ideal rope has zero thickness, zero mass, and it can't be stretched or compressed.

Sometimes we will allow use ropes that are ideal, except that they have mass in order to have a touch more realism.

Sometimes we let ideal ropes break. We do this if the tention exceeds some limit.

Why do we consider ideal ropes?

Ideal ropes approximate real ropes to some degree. Therefore, ideal ropes gives approximately correct results for real ropes.

Why not then not study real ropes?

Real ropes are too complex for simple analysis. This accords with the usual response in intro physics: real objects are too complex for simple analysis, and so must be left to occasions when you really need to analyzed them and have the tools to do so.

Of course, ideal ropes should have the correct limiting behavior for real ropes. This actually turns out to make ideal ropes somewhat complex in themselves, but still less so than real ropes. For one thing, they **CANNOT** be treated as point particles.

Since ideal ropes are zero-mass objects, all their analyzable motions are constrained to obey the motions dictated by objects with mass that fully obey the 2nd law. They must move at those objects dictate. The net force on an ideal rope and on any part of it while following the constrained motions must be zero. This follows from our discussion in the section on zero-mass systems (§ 6.7). If we allow a net force on zero-mass system, its acceleration is undefined. A constant net force on an object whose mass is going to zero causes the acceleration of the object to got infinity. But if the net force is zero, then finite accelerations with zero mass are consistent with the 2nd law.

What if the motions of the ideal rope are not constrained? We can't then formally

analyze them. But we qualitatively know what real ropes do in such cases and that fills in the blank left by ideal rope analysis.

For main example, if the rope is completely unconstrained and no external forces act on it, then it can be put in an arbitrary shape and it will stay that way—it's floppy.

The tension force of an ideal rope has no intrinsic formula. This is like the ideal normal force.

Therefore, we can only know the tension forces and tension by analyzing the external applied forces on the rope and the tension forces inside the rope itself, and making use of some principle of how the ideal rope forces act.

The ideal rope force principle is that the ideal rope forces can only resist a force, internal or external, that tries to stretch the rope. To be more precise, the only reaction forces the ideal rope can generate are ones that oppose stretching of the rope. The tension force is one of the ideal rope forces. But ideal ropes can also exert normal forces as we'll discuss below.

To keep track of the behaviors of constrained ideal ropes. Let's list the behaviors and the arguments for the behaviors in itemized form. The arguments are reasonably convincing, but not completely so. Complete convincing arguments are more than we want to get into here. The itemized list is:

1. Some region near the ideal point endpoint of a rope must align with an external force applied to the rope at that ideal point endpoint. This result follows from requirement of cancelling forces at each point in the rope and the ideal rope force principle. There must be rope tension force equal and opposite to the applied external force to cancel it at the endpoint. This reaction force on the part of the rope can only be generated by opposing stretching of the rope. Therefore the rope for some region must be aligned with its tension force at the endpoint, and thus with the applied force. The region of

alignment can be tiny. Any other external force applied near the endpoint can cause the rope to bend away form alignment if that external force is not itself aligned with endpoint external force. External forces that are aligned with the rope at the point they are applied we call parallel forces.

There's an aphorism: "You can't push on a rope." Well as it stands, it's not a fully correct statement. What we have just argued is that you can't push on the endpoint of an ideal rope: you can only pull on it.

2. If the only external forces that act on a rope are at the endpoints, then they must be equal and opposite so that the net force on the rope is zero. By item 1, the rope must align with the those two forces at the endpoints. In fact, the rope must aligned with those two forces everywhere. The internal forces in the rope cannot in an unstraight shape since such internal forces would not only oppose stretching of the rope. To keep any segment of the rope having zero net force on it, the tension forces at the two ends of the segment must be equal and opposite. Thus the tension at the two segment ends must be equal Since the segment is general, tension must be equal throughout the rope.

3. If parallel external forces are applied to a straight rope segment, they cause the the tension to vary. Consider an external parallel force applied at a point. The tension on the side of the point opposite parallel force must be higher than that side in the direction of the applied parallel force. This is so that the forces at the point cancel. In general tension increases in the direction of the applied parallel forces.

4. Say there is an external force applied at a point along a rope. The two branches of the rope that extend from the point, must generate two forces whose vector sum cancels the applied external force. These forces can only be generated by opposing stretching, and at least for a small region, the rope branches must be aligned with the canceling forces they generate. Between any two points where external force is applied, the rope must

be straight if no other non-parallel external forces are applied in the region between.

If you one only has external non-parallel forces applied at points, the rope would consist of straight segments. Points where the forces are applied at cusps. The internal forces of the rope are not defined inside these cusp points.

5. What if external forces is applied over continuous region and are everywhere normal to the rope. The rope must cancel them by generating, its own normal forces. It can only due this by bulging away from the applied force direction since the rope can only create a cancelling force by opposing stretching.

The most obvious situation is a rope wrapped around some smooth shaped rigid object. The rope is drawn taut about the object and exerts normal forces inward on the object and the object exterts normal forces out to cancel rope normal forces.

That a rope can generate normal forces should be no surprise. Loops of real ropes can be used to pull on things.

6. The differentially small normal force at each point along the rope points toward the local center of curvature. The center of curvature is the center of the osculating circle (kissing circle) that can be fitted to the curve to 2nd order in a Taylor series sense at the point. For a rope looped around a circular object, the object's edge is the osculating circle and the object's center the center of curvature.

The radius of curvature is the distance from a rope point to its center of curvature. If the rope goes straight, the radius of curvature goes to infinity. The radius of curvature can also go to infinity at isolated points. The normal force per unit length of the rope is zero where the radius of curvature is infinite.

The radius of curvature can become undefined at points. The normal force per unit length of the rope has a discontinuity at these points. This is not are problem for

using ideal ropes since the contribution of a point to the integrated normal force of the rope is zero.

7. The tension along smoothly curved rope is constant as long as no parallel external forces are applied. This is because over differentially small regions the rope is straight and by a previous argument has constant tension. One just extrapolates this constant tension over a finite region. This item actually probably needs a definitive argument.

8. If you have parallel forces along a curved rope the tension force must vary. In general tension increases in the direction of the applied parallel forces just as for the straight rope case.

The above list of behaviors is enough to analyze ideal ropes in the systems of most problems of intro physics.

We give some more details of ideal rope behaviors in § 12.2 below and a fairly full mathematical treatment of ideal rope behaviors in Appendix A.

12.2. Ideal Rope Details: Optional

Here we give some more details about the ideal rope results discussed in § 12.1.

After wrestling with my rigor conscience—and winning—I've decided not to derive key the ideal rope results in the main test of this lecture, but just present. A fairly full mathematical treatment of ideal rope tension, tension forces, and the normal forces is given in Appendix A.

The tension at any point s along the rope is given by

$$T = T_0 - F_{\mathrm{par}}(s) \ , \tag{61}$$

where T_0 is the tension at the start of the rope at $s = 0$ and F_{par} is the integrated applied parallel force per unit length along the rope from the start to s. By the 3rd law, T_0 equals the magnitude of the applied force at the start of this rope. This applied force at the start of the rope is antiparallel to the fiducial direction of s along the rope. It must be to keep the rope taut and be parallel to the region of the rope just near the rope start. The integrated applied parallel force per unit length along the rope from the start to s is given by

$$F_{\text{par}}(s) = \int_0^s f_{\text{par}}(s')\, ds' \, , \tag{62}$$

where $f_{\text{par}}(s)$ it the force per unit length parallel to the rope pointing in the fiducial direction of s along the rope.

Interpreting equation (61) isn't so hard.

Say $F_{\text{par}}(s)$ is zero everywhere. Then the tension is constant and equal to T_0. This means that at the end of the rope the tension is T_0. To keep the rope taut the applied force at the end must have magnitude T_0 and be parallel to the fiducial direction of s along the rope.

Consequently, a taut ideal rope transmits a force of magnitude T_0 from the start to the end if no parallel forces are ever applied to the rope.

What if $F_{\text{par}}(s) \neq 0$?

The tension varies. Positive $F_{\text{par}}(s)$ means the tension is lower than T_0. This just means that the applied force along the rope cancels some of the force applied at the start of the rope and the tension is reduced at s. Negative $F_{\text{par}}(s)$ means the tension is higher than T_0. This just means that some of the applied force along the rope adds to the force applied at the start of the rope and the tension is increased at s to maintain the balance of forces.

Recall all the external forces on the rope cancel. The rope (which is massless recall) can still be accelerated, but it's acceleration is determined by the system that the rope is part

of which has mass.

In our examples, we will usually assume $F_{\mathrm{par}}(s)$ is zero everywhere for simplicity. Thus, our examples have constant tension.

You may ask can tension go negative?

Yes, but then the rope is being compressed rather than stretched. To keep the rope from extreme buckling, one would have to confine it to a narrow pipe. For an ideal rope, the pipe would have to be infinitely narrow.

What of the normal force per unit length of taut rope? The normal force magnitude per unit length exerted by the rope on the curved surface at any general point s is

$$f_{\mathrm{nor}} = \frac{T}{r} \ , \tag{63}$$

where T is the tension at point s, the variable r is the radius of curvature at s, and the normal force per unit length points radially inward toward the center of curvature. The center of curvature is the center of a circle (called an osculating circle: i.e. kissing circle) that approximates the curve at s to 2nd order in a Taylor expansion sense. The normal force per unit length exerted by the curved surface on the rope is equal in magnitude to f_{nor}, but points radially outward by the 3rd law.

Actually finding the radius of curvature and the orientation of the osculating circle in general is hard, but in some cases its easy. Say the rope was wrapped around a circular pulley and was all in one plane. The radius of the pulley is the radius of curvature and the edge of the pulley is the osculating circle.

Note that if T goes to zero or r goes to infinity, then f_{nor} goes to zero. A straight ideal rope exerts no normal force and has no normal force exerted on it.

The normal force per unit length can have discontinuities at points for an ideal rope.

For example if you have two circular arcs of rope that join rotated with respect to each other. Such discontinuities are of no importance in analysis since they contribute nothing to the integrated force that the rope exerts.

12.3. Example: Rope Holding an Object

There is an object of mass m held from the ceiling by a rope. Nothing is moving.

There are no forces in the x direction, and so $a_x = 0$ which we already knew since nothing is moving—nuff said as the comic books use to say.

In the y direction, $F = ma$ gives

$$T - mg = ma_y = 0 \ , \tag{64}$$

where T is the tension force, $-mg$ is the gravity force, and $a_y = 0$ since we know everything is motionless.

So the 2nd law tells us that

$$T = mg \tag{65}$$

in this special case.

12.4. Example: Rope Pulling a Block

A rope is pulling a block on frictionless floor. The rope is horizontal.

In the y direction, we have

$$F_{\text{nor}} - mg = ma_y \ . \tag{66}$$

Since there is no motion in the y direction, $a_y = 0$ which implies that

$$F_{\text{nor}} = mg \ . \tag{67}$$

In the x direction, the only force acting is the rope. Applying $F = ma$ gives

$$T = ma_x \ , \tag{68}$$

where T is the tension force component in the x direction and also the tension since the rope is horizontal.

Say that we measure the tension somehow. Maybe with a fish scale—which is a spring scale that can be used for weighing fish—among other things. We can then solve for a_x.

What is a_x. You have 10 seconds. Go.

It is

$$a_x = \frac{T}{m} \ . \tag{69}$$

12.5. Example: Three Ropes Meet

Two ropes are attached to a ceiling: rope 1 that is at angle θ_1 from the vertical and rope 2 that is at angle θ_2 to the vertical.

Fig. 6.— A three rope system that supports a mass m. The ropes are all ideal ropes. Two ropes hang from the ceiling obliquely. They meet at joint. Rope 3 hangs straight down from the joint and mass is at the end of rope 3.

The two ropes join and a third rope that hangs down is attached at the joint. An object of mass m is attached to rope 3. All the ropes are taut—you can't push on a rope.

The system is in static equilibrium: i.e., nothing is moving.

The three-rope system is illustrated in Figure 6.

Find the tensions in the ropes.

What is T_3 in temrs of known quantities θ_1, θ_2 and m?

You have 30 seconds working individually or in groups. Go.

Well using the 2nd law

$$T_3 = mg . \tag{70}$$

That was easy because it was a one-dimensional problem.

As a zero-mass object, the net force on the joint must be zero as we discussed in § 6.7 no matter what acceleration is since the mass is zero.

From § 5.3, remember the class mantra:

"Newton's 2nd law is always true and it's always true component by component."

We apply the 2nd law in the x and y directions to the joint.

That is you can apply the $F = ma$ to the joint.

You have 1 minute to write down the expressions working individually or in groups. Go.

Behold:

$$\sum_i F_{x,i} = -T_1 \sin\theta_1 + T_2 \sin\theta_2 = 0 , \tag{71}$$

$$\sum_i F_{y,i} = T_1 \cos\theta_1 + T_2 \cos\theta_2 - T_3 = 0 , \tag{72}$$

where we've taken the positive x direction as direction of rope 2.

We have two unknowns T_1 and T_2 in two equations. In principle, we can solve for the unknowns. The only question is what is the most elegant way.

Well

$$
\begin{aligned}
-T_1 + T_2 \frac{\sin\theta_2}{\sin\theta_1} &= 0 \\
T_1 + T_2 \frac{\cos\theta_2}{\cos\theta_1} &= \frac{T_3}{\cos\theta_1} \ ,
\end{aligned}
$$

then add to get

$$
\begin{aligned}
T_2 \left(\frac{\sin\theta_2}{\sin\theta_1} + \frac{\cos\theta_2}{\cos\theta_1} \right) &= \frac{T_3}{\cos\theta_1} \\
T_2 &= \frac{T_3/\cos\theta_1}{\sin\theta_2/\sin\theta_1 + \cos\theta_2/\cos\theta_1} \\
T_2 &= \frac{T_3 \sin\theta_1}{\sin\theta_2 \cos\theta_1 + \cos\theta_2 \sin\theta_1} \\
T_2 &= \frac{T_3 \sin\theta_1}{\sin(\theta_1 + \theta_2)} \ .
\end{aligned}
\tag{73}
$$

Now

$$
T_1 = T_2 \frac{\sin\theta_2}{\sin\theta_1} = \frac{T_3 \sin\theta_2}{\sin(\theta_1 + \theta_2)} \ .
\tag{74}
$$

So the final analytic solutions are

$$
T_1 = \frac{mg \sin\theta_2}{\sin(\theta_1 + \theta_2)} \ , \qquad T_2 = \frac{mg \sin\theta_1}{\sin(\theta_1 + \theta_2)} \ , \qquad T_3 = mg \ .
\tag{75}
$$

Say, $mg = 122\,\text{N}$, $\theta_1 = 53°$, $\theta_2 = 37°$, then

$$
T_1 = 73.4\,\text{N} \ , \qquad T_2 = 97.4\,\text{N} \ , \qquad T_3 = 122\,\text{N} \ .
\tag{76}
$$

Conveniently, $\theta_1 + \theta_2 = 90°$ in this case, and so $\sin(\theta_1 + \theta_2) = 1$.

12.6. Example: Double-Incline-Pulley System and Atwood's Machine

You have double incline: i.e., a two-dimensional pyramid. The inclines are frictionless.

Incline 1 is at angle θ_1 from the horizontal. A block 1 of mass m_1 is on the incline.

Incline 2 is at angle θ_2 from the horizontal. A block 2 of mass m_2 is on the incline.

The blocks are joined by a taut ideal rope that runs parallel to the inclines and is looped over an ideal pulley. The rope and pulley have a no-slip condition between them: i.e., there's no slippage between them. The ideal pulley has no mass and accelerates with zero applied force maintaining the no-slip condition. Therefore, the rope exerts no parallel force (i.e., force component parallel to any rope segment) on the pulley since pulley will accelerate with any. But if the rope exerts no parallel force on the pulley, then the pulley by the 3rd law exerts to parallel force on the rope. This means that the rope tension is constant throughout its length.

The system is a not a system that can be treated as a single particle. But it's not so complex.

We want to solve for the acceleration components of the blocks along the inclines.

We need to set some conventions.

We give each block its own coordinate system. Experience tells us this is the good way.

We take up incline as the positive x direction for m_1 and down the incline as the positive direction for m_2.

Since rope is taut, the we have $a_1 = a_2$. Let's just call this value a for short.

Note the acceleration vectors are **NOT** equal since they are different directions, but the size of their x components relative to their own separate coordinate systems are equal because of the setup.

There is no motion in the y directions, and so we don't need to consider the y directions. Why is this? The inclines are rigid, and so there can be no motion into the inclines. If there

were any perturbation causing the blocks to lift off the inclines, gravity would pull them back. Such perturbations are then always damped out if small and here they are zero. So no motions in the y directions.

In the x directions, we have to apply $F = ma$.

This will give us equations of motion. The expression of "equation of motion" is used in various ways. I usually use it to mean a particular application of $F = ma$.

So apply $F = ma$ to the x directions to get the equations of motion for the two equations of motion, one for each block. Remember what forces act on the blocks?

You have 30 seconds working individually or in groups. Go.

Behold:

$$T - m_1 g \sin \theta_1 = m_1 a \ , \tag{77}$$

$$m_2 g \sin \theta_2 - T = m_2 a \ , \tag{78}$$

where the tension in both branches of the rope is the same because it and the pulley are ideal.

Now we have two equations in two unknowns a and T.

The clever way to solve for a is to add the equations of motion since this cancels out the unknown tension.

We get

$$m_2 g \sin \theta_2 - m_1 g \sin \theta_1 = (m_1 + m_2)a \tag{79}$$

which gives

$$a = g \left(\frac{m_2 \sin \theta_2 - m_1 \sin \theta_1}{m_1 + m_2} \right) \ . \tag{80}$$

Note that the acceleration can be positive, zero, or negative depending on the sizes of

$m_2 g \sin \theta_2$ and $m_1 g \sin \theta_1$. Note also that the acceleration is **NOT** independent of mass. This is because gravity isn't the only force causing acceleration: the tension force acts too. Nevertheless, in this special case only the relative size of the masses affect the acceleration. Moreover note also that the acceleration is constant.

So besides just acceleration under the force of gravity alone, there are other cases where constant accelerations can occur. Historically, this wasn't at all obvious I think. Maybe even Galileo never noticed cases like the double-incline-pulley system.

But which way the objects are sliding depends not only on acceleration, but on initial conditions. The objects could have a positive velocity for a negative acceleration or vice versa.

What about the T?

Solve for T any which way you can working alone or in groups. You have 1 minute.

Well divide equations (77) and (78) by the masses, subtract the second from the first, collect like terms in T, and solve for T. One gets

$$T = \frac{g(\sin \theta_1 + \sin \theta_2)}{1/m_1 + 1/m_2} = \frac{g m_1 m_2 (\sin \theta_1 + \sin \theta_2)}{m_1 + m_2} \ . \tag{81}$$

Notice tension is a constraint force and we only learn its value by making use of $F = ma$.

The collected solution is

$$a = g \left(\frac{m_2 \sin \theta_2 - m_1 \sin \theta_1}{m_1 + m_2} \right) \ , \qquad T = \frac{g m_1 m_2 (\sin \theta_1 + \sin \theta_2)}{m_1 + m_2} \ . \tag{82}$$

Actually, all kinds of special cases for a and T can be derived from equation (82).

The simplest is the case where there are no inclines: i.e., $\theta_1 = 0$ and $\theta_2 = 0$. In this case, $a = 0$ (nothing is moving) and $T = 0$ (nothing is being pulled).

If $\theta_1 = 90°$ and $\theta_2 = 90°$, then the system reduces to Atwood's machine. Here we have the collected solution

$$a = g \left(\frac{m_2 - m_1}{m_1 + m_2} \right) , \qquad T = \frac{2gm_1m_2}{m_1 + m_2} . \tag{83}$$

George Atwood (1745–1807) invented his eponymous machine in 1784. The device as a technological machine has been known since prehistory. But Atwood's idea was to use it to study the laws of motion. With Atwood's machine one can observe nearly constant accelerations that are arbitrarily small since the mass difference $m_2 - m_1$ can be made arbitrarily small.

Of course, real pulleys have some mass and including this changes the equations of motion. One can find those changes using rotational dynamics (see the lecture *ROTATIONAL DYNAMICS*). One still gets a constant acceleration for a pulley with mass. Other small effects due to mass of rope, friction, and air resistance are small and can decreased or accounted for with refinements. The tension in the ropes can be measured using fish scales (which are spring scales).

13.   FRICTION

There are several kinds of friction and in detail all are complex.

Here we'll just consider friction between macroscopically smooth, dry surfaces at rest with each other—which is static friction—or sliding over each other—which is kinetic friction. Sometimes we let lubrication creep in. The friction laws we introduce do work with lubrication to some approximation that is beyond our scope to consider. We briefly discuss other kinds of friction in § 13.4.

The friction just described is what one usually means when one says friction without

qualification: friction on smooth, dry surfaces.

We will idealize a bit by also assuming that the surfaces are perfectly rigid, and so we don't have to worry about sinking in effects.

There are approximate laws that are very accurate for friction in many cases. But, to emphasize, they are actually approximations. Friction is actually a complex and incompletely understood phenomenon (e.g., Tipler & Mosca 2008, 128)—and after all these centuries too.

But in intro treatments, one just treats the approximate laws as if they were exact for pedagogical purposes. One doesn't want to bother reiterating that they are approximations all the time especially when one wants to use them to solve problems.

The approximate laws are sometimes called Amontons's laws after discoverer Guillaume Amontons (1663–1705) (Wikipedia: Guillaume Amontons). Leonardo da Vinci (1452–1519) discovered them independently earlier, but the discovery was not effective since no one knew of it till centuries later.

Let's give an argument for them.

What actually causes friction between macroscopically smooth surfaces?

A form of chemical bonding. But friction as ordinarily considered is **NOT** a sticky force in the sense that it resists motion perpendicular to the surfaces of contact. It only acts parallel to the surfaces.

Of course, actually sticky forces can have friction effects, but that's not the kind of friction we are considering.

You might guess that this means friction should be proportional to contact area.

At the microscopic level this is roughly true.

But actually surfaces in contact macroscopically in contact are only in contact microscopically only for a fraction of the macroscopic contact area. Don't ask how we know—but, in fact, we can image the microscopic contacts these days I suppose. Even though the surfaces are macroscopically smooth, at the microscopic level they are rough and surface peaks and mounds keep them from full microscopic constant.

So one has approximately for static friction's upper bound (discussed in § 13.1 below) and kinetic friction (discussed in § 13.2 below)

$$F \propto f A_{\mathrm{macro}} \; , \tag{84}$$

where A_{macro} is the macroscopic contact area and f is the ratio of microscopic contact area to macroscopic contact area.

It turns out that

$$f \propto \frac{F_{\mathrm{nor}}}{A_{\mathrm{macro}}} \; , \tag{85}$$

where $F_{\mathrm{nor}}/A_{\mathrm{macro}}$ is the normal force per unit area. The more the surfaces are pressed together (i.e., the higher the normal force between them), the greater the microscopic contact.

Thus approximately

$$F \propto f A_{\mathrm{macro}} \propto \frac{F_{\mathrm{nor}}}{A_{\mathrm{macro}}} A_{\mathrm{macro}} = F_{\mathrm{nor}} \; . \tag{86}$$

This last equation essentially combines two of Amontons's laws: static friction's upper bound and kinetic friction are proportional to the normal force and are independent of the macroscopic contact area—of course, Amontons didn't put it like that—for one thing he wrote in French. The third Amontons's law is given in § 13.2.

A key point equation (86) is that F is independent on the macroscopic contact area. For example, put a non-cubic rectangular block (i.e., a right cuboid box) on a level surface. The normal force is determined by the gravitational force on the block. This normal force is

independent of what side the block rests on. Since the normal force is so independent, so is F. The fact that F is independent of macroscopic contact area is an immense simplification.

Note that if the normal force goes to zero, F goes to zero to even if the surfaces are still in macroscopic contact. This is unlike a sticky force. I imagine the distinction between friction and sticky forces breaks down sometimes. But we won't go into that.

In fact, we treat equation (86) as exact for pedagogical reasons.

Let's consider static and kinetic friction now.

13.1. Static Friction

Static friction applies when there is no sliding between the macroscopically smooth surfaces that are in contact.

For static friction, the friction force law for an object with a smooth surface on a fixed smooth surface is

$$F_{\text{st}} = \min[F_{\text{app}}, \mu_{\text{st}} F_{\text{nor}}] \, , \tag{87}$$

where F_{st} magnitude of the friction force on the object, F_{app} is the magnitude of the net force on the object parallel to the surface excluding the friction force itself. F_{nor} is the magnitude of the normal force between the surfaces, and μ_{st} is the coefficient of static friction. (The symbol μ is the small Greek letter mu.) The applied force is the net force on an object parallel to the surface, not counting friction itself.

First thing to note is that equation (87) is just a law about force magnitudes. The direction of the static friction force is opposite to the applied force and is perpendicular to the normal force.

What equation (87) means is that if the applied force magnitude is less than or equal to

$\mu_{\mathrm{st}}F_{\mathrm{nor}}$, then the static friction force is equal and opposite to the applied force and cancels it.

Thus, if the applied force magnitude exceeds $\mu_{\mathrm{st}}F_{\mathrm{nor}}$, then the object must move.

Equation (87) does **NOT** apply when the surfaces are sliding with respect to each other. When that happens, kinetic friction applies (§ 13.2)

One can see that static friction is a force of reaction—which sounds sort of Marxist. It is only as large as it needs to be to prevent motion, until, of course, it fails when its upper bound is exceeded. It is like the normal force in this sense: the normal force is as strong as it needs to be to prevent surface deformation, until, of course, it fails and the surface deforms.

The coefficient of static friction μ_{st} is a dimensionless physical parameter. It's dimensionless because it can be expressed as a ratio of two quantities of like dimension: static friction and normal force.

The static friction coefficient μ_{st} and the kinetic friction coefficient μ_{ki} (which we discuss in § 13.2) depends on the material of the two surfaces in contact. In general, the coefficients will also depend on temperature and pressure and on how the smooth surfaces are prepared— but we won't worry about those complications in this course.

Typically values of μ_{st} and μ_{st} are between of order 0.04 (for teflon on teflon) (e.g., Tipler & Mosca 2008, p.130) and of order 1.

Table 1 gives some representative values of static friction coefficients and kinetic friction coefficients.

Table 1.   Approximate Coefficients of Static Friction μ_{st} and Kinetic Friction μ_{ki}

Materials	μ_{st}	μ_{ki}
Copper on cast iron	1.1	0.3
Rubber on concrete (dry)	1.0	0.8
Glass on glass	0.9	0.4
Steel on steel	0.7	0.6
Aluminum on steel	0.61	0.47
Copper on steel	0.53	0.36
Brass on steel	0.5	0.4
Rubber on concrete (wet)	0.30	0.25
Wood on wood	0.25–0.5	0.2
Metal on metal (lubricated)	0.15	0.06
Waxed ski on snow (0° C)	0.10	0.05
Ice on ice	0.1	0.03
Teflon on teflon	0.04	0.04
Teflon on steel	0.04	0.04
Synovial joints in humans	0.01	0.003

Note. — The values are from Tipler & Mosca (2008, p. 130) and Serway & Jewett (2008, p. 121). Although we are primarily thinking of dry surface friction in our discussion, friction between wet surfaces can be treated too using the approximate laws of friction in some cases. Thus, there are coefficients for some wet cases too in the table.

The coefficients of friction in principle should be calculable from material properties. But in practice I believe they are usually measured. The calculations may not be sufficiently accurate. Friction is actually a complex and incompletely understood phenomenon (e.g., Tipler & Mosca 2008, 128).

A simple way to determine the static friction coefficient is to put an object with a smooth surface on an inclined plane with a variable angle θ.

If the object stays at rest for a given angle θ, static friction applies. Let's analyze the system using Newton's 2nd law and remember:

"Newton's 2nd law is always true and it's always true component by component."

We set the x direction as positive down the incline and the y direction perpendicular to the incline and the positive direction outward.

What is the normal force on the object? You have 10 seconds. Go.

From the y direction application of the 2nd law, we know that the normal force magnitude is given by

$$F_{\mathrm{nor}} = mg \cos \theta \ , \tag{88}$$

where m is the mass of the object.

From the x direction application of the 2nd law, we know that the

$$F_{x,\mathrm{grav}} + F_{\mathrm{st}} = mg \sin \theta - \min[mg \sin \theta, \mu_{\mathrm{st}} mg \cos \theta] = 0 \ . \tag{89}$$

We see that if $mg \sin \theta \leq \mu_{\mathrm{st}} mg \cos \theta$ the last equation is valid. If $mg \sin \theta > \mu_{\mathrm{st}} mg \cos \theta$ is invalid, and the object will start to slide. When the object is sliding, kinetic friction applies (§ 13.2).

The function $mg \sin \theta$ rising monotonically for the range 0 to 90° and the static friction upper bound function $\mu_{st} mg \cos \theta$ falls monotonically for the same range. See Figure 7.

Because of the monotonic behaviors, there is only one angle where the functions are equal and at that point the 2nd law gives

$$mg \sin \theta - \mu_{st} mg \cos \theta = 0 . \tag{90}$$

If we solve for μ_{st}, we find that

$$\mu_{st} = \tan \theta . \tag{91}$$

One can find the solution for the upper limit angle by inverting this formula:

$$\theta = \tan^{-1}(\mu_{st}) . \tag{92}$$

The solutions for μ_{st} and θ are independent of mass and gravity. The μ_{st} solution just depends on the upper limit angle θ and the upper limit angle θ solution just depends on μ_{st}. This seems remarkable at first, but it's just a consequence of the fact that the two forces we use in the solution for μ_{st} (i.e., the applied force parallel to the surfaces and the normal force) are both proportional to gravity in this case. The gravity force mg cancels out leaving just μ_{st} and θ in a relationship.

Fig. 7.— Functions $mg \sin \theta$ and $\mu_{st} mg \cos \theta$.

For small μ_{st}, one has

$$\mu_{\mathrm{st}} = \tan\theta \approx \theta \qquad \text{and} \qquad \theta = \tan^{-1}(\mu_{\mathrm{st}}) \approx \mu_{\mathrm{st}} \tag{93}$$

using small-angle approximations for θ which is measured in radians.

For, example in the case of teflon on teflon, sliding would start to occur at $\theta \approx 0.04\,\mathrm{rad} \approx 2.3°$.

By the by, the small angle approximation for tangent is 1st order good in small θ which means that it becomes effectively exact in the limit that θ^2 is negligibly small compared to θ. Table 2 illustrates the accuracy of the small angle approximation for tangent.

Table 2.   Accuracy of the Small Angle Approximation for $\sin\theta$

θ (degrees)	θ	$\tan\theta$	Relative Error $\dfrac{\theta - \tan\theta}{\tan\theta}$
0	0	0	0
10	0.17453	0.17633	-0.0101746
20	0.34907	0.36397	-0.0409495
30	0.52360	0.57735	-0.0931003
40	0.69813	0.83910	-0.1679990
50	0.87266	1.19175	-0.2677474
60	1.04720	1.73205	-0.3954002
70	1.22173	2.74748	-0.5553265
80	1.39626	5.67128	-0.7538011
90	1.57080	∞	∞

Note. — The calculations were done with a double precision fortran-95 program.

13.2.   Kinetic Friction

The approximate law for kinetic friction F_{ki} for surface sliding relative to each other is

$$F_{\text{ki}} = \mu_{\text{ki}} F_{\text{nor}} \ , \tag{94}$$

where μ_{ki} is the kinetic friction coefficient and F_{nor} is the normal force between the surfaces.

Note the law only relates magnitudes. The kinetic friction is parallel to the surfaces and the normal force is perpendicular.

The direction of kinetic friction is opposite to the direction of motion. Remember static friction is opposite to the direction of the applied force on the object parallel to the surfaces.

For pedagogical reasons, we treat this law as exact even though it is approximate. Recall we treated the static friction law as exact for pedagogical reasons even though it is approximate.

A remarkable feature of equation (94) is that the force is independent of the relative speed of the surfaces. This velocity independence is, in fact, Amontons's 3rd law (Wikipedia: Guillaume Amontons). Figure 8 shows ideal Amontons's laws friction as a function of applied force and as a function of velocity.

Fig. 8.— Ideal Amontons's laws friction as a function of applied force and as a function of velocity.

Certainly not all resistance laws are speed independent. The resistance of fluids to motion through them (i.e., drag) tends to be either linear in relative speed or quadratic in relative speed.

From Table 1, one can see that kinetic friction coefficients are almost always distinctly smaller than the static friction coefficients. The vague reason given for this is that bonds that create friction have less time to form for kinetic friction, and so some bonding does not take place that does for static friction.

One notes that teflon on teflon and teflon on steel have equal static and kinetic coefficients to within significant figures.

Are there any cases where the static friction coefficient is less than the kinetic friction coefficient. Yours truly knows of no such cases—but who knows, there may be some. There almost always is some weird material that violates common material laws.

What do the smaller kinetic friction coefficients mean in practice?

To be concrete, let's consider the motion of a car. The outside forces that control a car's motion are friction, the normal force, and gravity. All are obviously important. The main friction force is actually static friction. At the point of contact between wheels and ground, the wheels and ground are at rest. On level ground, cars that are not sliding use only static friction to speed up, slow down, and turn corners. If the car starts to slide the situation is different Instead of having static friction whose size you effectively control in order to control the car, you have kinetic friction that just opposes the direction of motion of the car. So when sliding, you have lost control.

Now consider the case of trying to stop on ice. Say you can stop without sliding starting from an initial speed. But now you have the same speed, but are sliding. The kinetic friction opposing your direction of sliding could be less than the static friction you used to stop the

car in the sliding. This can happen since $\mu_{st} > \mu_{ki}$ as usual for tires on ice—though exact values seem elusive even on the web. Let's assume the kinetic friction used to stop was less than the static friction used to stop. In this case, it would take longer and you would go a longer distance before stopping sliding and relying on kinetic friction.

For another example, you may be able park on icy incline because the applied force down the slope $mg \sin \theta$ is smaller than the maximum force allowed by the static friction force. But if the your car came on to the incline sliding and you tried to stop, it may not be possible because the kinetic friction force is smaller than the static friction force. Actually sliding down icy slopes is all too common in Moscow, Idaho.

13.3. Example: Double-Incline-Pulley System with Kinetic Friction

Let's redo our double-incline-pulley system from § 12.6, but now including kinetic friction.

By including kinetic friction, we are assuming that the objects are in motion.

Write equations of motion for the two objects for the respective x directions assuming that object 1 is sliding uphill and object 2 is sliding downhill.

Remember equation of motion means a particular application of the 2nd law among other things and that's what it means here.

You have 1 minute working in groups or individually. Go.

In this case, the equations of motions for the two x directions are

$$T - m_1 g \sin \theta_1 - \mu_{ki} m_1 g \cos \theta_1 = m_1 a \ , \tag{95}$$

$$m_2 g \sin \theta_2 - T - \mu_{ki} m_2 g \cos \theta_2 = m_2 a \ . \tag{96}$$

Now for a subtle point. By including the friction terms as we do, we are implicitly assuming that the objects are moving and that their velocities are positive since kinetic friction always opposes the direction of motion.

The acceleration a can be any of positive, negative, or zero, but our analysis is wrong if velocity is or becomes zero or negative. One just has to do the analysis more carefully if these bad things happen.

Note that velocity is determined not only by acceleration, but by initial conditions.

Our two unknowns are again a and T. We won't bother solving for T.

It's pretty easy to solve for a. Just add equations (95) and (96) and T cancels out. Then just rearrange to get the solution for a.

One gets constant acceleration

$$a = g \left[\frac{m_2(\sin\theta_2 - \mu_{ki}\cos\theta_2) - m_1(\sin\theta_1 + \mu_{ki}\cos\theta_1)}{m_1 + m_2} \right] . \tag{97}$$

Note that the acceleration does depend on the masses. This is because gravity isn't the only force determining the acceleration: the tension force and friction help to determine the acceleration too. Nevertheless, in this special case only the relative size of the masses affect the acceleration. The fact that the acceleration is constant is interesting. We see that besides acceleration under the force of gravity alone, there are other cases (even with friction turned on) where constant accelerations can occur.

An interesting special case is when $\theta_1 = \theta_2 = 0°$: i.e., the system is on level ground.

In this case,

$$a = -\mu_{ki}g . \tag{98}$$

The masses have canceled out and the acceleration is negative.

In order for this result to be valid, the velocity has to be positive just as it does for the

general result.

Equation (98) is just the acceleration you would get for any object sliding on level ground with only kinetic friction acting.

Why do the masses cancel out for equation (98).

Well kinetic friction is not intrinsically mass dependent. But if the normal force is linearly dependent on mass, then equation (94) makes kinetic friction linearly dependent on mass. This leads to the cancellation of mass in equation (98). The cancellation is due to the special nature of the system, and not only to the intrinsic nature of kinetic friction.

If all the forces that can accelerate an object are linear dependent on mass, then mass cancels out of $F = ma$.

13.4.   Other Frictions

There are other kinds of friction besides the Amontons's laws friction considered in this section.

For example, if surfaces in contact are macroscopically rough, then there can be normal forces parallel to the surfaces. Such forces would provide a static and kinetic friction. If the surfaces are in relative motion, the macroscopic roughness would tend to be worn down.

Another kind of friction is rolling friction. This occurs for a roller (a ball or cylinder) rolling on a surface. Both roller and surface can deform a bit under the normal force between them. The continual deformation and rebound causes kinetic energy losses inside the materials due to internal resistance (which can also be considered a kind of friction). Rolling friction between rigid roller and surface is usually much smaller than kinetic friction of sliding surfaces. This is well known from everyday life. A rigid ball rolling on a level rigid

surface rolls "forever".

Then there is cold welding between metals. This actually an extreme form of the chemical bonding that gives rise to static friction. The metal surfaces have to cleaned of any oxide layer, made to fit each other macroscopically, and pressed into contact. Strong bonding forms. The upper limit of cold welding is when the metals fuse and there is no macroscopic interface left.

Fluid drag (AKA fluid resistance) can also be viewed as a kind of friction although one usually doesn't usually call it friction. Drag is the resistive force that a fluid exerts on a body opposite to the bodies relative motion through the fluid. Unlike ideal kinetic friction between fluid surfaces, drag is velocity dependent and goes to zero for zero relative velocity. We consider drag in the lecture *Newtonian Physics II.*

There are other kinds of resistive forces that can be described as friction in some sense. But that's enough for now.

ACKNOWLEDGMENTS

A.   TENSION IN AN IDEAL ROPE

Now for a rather elaborate derivation which we do for a darn good reason—which is to understand what is really happening for an ideal rope.

First we have to digress into vector calculus and differential geometry in order to understand radius of curvature.

A.1.   Radius of Curvature

A path is a vector displacement that traces a curve in space as a path parameter increases: e.g., for three-dimensional space

$$\vec{r} = (x(t), y(t), z(t)) \ , \tag{A1}$$

where t is the path parameter. The path parameter could be time, but in pure geometry it is just a parameter whose increase moves you along the path in one direction.

At any point on a sufficiently smooth path in space, one can fit a circle that is tangent to the path and agrees with the path to 2nd order in the path parameter.

In this case, tangent means that the directions of curve and circle are aligned. The 2nd order agreement means that path and circle expanded in in Taylor series agree to 2nd order. We will show how to get 2nd order agreement for a special case in § A.1.1.

To be sufficiently smooth the path's second derivative with respect to a path parameter must exist at that point where the osculating circle is fitted. The zeroth derivative is the path itself and the 1st derivative is the a vector that points in the instantaneous direction of the path as the path parameter increases. If the path is actually the displacement increasing with time, the 1st derivative is velocity and the 2nd is acceleration.

In differential geometry, the fitted circle is called an ocsculating circle (Wikipedia: Osculating circle) which means kissing circle: it just kisses the curve. The curve and circle may coincide for more than a point which means the curve actually follows the circle for a finite piece of arc length. The radius of the osculating circle is the radius of curvature (Wikipedia: Radius of curvature (applications)).

In the subsequent subsubsections, we investigate how to find a radius of curvature for a path and infinities and discontinuities in the radius of curvature as function of path

parameter.

These topics have relevance for the subject of tension in an ideal rope. The ideal rope follows a path through space. When it has tension the radius of curvature as function of this path is needed quantity. The radius of curvature is needed find the normal force magnitude per unit length exert by an ideal rope under tension.

We allow infinities and discontinuities in the radius of curvature for the ideal rope and so need to understand their effects on the ideal rope system.

Ideal ropes approximate real ropes and so the radius of curvature topics we discuss are relevant to reality. The infinities in the radius of curvature do happen for real ropes. The discontinuities in the radius of curvature probably don't happen in real ropes, but this may depend on how a discontinuity is identified for real ropes. But in any case, a real rope can show rapid change in radius of curvature (e.g., when it is bent around a sharp corner but is straight on either side). So discontinuities approximate real rope behavior.

A.1.1. Finding Radius of Curvature

How does one find the osculating circle and radius of curvature?

We won't look at the problem in general.

Consider a two-dimensional space and locate the origin at a point on path where one wants to find the osculating circle and radius of curvature. Make the x axis tangent to the path at the origin.

Let the path parameter be x itself. The path is then specified by $(x, f(x))$, where y coordinate of the path is a function of x. Now let's expand $f(x)$ in Taylor's series about the

origin:

$$f(x) = \sum_{\ell=2}^{\infty} \frac{x^\ell}{\ell!} f^{(\ell)}(0) = \frac{x^2}{2} f^{(2)}(0) + \dots , \tag{A2}$$

where the $\ell = 0$ and $\ell = 1$ terms are zero by the choice of origin and x axis and where $f^{(2)}(0) \neq 0$.

Now consider a circle whose center is at $(0, \pm r)$, where r is also the circle radius, and the upper case is for when $f(x)$ opens upward (i.e., $f^{(2)}(0) > 0$) and the lower case is for when $f(x)$ opens downward (i.e., $f^{(2)}(0) < 0$). The circle equation circle is

$$x^2 + (y \mp r)^2 = r^2 . \tag{A3}$$

The circle touches the origin note. Solving for y as a function of x for the lower half circle, we get

$$y = \pm r \left[1 - \sqrt{1 - \left(\frac{x}{r}\right)^2} \right] \tag{A4}$$

which Taylor's expanding about the origin gives

$$y = \pm \frac{1}{2} \frac{x^2}{r} + \dots . \tag{A5}$$

We now see that path and the circle agree to 2nd order in the path parameter with the radius chosen according to the formula

$$r = \frac{1}{|f^{(2)}|} . \tag{A6}$$

Both path and circle have the path formula $(x, \pm x^2/(2r))$ to 2nd order in x.

Since we have 2nd order agreement in the path parameter for our choice of circle and radius, the chosen circle is the osculating circle and chosen radius is the radius of curvature.

The Wikipedia article of radius of curvature applications (Wikepdia: Radius of curvature (applications)) gives the general formula for the radius of curvature, but alas no general prescription for fully specifying the osculating circle.

We will specialize the general formula for two-dimensional cases with x as the path parameter $y = f(x)$: i.e., the path is $(x, f(x))$, the 1st derivative is $(1, f')$, and the 2nd derivative is $(0, f'')$. The general formula for radius of curvature specializes to

$$r = \frac{(1 + f'^2)^{3/2}}{\sqrt{f''^2}} \; , \tag{A7}$$

where the derivatives of $f(x)$ are evaluated at a general x and where we've switched to using the prime notation to indicate 1st and 2nd derivatives.

For our special choice of origin and x axis in the example above, we have $f' = 0$ and we recover equation (A6) from equation (A7).

A.2. Infinities in the Radius of Curvature

What happens if the 2nd derivative of the path is zero?

The radius goes to infinity.

This is shown by the general formula given by Wikipedia (Wikepdia: Radius of curvature (applications)) and by our specialized formula equation (A7) where a zero 2nd derivative implies $f'' = 0$

There are two cases of the 2nd derivative going to zero.

The first case is when it is zero only at single point on the path. In this case, radius of curvature as function of the path parameter has an infinity at that path parameter value that gives that point on the path.

The second case is where the 2nd derivative is zero over a finite region of the path. In this case, the 1st derivative is a constant over the region. Recall that the 1st derivative gives the direction the of path increase at the point of evaluation. So if the 1st derivative is a constant over the region, the path is a straight line over the region.

Both these cases of 2nd derivative going to zero can occur for ideal rope with tension. In both cases, normal force exerted by the rope goes to zero when the 2nd derivative goes to zero. We will show this explicitly in § A.3 below.

A.2.1. Discontinuities in the Radius of Curvature

There are two kinds of discontinuities in the radius of curvature to consider. We will only consider discontinuities at isolated points on the path. We won't consider discontinuities in path itself since those arn't relevant to ideal rope problems.

The first kind of discontinuity is where 1st derivative of the path does not exist at a point because the path has a cusp at a point: i.e., the path direction changes at single point.

Thus, the 1st derivative has a discontinuity at the point and the 2nd derivative an infinity. The is shown by the general formula radius of curvature given by Wikipedia (Wikepdia: Radius of curvature (applications)) and our specialized formula equation (A7) both show that if 2nd derivative goes to infinity, the radius of curvature goes to zero. Before and after the point the radius of curvature is non-zero and its zero value at the point is a discontinuity.

For an ideal rope this point of zero radius of curvature is the rope bending at point around a perfectly sharp corner.

Actually, what happens at the discontinuity to the tension as it varies along the rope and normal force magnitude per unit length exerted by the rope at this point is indeterminate without more specifications for the systems. It takes us to far astray to go into that now. In any case, the ideal rope going around an perfect point corner may be too ideal to correspond to reality.

The second kind of discontinuity is where the 1st derivative of the path exists, but the

2nd doesn't at some point. So the path has no cusp at the point and the path direction doesn't change at the point.

At the point the radius of curvature changes discontinuously, but does not go to zero.

We have no reason to demand that the variation in tension change discontinuously at the discontinuity. In fact, the correct idealization of reality would say it doesn't.

The discontinuity in radius of curvature does cause a discontinuity in the in the normal force per unit length that the rope exerts. But since the undefined normal force per unit length that occurs at a discontinuity is only at a point, it makes no contribution to the actual normal force the rope exerts.

So discontinuities of the second kind are innocuous. We have no ambiguities in understanding the ideal rope system because of them and thus they can be allowed without worry about them causing unphysical behavior in real ropes.

We won't consider discontinuities of the second kind in general. We'll ust look at a couple of examples to get a feel for them.

Consider a path that is a straight line that suddenly changes at a point into a circle without a discontinuity in position or direction. The path's zeroth and 1st derivatives are exist at the change point, but not the 2nd derivative. To show this concretely, consider the line and circle specified by, respectively,

$$\vec{r} = r_0(1, \tan\theta) \qquad \text{and} \qquad \vec{r} = r_0(\cos\theta, \sin\theta) \ , \tag{A8}$$

where θ is the path parameter. Let's say the path follows the line for $\theta < 0$ and the circle for $\theta \geq 0$. The zeroth, 1st, and 2nd derivative for the line are, respectively,

$$\vec{r} = r_0(1, \tan\theta) \ , \qquad \frac{d\vec{r}}{d\theta} = r_0\left(0, \frac{1}{\cos^2\theta}\right) \ , \qquad \frac{d^2\vec{r}}{d\theta^2} = r_0\left(0, \frac{2\sin\theta}{\cos^3\theta}\right) \ . \tag{A9}$$

The zeroth, 1st, and 2nd derivative for the circle are, respectively,

$$\vec{r} = r_0(\cos\theta, \sin\theta) \ , \quad \frac{d\vec{r}}{d\theta} = r_0(-\sin\theta, \cos\theta) \ , \quad \frac{d^2\vec{r}}{d\theta^2} = -r_0(\cos\theta, \sin\theta) \ . \tag{A10}$$

For $\theta = 0$, for line quantities we get

$$\vec{r} = r_0(1, 0) \ , \quad \frac{d\vec{r}}{d\theta} = r_0(0, 1) \ , \quad \frac{d^2\vec{r}}{d\theta^2} = r_0(0, 0) \tag{A11}$$

and for circle quantities,

$$\vec{r} = r_0(1, 0) \ , \quad \frac{d\vec{r}}{d\theta} = r_0(0, 1) \ , \quad \frac{d^2\vec{r}}{d\theta^2} = -r_0(1, 0) \ . \tag{A12}$$

We see the two sets of quantities at $\theta = 0$ agree for the zeroth and 1st derivative, but not for the 2nd derivative. Thus, the 2nd derivative, osculating circle, and radius of curvature does not exist at the change point. Our special choice coordinates and path parameter are not limiting. Any choice would lead to the same result.

Let's consider a three-dimensional case.

Start with a smooth curve in that lies in a plane. Say the radius of curvature is continuous everywhere ????

A.3. Tension

Let's consider an ideal rope that follows a curved path through space.

The path and path's direction at each point are continuous.

It is a smooth curved path.

The tension in the rope is T. It is not constant in general and for the sake of generality, we allow tension T be positive, zero, or negative. Nothing in the developments that follow require positive tension. We take up the consideration of negative tension in § A.5.1.

Now the region about a general point on a curved path can be approximated as part of circle in a plane centered on a center of curvature. (I think this can always be done even for a general smooth curve in three-dimensional space.) Let the circle have radius r. If the curve is actually straight at this general point or the point is actually an inflection point, the center of curvature is at infinity and $r = \infty$. The circle approximation is first order good in small angular deviation of a radius vector from the general point. But we won't proof that here.

Now let the center be the origin of an x axis and let the general point be at on the positive side of the x axis.

Let θ be an angle measured from the positive x axis small enough that the actual curve approximates the circle to arbitrary accuracy from 0 to θ. See Figure 9.

Let's apply the 2nd law to the rope segment from 0 to θ and from § 5.3, remember the class mantra::

"Newton's 2nd law is always true and it's always true component by component."

But note we are applying the 2nd law to an object which are **NOT** considering as a particle. This is valid as we'll discuss in the lecture *SYSTEMS OF PARTICLES AND MOMENTUM.*

Fig. 9.— An ideal rope's curved path approximated by a circle.

The acceleration for the non-particle is that of the center of mass as it turns out. In the present case, by taking limits that allow us to skirt the issue of where the center-of-mass is.

First, consider the x direction:

$$-T \sin \theta + F_x = ma_x \, , \tag{A13}$$

where T is the tension at θ, F_x is some other applied force in the x direction, m is the mass of the rope segment, and a_x is the segment's acceleration.

The tension forces recall must be aligned with the rope. The $\sin \theta$ factor gives the correct component of the tension forces for the x direction. The tension force at angle zero has zero component in the x direction and so does not appear in equation (A13).

Since our rope is ideal $m = 0$ and we have

$$-T \sin \theta + F_x = 0 \, . \tag{A14}$$

Note θ is sufficiently small that $F_x \geq 0$ for the equation to be valid. If the rope is being accelerated, then the acceleration must be determined the system the rope is attached too. Without that system, the acceleration would be undefined.

Now consider the y direction:

$$T \cos \theta - T_0 + F_y = ma_y \, , \tag{A15}$$

where T_0 is the tension at zero angle.

The tension forces recall must be aligned with the rope. The cosine factors give the correct components of the tension forces for the y direction. For zero angle, the cosine factor is 1, of course.

Since our rope is ideal $m = 0$ and we have

$$T \cos \theta - T_0 + F_y = 0 \, . \tag{A16}$$

If the rope is being accelerated, then again the acceleration must be determined the system the rope is attached too. Without that system, the acceleration would be undefined.

Now we want to find the ratio $F_x/(r\theta)$ (with θ measured in radians) in the limit that θ goes to zero. This is the normal force per unit length at the general point. Let's call it f_{nor}. We find

$$f_{\text{nor}} = \lim_{\theta \to 0} \frac{F_x}{(r\theta)} = \lim_{\theta \to 0} \frac{T \sin \theta}{(r\theta)} = \frac{T_0}{r} \, , \tag{A17}$$

where we have used the well known calculus result that

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \, . \tag{A18}$$

Next we want to find tension varies with θ at the general point. Differentiating (A16) with respect to θ gives

$$\frac{dT}{d\theta} \cos \theta - T \sin \theta + \frac{dF_y}{d\theta} = 0 \, . \tag{A19}$$

Evaluating at $\theta = 0$ and rearranging gives

$$\frac{dT}{d\theta} = -\frac{dF_y}{d\theta} \, . \tag{A20}$$

We now see that at the general point F_y is a force parallel the rope's path. Let's rename F_y to F_{par}, where "par" is for parallel.

To get general results, let s be distance along the rope which take to be increasing in the positive y direction at the general point. We now note that at the general point

$$ds = d(r\theta) = r \, d\theta + \theta \, dr = r \, d\theta \, , \tag{A21}$$

where $dr = 0$ at the general point since by definition along the path at the general point the path is a circle to first order.

Finally, we get

$$f_{\text{nor}} = \frac{T}{r} \tag{A22}$$

and

$$\frac{dT}{ds} \;=\; -\frac{dF_{\mathrm{par}}}{ds} = -f_{\mathrm{par}} \;, \tag{A23}$$

where f_{par} is the force per unit length and where we have dropped the subscript 0 from equation (A22) since we now just mean T and r are evaluated at a general s: likewise equation (A23) is evaluated at a general s.

Equations (A22) and (A23) are what we've been working for. They are exact, I believe, since the circle path approximation is exact at the general point where we evaluated them.

Let's explicate them a bit.

A.4. Equation $f_{\mathrm{nor}} = \dfrac{T}{r}$ Explicated

Equation (A22) gives the normal force per unit length exerted on the rope by some surface and, by the 3rd law, by the rope on that surface. The direction of the normal force on the rope is away from the local center of curvature and the normal force the rope exerts is toward the local center of curvature.

We see that if T is zero (i.e., tension is zero) or r infinite (the rope is straight), then the normal force per unit length is zero.

Our derivation oddly enough allows for r and therefore f_{nor} to change discontinuously. This happens for example at the point where are rope changes from being straight to just bending around a curve surface (such as that of a pulley). In our derivation, we just take different radii for the different sides of the general point. The tension can't change discontinuously at such a point unless F_{par} does (see just below).

Since you can't push on an ideal rope, it seems odd at first that a rope can exert a normal force per unit length. But on the other hand, all life experience shows it does since

you can pull things with a rope loop.

A.5. Equation $\dfrac{dT}{ds} = -\dfrac{dF_{\mathrm{par}}}{ds}$ Explicated

Equation (A23) gives the rate of change of the tension with path length due to the integrated applied parallel force component F_{par} which has its positive direction in the direction of increasing s. If F_{par} increases with s, tension decreases. One can view this as tension decreasing to accommodate increasing F_{par}. Discontinuous changes in F_{par} are allowed too. One simply views them as the limiting case of very rapid changes in F_{par}. If F_{par} changes discontinuously from by ΔF_{par}, then the discontinuous change in tension is

$$\Delta T = -\Delta F_{\mathrm{par}} \ . \tag{A24}$$

Equation (A23) also shows us how ropes transmit force.

If we integrate equation (A23) from the beginning to s of the rope, we find

$$T(s) - T_0 = \int_0^s \frac{dT}{ds}\, ds = -\int_0^S f_{\mathrm{par}}\, ds = F_{\mathrm{par}}(s) \tag{A25}$$

where $T(s)$ is the tension at s, T_0 is the tension at the start of the rope, $F_{\mathrm{par}}(s)$ is the integrated parallel force component to point s, and $F_{\mathrm{par}}(0) = 0$. Thus, we can write

$$T(s) = T_0 - F_{\mathrm{par}}(s) \ . \tag{A26}$$

Note that T_0 must be equal the magnitude of the applied force at the start of the rope. This applied force pulls antiparallel to the rope.

Say the end of the rope is s_{max}. The applied force at the end of the rope must equal $T(s_{\mathrm{max}})$ in magnitude and point parallel to the rope.

That there are applied forces at the start and end is a given condition. If these are zero, then rope has zero tension at the start and end.

Note if $F_{par}(s_{max}) = 0$, then $T(s_{max}) = T_0$, and the starting applied force magnitude is transmitted to the end of the rope where the rope pulls on whatever source of the end applied force with a force of magnitude T_0. This is the ideal case of a transmission of force by a rope. The transmission can be around curves.

A.5.1. Negative Tension Explicated

The formalism developed above allows the tension to go negative.

What the heck does that mean?

If the tension goes negative, the rope resists compression along its length and there must be an applied external center-attracting force (rather than an applied external center-repelling force) to keep them a circular path. (Checking back over our developments shows that this makes sense.) For positive tension, the rope is stable when straight and if unstraight would return to straightness if the external applied forces turned off. For negative tension, the rope is unstable when straight: i.e., any perturbation from exact axial symmetry along the rope would cause buckling. When unstraight, the rope would buckle with any applied force that tried to compress it. The buckling behavior is ill-defined for ideal rope since we have no defined fully buckled state for an ideal rope—the fully buckled state for a real rope isn't known without careful specifications for each case. The stability and instability statements just follow from everyday observations and not mathematical proof—but that could be done to I suppose.

Real ropes actually actually resist compression and actually do behave like the ideal rope under compression.

So the generalization of the ideal rope to negative tension cases is valid.

But negative tension cases may not have so much use, because of the instability to undefined buckling of both ideal and real ropes.

To see this note that in two dimensions, forces needed keep the rope under positive tension on a curve a fairly easy to arrange: just a single solid curved surface will do: the instability toward straightness is stopped. But for negative tension because of instability to ill-defined buckling, you would need a two-dimensional pipe to keep real rope from buckling: inhomogeneities in the rope could cause it to buckle away from just a curved surface that would prevent an ideal rope from buckling. If the real rope can buckle easily where an ideal wouldn't, then the ideal rope isn't much use as model for reality.

In three dimensions, it's a bit trickier. For positive tension, one often needs a groove for the rope to run through like those on pulleys to prevent the instability toward straightness. Even though real pulleys are semi-two-dimensional, the groove prevents the rope from slipping off in practical applications. For negative tension, because of the instability to buckling for both real and ideal ropes would probably have to be confined to pipes.

Now if you are going to confine a rope to a pipe, then you might as well dispense with ropes and use the pipe to hold a fluid that can be used to transmit a force—in other words use hydraulics.

Come to think of it, rope and pulley systems are probably of most use when you have transmit forces that cause rotation and in particular rotation at high speeds.

Hydraulic systems are more suited for straight pushing needs, but they are a bit more flexible than rope and pulley systems and probably can be used to transmit much larger forces easily.

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