INTRODUCTION TO INTRODUCTORY PHYSICS

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ABSTRACT

Oh, the poetical-tragical-comical-historical-pastoral-philosophical introduction to physics. A word on systems, environments, models, and idealization. Some stuff on quantities, base units, other units, and unit conversions. Two words on dimensional analysis. A bit on significant figures. Some discussion of order-of-magnitude calculations, 1-digit calculations, and no more.

Subject headings: philosophy — systems/environments/models/idealization — units — base units — unit conversions — dimensional analysis — significant figures — order-of-magnitude calculations — 1-digit calculations

1. INTRODUCTION TO THE INTRODUCTION

To be brief, **PHYSICS** is the science of matter and motion.

To be slightly less brief, **PHYSICS** is the science of matter and motion, time and space.

It's an empirical science: it studies things in the physical world that can be known through observation and experience. Maybe all sciences are really empirical—but usually

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one says pure math isn't since it studies abstract relationships—but those relationships exist to be discovered and are discovered by experience.

PHYSICS in its modern form is a mathematical science—and soon we'll start doing math, and it won't let up till the end of the semester.

This was not always true. From the Greek Pre-Socratic philosophers (circa 600–400 BCE) until circa 1600, physics was considered by most scholars to be a qualitative science that gave a philosophically reasonable account of matter and motion, time and space. In the Medieval Islamic and European contexts, the physics of Aristotle (384–322 BCE)—Aristotelian physics—was considered the dominant version. But even in ancient times, some mathematical physics existed: in astronomy (counted as part of physics) since prehistory depending how one counts things and in other physics fields (as we now recognize them) starting arguably with Archimedes (circa 287–212 BCE). Thereafter, only relatively small additions were made to mathematical physics—outside of astronomy—until the time of Galileo (1564–1642) whose life coincided with the Scientific Revolution. The work of Galileo, others, and Newton made physics a deeply mathematical science.

Nowadays, much of **PHYSICS** is embodied in mathematical laws. These laws relate the physical quantities—which are things that can be measured or calculated. The laws allow us to understand **SYSTEMS**: i.e., predict their nature and their past and future evolution.

A **SYSTEM** is any particular set of objects which we are studying at the moment—but we'll elaborate more on systems in § 3.

To be a physical law, a formula must apply to a wide variety of cases. In fact, one usually says it must apply nearly exactly in some well defined realm of physics. From physical laws, many general formulae are derivable and infinitely many special case formulae.

There is no physical law or general formula about this chair—the one in this room—

sitting on the floor. But there are physical laws to describe the forces on it and why it is at rest and how it would move if a **NET FORCE** were applied to it.

Understanding particular systems involves using general formulae and the peculiar features of the systems. These peculiar features are often called boundary conditions or initial conditions (which are actually boundary conditions in the time dimension).

The ultimate goal of physics—well one of them anyway—is to achieve a true—a truly true—exact—exactly exact—theory of everything—TOE as we call it sometimes. We don't have TOE yet—maybe tomorrow, maybe next year, maybe never.

TOE would be the **FUNDAMENTAL PHYSICAL LAW**. From TOE, all known physical laws and general formulae could be derived. In fact, some folks demand that TOE explain the existence of the universe too. Which is asking an awful lot.

Actually, I don't think TOE will explain everything uniquely. I think that there are emergent principles like the 2nd law of thermodynamics in physics itself and evolution in biology which are independent of physics. They could exist in alternative universes—which might exist. So I think theory of everything is a misnomer.

The faith is that TOE will be a simple theory in that it will have very few basic fundamental laws or axioms and few or even no free parameters. It's not likely to be simple in an everyday life sense—you'd likely need three years of study to understand it.

Physical laws which can be derived are not fundamental physical laws—in the most fundamental sense of fundamental.

But the term "**FUNDAMENTAL**" is used in various senses. In a first sense, only TOE is fundamental: i.e., the true exact physics of the universe. In second sense, the currently known most basic laws are fundamental and they will cease to be fundamental if superceded—the standard model of particle physics is an example of current fundamental law. In a third sense, fundamental can be used to describe laws that fully describe some limited realm of physics—Newtonian physics is fundamental in the realm of **CLASSICAL PHYSICS** which we discuss below. Newtonian physics was once considered fundamental in the second sense and was a candidate for being fundamental in the first sense—but those days are no more.

Context must decide what meaning of "**FUNDAMENTAL**" applies.

Physics which is not fundamental physics can be called applied physics.

But the term applied physics is used variously too. Some would only call physics directly used in some technology applied physics.

I don't think one has to be too prescriptive about the term "applied physics".

2. CLASSICAL PHYSICS AND MODERN PHYSICS

Classical physics is essentially the well-established physics current circa 1900 or a little before. The principles of classical physics have not changed since then—and can't by definition. But application areas and computational techniques for classical physics continue to advance.

Modern physics in one sense is all the physics that developed since that time.

In a second sense, modern physics is the physics that developed only in the time frame from circa 1900 to circa 1960—it's modern in the sense that Picasso is a modern painter and Hemingway, a modern writer. The second sense is largely used in pedagogy—there are courses and textbooks that are modern in the second sense. In the following, I mean modern physics in the first sense.

Loosely speaking classical physics consists of Newtonian physics (which is the primary topic of a 1st semester of intro physics), classical or Maxwellian electrodynamics, and classical thermodynamics.

Introductory physics courses are largely about classical physics with some bits of modern physics added.

One would have to be an obscurantist purist not to include bits of modern physics even when teaching essentially classical physics. We know there are electrons, protons, neutrons, atoms, and molecules, and that many things can only be adequately explained in terms of them and other bits of modern physics. So we do introduce bits of modern physics as helpful explanations even in topics that are mostly classical. And, of course, modern physics topics at some level can be included in introductory physics and some introductory physics textbooks include them—at least in extended versions.

Loosely speaking, modern physics consists of quantum mechanics (including quantum field theory), special relativity general relativity, and more esoteric realms we will not mention.

Quantum mechanics is needed to understand small systems: of order molecular size and smaller. Such small systems are called microscopic in physics jargon—even though they are much smaller than can be observed with a traditional optical microscope. Above the microscopic realm is the macroscopic realm. Sometimes people insert a mesoscopic realm in between. There are no hard boundaries between these realms. People use the terms microscopic, mesoscopic, and macroscopic loosely. They are not intended to be strictly defined. Every field needs some elastic terminology.

Special relativity is needed to understand systems with **RELATIVE SPEEDS** approaching the vacuum speed of light. The vacuum speed of light is the ultimate physical speed—the highest speed at which information can be transferred. Actually, some qualification is needed to this description of the vacuum speed of light because of quantum mechanical and general relativistic effects—but we won't go into all that. Special relativity upsets some of our classical physics notions. Most notably time flow and length become dependent on the frame of reference in which they are measured. This leads to some very striking violations of the everyday life sense about how reality works. But those violations are experimentally verified. We just don't observe them in everyday life because in everyday life **RELATIVE SPEEDS** are usually much smaller than the vacuum speed of light—except for the speed of light itself—but that's a special case.

General relativity is essentially a theory of gravity and is needed to understand systems with strong gravity (like black holes) and the cosmos as a whole.

The realm where modern physics is **NOT** needed for an adequate understanding is the realm of classical physics. The center of this realm is can be called the classical limit in which all classical physics is exactly true. The classical limit can't actually be reached in nature and one can't come arbitrarily close to in all senses. For example, getting larger and larger takes you further from the quantum mechanical realm and for awhile—a long while—makes things more classical, but, eventually you get closer to the cosmological realm where classical physics does not apply again. But between all the modern physics realms, one can still still imagine the classical limit as an ideal point where all of classical physics holds exactly.

There are no hard lines to the realms of physics.

As you move away from the classical limit—going too small, too fast, too strong a gravity field, too big—classical physics progressively fails.

But within a broad region of behavior, classical physics is entirely adequate—which means that errors due to the failure of classical physics are negligible compared to the errors in measurements. Classical physics can be called a true approximate theory. Within it's realm of validity it's never wrong and no one believes it will be proven wrong in any circumstance. Perhaps, we will never know the true fundamental theory, but we can know true approximate theories.

Since classical physics is usually much easier to deal with than modern physics, one uses classical physics in the realm of classical physics.

Because it's eminently useful and much simpler than modern physics, one teaches mostly classical physics first. For many fields, particularly many engineering specialties, classical physics is all one needs in one's professional work.

Many concepts of classical physics get used or generalized in modern physics: this is another reason for studying classical physics first.

The classical limit of modern physics is classical physics although I believe most people say this has not been completely proven. There is still controversy about how classical behavior emerges from quantum mechanics.

Modern physics is, of course, not complete: i.e., it's not the fundamental theory—at least we have strong reasons for believing that. One of the strongest reasons is that general relativity is not consistent quantum mechanics. Quantum mechanics is such a powerful, well verified theory that is difficult to believe that it is fundamentally wrong. So the belief is that there is a quantum theory of gravity of which general relativity is the macroscopic limit or at least an approximation to that limit. Quantum theories of gravity exist, but none have yet become well established.

2.1. Physics at the End of the 19th Century: Optional

Toward the end of the 19th century, some folks thought that physics was nearly complete. One prize witness is Lord Kelvin (1824–1907), who in 1900 said:

> "There is nothing new to be discovered in physics now. All that remains is more and more precise measurement."

Kelvin was one of the great developers of classical thermodynamics—but in 1900, he was no longer in his prime. Of course, perhaps all he meant was that at the moment he didn't see anything new in physics. He may have allowed that new developments would change things.

In any case, how general the end-of-physics thinking was I don't know.

But I find it hard to believe that it could have been general among the brightest, active scientists. There are always lots of unsolved problems and mysteries if you are intensely engaged in an important field of research.

But we can advance specific reasons, why it active scientists circa 1900 would believe that physics was far from compete.

First, classical physics does not explain the properties of materials: e.g., all chemical properties. There are jillions of material properties and in a classical physics perspective (as we now understand it), they are all just givens: fundamental properties. But the plethora of fundamental properties conflicts with idea current in physics from the time of the ancient Greeks that fundamental physics should somehow be simple with a limited number of axioms required to explain everything. So I think that many 1900 physicists must have believed that there was a lot more physics to come.

Second, it was appreciated—as it always should be—that outside of the realm in which theories were experimentally verified, they could fail. There were also particular problems with classical physics. A striking one is that Maxwellian or classical electromagnetism taken as an exactly true classical theory is inconsistent with Newtonian physics. Some may have thought this was a minor problem that had a simple solution. But it turned out that special relativity was needed to replace Newtonian physics. Special relativity and classical electromagnetism are consistent.

If we ever found the fundamental physical theory, that would be the end of fundamental physics. But how would we know it when we found it?

3. SYSTEMS, ENVIRONMENTS, MODELS, IDEALIZATION

As discussed in § 1, a **SYSTEM** is any set of objects that one is studying.

Everything else in the universe is the **ENVIRONMENT** in physics jargon.

So we divide the universe into **SYSTEM** and **ENVIRONMENT**.

The system-environment conception is actually general in science and in everyday life.

Since the system is the subject of study, one only needs to understand the environment insofar as it affects the system. This is a great simplification—one doesn't have to know everything in the **UNIVERSE** to know something about a small part.

If the environment cannot affect the system at all, the system is a **CLOSED SYSTEM**: otherwise it is an **OPEN SYSTEM**.

Between the system and the environment is the **BOUNDARY**. The boundary may be a real physical surface or change of conditions or may just be an arbitrary surface we define in space.

Actual real systems are often immensely complex at least in detail.

But often much of that complexity is beyond what you want to know about the system.

So the system can be approximated by a **MODEL** that is simpler than the actual reality, but accurate enough—realistic enough—to yield what you what to know about the system. The environment can likewise be modeled.

In fact, it's true to say that you must always model real systems and environments. The model can be very simple or it can be very complex and, one hopes, very realistic if very complex. You can always approach complex reality more and more closely in principle. Usually, the model for the environment can be simpler than for the system. For a closed system, you don't need to model the environment at all—though you may have to model the boundary.

You hope that at some point in making your system-environment model more complicated, it becomes realistic enough to give you the understanding of the accuracy you desire.

The process of modeling is often one of **IDEALIZATION**. You study an ideal system that lacks many of the complications of the real system.

For example, in intro physics, we often neglect resistive forces like friction and air resistance. Those forces are often hard to deal with.

For another example, the completely closed system is an idealization. No system is ever really completely detached from the rest of the universe.

In intro physics, the level of idealization is often very high. This is to make problems simple enough that they are tractable for intro physics students—and instructors too.

The **IDEALIZED PROBLEMS** illustrate physical law which is one of their main functions. In some cases, the calculated solutions will be quite accurate. In other cases, the calculated solutions will not be all that accurate.

But usually you can't solve more realistic problems, until you can solve the idealized ones.

One particular idealization we will often make is to model objects as particles. The term "particle" has several meanings in physics. In one meaning, a particle is just a point object: it has no size or internal structure. In a second meaning, a particle is a just small discrete object. In a third meaning, a particle is a quantum mechanical particle (e.g., electron, proton, neutron, etc.). Quantum mechanical particles obey quantum mechanics and not classical physics and the fundamental ones may be actual point particles, but we don't know for sure. Even though they are outside of classical physics, we often have to refer to quantum mechanical particles. In a fourth meaing, a particle is an object whose size and internal structure can be neglected even if it isn't a small object. So a car, a human, and the ever-popular nondescript block can all be regarded as particles.

The fourth meaning of particle that we use in our idealization of objects as particles.

This particle idealization is actually very realistic for certain, but not all, purposes. The justification for the realism of the particle idealization is the use of the concept of center of mass. A center of mass is a mass-weighted mean position of an object. It's just a point in space, and so is point-particle-like: no size, no internal structure. The motion of the center of mass of an object is usually what we mean when discussing the motion of the object treated as a particle. The center-of-mass concept is dealt with in the lectures *NEWTONIAN PHYSICS I* and *SYSTEMS OF PARTICLES AND MOMENTUM.*

4. QUANTITIES AND UNITS

Much of physics is in the relationships between between **PHYSICAL QUANTITIES**. The relationships may be physical laws or formulae derived from physical laws.

Physical quantities are measurable or calculable things of relevance in physics.

Now some quantities are **DISCRETE**: i.e. they come in discrete amounts. Those quantities can be measured exactly.

For a non-physical example, consider sheep. If there are three sheep in a field, then there are exactly three sheep in the field.

For an example from physics, the number of atoms in a container is a discrete quantity—but only in some cases anyway since it is possible to have atoms partially in and partially out. It may be hard to measure how many atoms are there—in fact very hard—but there is an exact number of them—in some cases anyway. Counts of this nature can actually be done—in some cases anyway.

But many quantities are **CONTINUOUS** at least at the macroscopic level. By continuous, we mean that the quantity can have any real number value: e.g., length and time.

For discrete quantities, the discrete amounts themselves form natural standard units for measuring the quantity. So there is no problem in principle.

But there is a problem with atoms, molecules, and other microscopic entities at the macroscopic level. These entities are discrete, but we usually can't count them exactly at the macroscopic level. The problem is partially dealt with the concepts of amount of substance and moles. The problem is also partially dealt with by making the **CONTINUUM APPROXIMATION** for microscopically discrete entities. We assume that they are continuous quantities.

But then how do we quantify **CONTINUOUS QUANTITIES** that can't be counted exactly in practice in the real world.

Nature has **NOT** been altogether kind. It has not provided us with standard objects at the macroscopic level that are absolutely identical. We know this empirically and in modern physics we theoretically understand why it must be true. Therefore there is no macroscopic object that can be used as an ideal standard size—an ideal standard unit—for any continuous quantity.

In the past, people agreed on some object or thing that provided a standard unit. In the long ago past, that standard object was only very approximately standard.

For example, human body parts have long been used to define standard units. For specific example, the human foot has probably been used as measure of length since long ago in prehistory. Since most humans have a foot that is not very different in length from the typical human foot length, using an actual foot as measuring device is not so bad—we still sometimes use it that way. One person's ten feet is near enough to another's for many practical purposes.

But as society became more complex demands for precision increased and more standardized standard objects than any random person's foot was needed. Eventually, a standard foot became defined. The modern US foot is defined to be exactly 0.3048 meters (Wikipedia: Foot (length)). The US is metricating—slowly: see below.

The need for standard units leads us to consider the Système International (SI) or metric system of units. The **SI SYSTEM** was first arose in Revolutionary France in the 1790s and has been much developed since then (Wikipedia: International System of Units).

The **SI SYSTEM** is the standard unit system for all of modern science, much of modern engineering, and for standard use in all but three countries it seems: Liberia, Myanmar, and the United States of America (Wikipedia: International System of Units).

There are SI units for all standard physical quantities. But only seven of these units are **BASE UNITS**. Base units are those for which we require a standard procedure to experimentally determine them. The procedure is in fact the definition of the base units. We will consider discuss the base units below—but some of them only optionally.

The program of modern metrology—which is not modern meteorology—is to find base unit definitions that in theory are exact so that the base unit never varies in size in theory. Except for mass (as we discuss below in § 4.2), the modern program has been fulfilled. Actual experimental determinations of base units can never be exact at least at the macroscopic level even if the base units are exact in theory, but there is no macroscopic theoretical limit to how accurately they can be determined if they are exact in theory.

You can always try to determine them more accurately in practice than has been done before.

Of course, most measurements with the base units (and all other units) do not use the exact base unit definition directly, but use some measuring device that is calibrated at some remove by the exact base unit definition.

The SI base units are given in Table 1. The base units of most immediate interest us are the meter for length, the kilogram for mass, and the second for time.

Table 1. SI Base Units

Name	Symbol	Quantity
meter	m	length
kilogram	kg	mass
second	s	time
ampere	A	electric current
kelvin	K	Kelvin or thermodynamic temperature
mole	mol	amount of substance
candela	cd	luminous intensity

Note. — The entries are from Wikipedia's article "SI Base Unit" which also gives the base unit definitions.

What of the **FIDUCIAL UNITS** (i.e., reference units) for quantities that do not have base units?

Exact physical formulae relate those quantities to the quantities that have base units.

So the fiducial units for those quantities without base units can be related exactly to the base units.

The relationships give the non-base fiducial units in terms of the base units. We say that the non-base fiducial units are **DERIVED UNITS**.

For example, speed is the ratio of length to the time it takes an object to traverse that length. So the SI unit of speed is the meter per second or m/s.

Some quantities without base units have special names (and also special symbols, of course) for their fiducial units. Speed is an example of quantity which does **NOT** have a special name for it's fiducial unit. The speed unit is just the meter per second.

An example quantity whose fiducial unit does have a special name is energy. The fiducial unit is the joule with symbol J. The joule is a $\mathrm{kg\,m^2/s^2}$. We get to energy and joules in the lecture *ENERGY*. But most people have a good understanding of **ENERGY** anyway. It's the stuff that allows you do stuff. To be more elaborate, yours truly likes the definition "Energy is the transformable and conserved universal essence of structure". The "transformable" part indicates that stuff can be done with energy. In fact, there is no short fully adequate definition of energy.

The fiducial units (both base and derived) are actually called the MKS units where MKS stands for meters kilograms seconds. The MKS units are a subset of all SI units as we'll discuss below.

Why do we have just 7 base units?

Because all standard physical quantities can be measured in those base units or in the units derived from them.

Are there quantities that can't be measured in the units of the standard physical quantities?

Sure. For a super-trivial example, the amount of oranges in pile. One would naturally measure this amount in the unit of an orange. But the units for such special quantities have no general utility. For example, the amount of oranges in most systems is obviously zero.

Are the 7 base units unique choices?

No. Some of the fiducial derived units could be accepted by convention as base units replacing some of the current base units. The replaced base units could then be derived from the new set of base units. For example, the unit of current the ampere is a base unit. A unit derived from it is the unit of charge, coulomb (C), which is an ampere times a second (A s). One could define the coulomb as a base unit and then the ampere would be a derived unit: the Coulomb per second (C/s).

Why do we use the 7 base units that we do if their choice is not dictated by physical principle?

The current 7 base units have definitions that are judged in some sense to be convenient for accurate and precise measurement of the base units. It's possible that the choices may change in the future.

Who governs the SI system? The organizations created by the Metre Convention (Wikipedia: Metre Convention)—and it's metre, not meter. The US is a signer of the convention and so are most other large countries (and some minor ones too). I imagine the national bureaus of standards of these countries are involved in these organizations. The US national bureau of standards is National Institute of Standards and Technology

(NIST)—which until 1988 was conveniently called the National Bureau of Standards (NBS).

As well as the MKS units, SI includes units that are these units multiplied by standard powers of ten. The standard powers of ten are denoted by standard prefixes given to the fiducial unit. The standard powers of ten and prefixes are given in Table 2. Units created by standard powers of ten are also derived units, of course.

Table 2. SI Prefixes for Powers of Ten

Power of Ten	Prefix	Abbreviation
10^{-24}	yocto	y
10^{-21}	zepto	z
10^{-18}	atto	a
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	μ
10^{-3}	milli	m
10^{-2}	centi	c
10^{-1}	deci	d
10^{0}		
10^{1}	deca	da
10^{2}	hecto	h
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T
10^{15}	peta	P
10^{18}	exa	E
10^{21}	zetta	Z
10^{24}	yotta	Y

Note. — There are plans afoot to extend the standard powers of ten to include 10^{-27} (prefix: harpo abbreviated ha) and 10^{27} (prefix: groucho abbreviated Gr). But this is still controversial.

One anomaly with the powers-of-ten units is that the prefixed kilogram is the fiducial unit of mass and not the unprefixed gram. Another anomaly is that 10^3 kg is usually called a tonne (or metric ton) and not a megagram (Mg).

Not all the prefixed units are in common use. A prefixed unit tends to be used in fields where it is a convenient size for the quantities of that field. For example, in nanotechnology the nanometer (nm) is commonly used.

In fact, the prefixed units are often only used for mental convenience in thinking about quantities, not in calculations. In most calculations, one uses only MKS units. The reason is that one doesn't have to do any unit conversions (see § 5) or keep track of units. If only MKS units are used with the inputs, then all the results of the calculations will be in MKS units.

If you have a problem where the given values are not in MKS units, the usual procedure is to convert them to MKS units before doing anything else, do all the essential calculations, and then if necessary convert any final results to any units one wants to see them in.

In some fields, one actually uses another subset of SI units in calculations instead of MKS. The other subset is the CGS units where CGS stands for centimeters grams seconds. CGS units are commonly used in astronomy—and so I'm pretty familiar with them.

As well as official SI units, some other units are in common scientific use. The first example that comes to mind is the angstrom (Å) which is 10^{-10} m. It has a continuing vogue because the atoms have a size scale that is of order 1 Å. Another example is that in many fields the Celsius temperature scale is used rather than the Kelvin temperature scale which is the official SI temperature scale. The Celsius degree (symbol C) and Kelvin degree have the same size, but the zero temperature is different for the two scales: in the Celsius scale, it is approximately the freezing point of water and in the Kelvin scale, absolute zero. The zero of the Celsius scale is $0°C = 273.15\,K$, where the degree symbol is not used for the Kelvin scale by convention.

Of course, nonstandard units are used all the time for special purposes. If those nonstandard units are suggested by the nature of the system being studied, then they are often called natural units. For example, in astronomy, the Sun's mass is a natural unit for measuring stellar masses. This use is mainly for thinking about stellar masses. For calculations, one almost always uses an SI mass unit, the kilogram or (if you are astronomer) the gram. In fact, many nonstandard or natural units are used primarily for convenience in thinking about quantities.

There are non-SI conventional unit systems, but mostly they have fallen out of general or primary use. The great exception is that for everyday and many commercial purposes, the US uses United States Customary System which in the US we usually call British units since they are approximately the units the British used to use (Wikipedia: United States customary units). You know about these—feet, furlongs, hogsheads, and so on. Almost no one else in the world uses British (or British-like) units for standard units anymore. However, Liberia and Myanmar apparently still use them as their standard units (Wikipedia: International System of Units).

Actually, the US is supposedly abandoning the United States Customary System for SI—a process called metrication. The Omnibus Trade and Competitiveness Act of 1988 says that the United States government designates the SI system as "the preferred system of weights and measures for U.S. trade and commerce". But metrication in the US is proceeding so slowly that to most folks it seems a whole lot like no progress at all.

We could at least get rid of Fahrenheit. I'd suggest getting rid of the Celsius scale too. We should just use the Kelvin scale for everything all the time—but no one ever listens to me.

In the following subsections, we discuss the quantities length, mass, and time and their base units. Optional subsections on the other quantities with base units and their base units also follow.

4.1. Length

Quantifiable space is such a basic element of existence that it defies easy definition. We won't try.

One can say that **LENGTH** is a measure of space a 1-dimensional curve in space. Maybe that helps.

We usually think of space as being the 3-dimensional Euclidean space of Euclidean geometry. The space of universe over short distances and far from strong sources of gravity approximates Euclidean space to high accuracy. There are other kinds of spaces: e.g., the curved unbounded, but finite space of the surface of a sphere.

The SI base unit of length is the meter.

Originally the meter was defined to be $1/10^7$ of the north-pole-to-equator distance along a meridian. But this definition isn't so good since it turned out to be difficult to measure the meter this way to the desired accuracy. Also that distance isn't fixed since the Earth flexes under Earth tides and other slower motions. But modern definitions are roughly consistent with this definition. This is why the meridional circumference of the Earth is $4.000786 \times 10^7\,\text{m} = 40078.6\,\text{km}$ to this day.

From 1889 to 1960, the meter was defined to be the distance between two marks on platinum-iridium bar (the prototype meter bar) that was and is kept in France at Sèvres near Paris. Why Paris? Well why not? It beats Peoria.

But the prototype meter bar is artifact. Its length can change with environmental conditions and with cleaning and caretaking. And you have to go to France to check your meter stick against it.

After some intermediate stages, we arrived at the modern meter definition in 1983. The modern meter is defined to be exactly the distance traveled by light in a vacuum in 1/299792458 s.

This means that the vacuum speed of light is exactly 299792458 m/s by definition.

The meter is thus based on the vacuum speed of light and the definition of the second which we consider in § 4.3. The second is defined using a clock that is in theory exact.

The choice of the number 299792458 in the definition is for consistency with the older definition of the prototype meter bar.

Why is this a good definition?

Aside from corrections to general relativity which can be made negligible, the vacuum speed of light is in the theory of special relativity an exactly constant value. Special relativity is an extremely well verified theory. Aside for corrections to general relativity, it has never been found to be wrong to any degree. Thus, we believe the constancy of the vacuum speed of light (aside from general relativity corrections) to be exact to a higher degree than we can measure. Perhaps, it is truly exactly constant (aside from general relativity corrections). Thus, people determining the meter from the modern definition are limited in accuracy in theory only by their experimental technique, their corrections for general relativity, and the accuracy of their second determination.

And they don't have to go to Paris to make their determination.

Thus, the modern meter definition accords with the program of modern metrology to find base unit definitions that in theory are exact.

Actually, one could ask why not choose the meter per second as the base unit and thus make speed a base unit quantity. Well except for saying so, we sort of have done that already. Probably people are just more use to thinking of length as base unit quantity. Also it's hard to use light directly to measure other speeds. It can be used most directly to measure distances. So keeping length as the base unit quantity probably seems best.

4.2. Mass

MASS as we'll discuss in the lecture *NEWTONIAN PHYSICS I*, is the resistance to acceleration of a net force.

Mass is also sometimes defined as the quantity of matter. This definition is useful for many purposes since mass (in the sense of the first definition) scales nearly exactly with the number of atoms in an object of one element or the number of molecules in an object consisting of one kind of molecule. But the first definition is the primary definition and the one we mean usually in intro physics.

We won't go into procedures for measuring mass here. Simple ones turn up in or at least become obvious extensions of the material in the lectures *NEWTONIAN PHYSICS I* and *NEWTONIAN PHYSICS II*.

The base unit of mass is the kilogram.

It is defined to the mass of platinum-iridium cylinder kept in Sévres along with the old prototype meter bar (Wikipedia: Kilogram). This cylinder is the prototype kilogram. It's pretty small: the height and diameter are 3.917 cm (i.e., about 1.542 inches).

The trouble with using an artifact for the kilogram definition is that it is subject to change. The environment (which includes cleaning and caretaking) can add and subtract minute amounts of material. And, of course, the object could be damaged or lost—say in war.

So metrologists would like replace the prototype kilogram cylinder definition with a definition based on some exact feature of nature. There are ideas for doing this, but so far they havn't been good enough.

But that may change very soon. There are plans to convert to an exact physics definition of the kilogram by 2011. But we won't go into that now—maybe then.

There is, however, an exact definition for mass on the microscopic scale.

The atomic mass unit or AMU (symbol u) is defined to be the mass of an unperturbed carbon-12 atom. Note when the AMU is used as physical constant rather than as as unit, it is often given the symbol a. Unperturbed means the atoms are unbound, are free of external forces, and are in their ground state (i.e., their lowest energy state). One can't practically reach exact unperturbedness, but there is no limit on how closely one can approach it. Carbon-12 is the isotope of carbon with 6 protons and 6 neutrons.

In quantum mechanical theory, all unperturbed carbon-12 atoms are absolutely identical. Actually, all unperturbed microscopic particles of a given type are absolutely identical. Quantum mechanics is such a well verified theory that this identically property is believed to more exact than any measurement we can do to disprove it. In fact, it makes sense to believe that the identicality property is truly exact.

The carbon-12 atom is chosen to define the AMU for some reason of experimental convenience I imagine.

The AMU is used in the measurements of the masses of microscopic particles.

Why can't we use the carbon-12 atom to define the kilogram?

At present, the accuracy of measuring macroscopic objects on the scale of carbon-12 masses is not sufficiently accurate. Maybe this will change one day. Maybe one day the kilogram will be defined as so many AMUs (see also § 4.6). But at the moment, we are stuck with the cylinder in Sèvres.

4.3. Time

TIME is actually a pretty hard thing to define.

Objects occupy different places in a sequence—time passes.

Actually there is a continuum of positions occupied in a sequence—time passes.

Some systems go through repeated motions which we call periodic motions—time passes.

Some periodic motion systems seem so exact in their spatial evolution and so synchronizable that they were recognized as clocks and used to measure time—how many periods of the clock does it take for such and such to happen.

There are many historical clocks.

The sequence of days, lunar months, solar years, and other astronomical repeating events.

In fact, these historical astronomical clocks seemed to repeat so exactly that they were taken to repeat exactly regularly and to measure time itself. Of course, many of the astronomical clocks were known to vary when compared to other astronomical clocks: daytime and nighttime vary in length compared to the day, etc. But as time passed, those variations it seemed could always be found to be part of a larger, more regular cycle.

Other clocks like the passing of human lives and other life cycles and the beating of the heart seemed too irregular and individual in behavior and only approximately kept pace with the astronomical clocks, and so were not taken as measurers of time, except approximately.

Artificial clocks (water, sand, and mechanical clocks) were invented that were synchronizable with the best astronomical clocks. Various irregularities in their periods compared to those of the astronomical clocks could be removed by refinements. Of course, such clocks had to be maintained and supplied with some source of energy—although that it was energy that was needed was not well understood until the 19th century.

In Newtonian physics when it came along in the 17th century, there is a time parameter which we just call "time". And in Newtonian physics, there are ideal periodic systems that should repeat in equal periods of time. Newton and everyone else were not at all surprised—it was built into their preconceptions—that real artificial clocks should approximately keep time (i.e., measure time) according to Newtonian physics and that ideal artificial clocks should keep time exactly.

But no artificial clock is ideal. Even the best astronomical clocks could not keep time exactly in principle—Newtonian physics showed this—but the deviations were not measurable until well after Newton.

Newton himself did wonder if time flowed equally in all places and times: i.e., did even ideal clocks in different places and times keep the same time if they could somehow be compared. There was no absolute proof that they did, but it was the simplest hypothesis that they did and nothing then contradicted that.

In the 20th century, relativity and confirming experiments proved that time is reference frame dependent. Even ideal clocks in frames moving with respect to each other and in different gravity fields desynchronize in a predictable way. The effects were too small to detect in laboratory situations before the 20th century.

In modern cosmology, there is, in fact, a theoretical cosmic time which is consistent with our observations. This cosmic time is the time in frames of reference that participate in the mean expansion of the universe since the big bang. We can to some accuracy measure cosmic time. In fact, the ordinary time of our reference frame on Earth is not very different from cosmic time. The time since the big bang in both cosmic time and Earth frame time according to best modern theory and measurements is about 13.7 billion years (or 13.7 gigayears). This value is may change actually with improvement in theory and/or observations.

But how do measure time to highest accuracy in the modern world.

We use atomic clocks.

The practical aspects of them, we won't discuss. If you want some details, see the Wikipedia article *Atomic clock*.

In quantum mechanical theory, such clocks when unperturbed keep exactly regular time. One can't practically reach exact unperturbedness, but there is no theoretical limit on how closely one can approach it. The best atomic clocks are those deemed to be the most exact and convenient.

The base unit of time is the second.

It is defined (and therefore measured) to be exactly 9192631770 periods of oscillation of the electromagnetic radiation from a particular transition (emission channel) of the caesium-133 atom. (The 133 indicates the isotope of caesium with 55 protons and 78 neutrons).

Why this particular atom and transition? Some reason of experimental convenience that is beyond me.

Why 9192631770 periods?

To keep the modern definition of the second roughly consistent with the historical definition of the second which was 1/86400 of the mean solar day. Actually, the mean solar day increases in time as people knew by comparison to other more exact astronomical clocks, I believe, even before atomic clocks came along.

4.4. Electrical Current: Optional

This is best left to when electromagnetism is being studied.

4.5. Temperature: Optional

This is best left to when thermodynamics is being studied.

4.6. Amount of Substance: Optional

Amount of substance is a somewhat weird thing.

Formally, it's a quantity that is proportional to the number of of elementary entities in a sample (Wikipedia: Amount of substance). The elementary entities could be electrons, atoms, molecules, neutrons, or anything customarily regarded as a microscopic particle.

The SI base unit is the amount of substance in exactly 12 g of unperturbed carbon-12 atoms. It's impossible to really have 12 g of carbon-12 in this state, but it's an ideal limit one can approach. The unit is called a mole (abbreviation mol).

There is, of course, a number value for the number of atoms in exactly 12 g of unperturbed carbon-12 atoms. That number value is Avogadro's number

$$N_{\rm A} = 6.02214179(30) \times 10^{23}\,\text{particles/mol}\ , \tag{1}$$

which is customary defined to have units of particles per mole. Contrary to popular belief, the mole is not defined to be Avogadro's number. At present Avogadro's number is an experimentally determined value.

But the day may come when Avogadro's number is defined to be an exact value and the gram will be defined to be exactly the mass of 1/12 of a mole of unperturbed carbon-12 atoms. The kilogram would then be exactly 1000/12 moles of unperturbed carbon-12 atoms (Wikipedia: Mole (unit)). Then we could dispense with that prototype kilogram cylinder in Sèvres, France.

At present though, it seems that determining the kilogram experimentally by this definition is not as accurate as determining it from the prototype kilogram cylinder.

But how does one use the mole at our level without going into refinements about unperturbed and perturbed atoms?

Say you have a sample of mass m of some substance made up of a single kind of microscopic particle. Let m be measured in grams. The number of entities N in the sample is

$$N = \frac{m}{Aa} = \left(\frac{m}{A \times 1\,\mathrm{g/mol}}\right)\left(\frac{1\,\mathrm{g/mol}}{a}\right) \ , \tag{2}$$

where A is the mass of the entity in terms of AMUs, a is the AMU in grams, and

$$\frac{1\,\mathrm{g/mol}}{a} = \frac{12\,\mathrm{g/mol}}{12a} = N_{\mathrm{A}} \tag{3}$$

is Avogadro's number. So the number of moles in the sample is

$$N_{\mathrm{mol}} = \frac{N}{N_{\mathrm{A}}} = \frac{m}{A \times 1\,\mathrm{g/mol}} \ . \tag{4}$$

The quantity $A \times 1\,\mathrm{g/mol}$ is called the molar mass or in older terminology the gram atomic weight for atoms and the gram molecular weight for molecules, but in both cases its actually a mass, not a weight.

Note you don't need to know a Avogadro's number at all to find the amount of substance $N_{\rm mol}$, just the molar mass of the particles in the sample. To find the number of particles in the sample, you do need to know Avogadro's number since

$$N = \frac{m}{Aa} = \left(\frac{m}{A \times 1\,{\rm g/mol}}\right)\left(\frac{1\,{\rm g/mol}}{a}\right) = N_{\rm mol} N_{\rm A} \ . \tag{5}$$

4.7. Luminous Intensity: Optional

Luminous intensity is the quantity of human sensitivity to light.

The formula definition is

$$I_V = 683 \int_0^\infty I_\lambda V(\lambda)\, d\lambda \ , \tag{6}$$

where 683 is an exact traditional scaling constant by definition (with units of candela second steradian per joule: i.e. candela $({\rm s\,sr})/{\rm J}$), I_λ is the specific intensity which is energy per unit time per unit wavelength per unit steradian (with units of $\rm J/(s\,nm\,sr)$), λ is wavelength (with units of nanometers), the integral is over all wavelength as indicated by the limits, and $V(\lambda)$ (which is unitless or dimensionless (see § 6)) is CIE standard luminosity function as adopted in 1924.

The units of I_V work out to be in candelas—and this is by design. We won't go into the details of the procedure for officially determining (i.e., defining) the candela: see Greene (e.g., 2003).

CIE is the Commission Internationale de l'Éclairage. Their official vision curve $V(\lambda)$ is fiducial average of the normal human sensitivity to light under bright or photopic conditions. It's a fiducial average since it was determined from a limited number of human specimens long ago and has never been updated it seems. So it's not an average for any particular human population nor does it account for any evolution of human population. Still its pretty close to the vision of just about anyone with normal vision.

What $V(\lambda)$ looks like is a curve that rises from zero at 380 nm (the violet end of human sensitivity), reaches a smooth peak of 1 at 555 nm (which is yellow light), and then falls to zero at 780 nm (Wikipedia: Luminosity function).

What does $V(\lambda)$ mean? Well to obtain the same subjective human response to light of arbitrary λ as at 555 nm to $I_{\lambda=555\,\mathrm{nm}}$, the specific intensity of the light must $1/V(\lambda)$ times the $I_{\lambda=555\,\mathrm{nm}}$. That's what it means.

Note that at the ends of the curve $1/V(\lambda)$ goes to infinity and the description of its meaning fails.

Of course, subjective human response is a tricky concept and special procedures are used to define how to measure that. Since subjective human response varies from person to person and even for a person depending on many things, subjective human response is probably not all that constant. The standard luminosity function $V(\lambda)$ only gives the fiducial human response as established in 1924.

It actually isn't a perfectly good. For example, humans can see into infrared as far as about 900 nm for a sufficiently bright source (e.g., Pedrotti & Pedrotti 1998). It's no revelation though—the light just looks deep red. It may not be all that safe to look at such bright sources—I don't know.

There has been a debate as to whether or not the candela should be a base unit. It is after all unit defined for the very special quantity of human sensitivity to light, and does not have wide applicability like other base units. But traditionally its a base unit and a base unit it remains.

5. UNIT CONVERSIONS

A unit represents a value and it can be treated liked an algebraic variable—it is an algebraic variable whose value is known.

So unit conversions are really easy to do although often tedious. They are just algebra with variables whose value you know actually, but never explicitly show.

I like using the concept of **FACTORS OF UNITY** in doing conversions. Factors of unity are best explained by examples.

And so for example, consider the known equation

$$1\,\mathrm{km} = 1000\,\mathrm{m}\ . \tag{7}$$

This equation is exact by definition. You can never measure a length exactly, but you can define two unit lengths to have an exact relationship. The units kilometers and meters are just symbols standing for amounts that are multiplied by numbers. If you divide one side by the other, you have a factor of unity. Say

$$1 = \frac{1000\,\mathrm{m}}{1\,\mathrm{km}}\ . \tag{8}$$

Both the left-hand side and the right-hand side are 1 in value. The right-hand side is a factor of unity in the jargon I use.

You can always multiply anything by 1 without changing its actual value. So say you want to convert 7 km to its value in meters. Multiply it by the factor of unity: i.e.,

$$7\,\mathrm{km} = 7\,\mathrm{km} \times 1 = 7\,\mathrm{km} \times \frac{1000\,\mathrm{m}}{1\,\mathrm{km}} = 7000\,\mathrm{m}\ , \tag{9}$$

where the kilometer variable has been canceled out.

Let's do a tougher example. Let's convert 10 m/s into miles per hour.

By the by, what's important in the human context about 10 m/s?

It's about as fast as a human can run—Usain Bolt can do about 10.4 m/s.

Of course, one has to be an Olympic class sprinter to run that fast.

OK, the conversion is

$$10\,\mathrm{m/s} = 10\,\mathrm{m/s} \times 1 \times 1 = 10\,\mathrm{m/s} \times \frac{1\,\mathrm{mi}}{1609.344\,\mathrm{m}} \times \frac{3600\,\mathrm{s}}{1\,\mathrm{h}} = 22.36936\ldots\mathrm{mi/h} \approx 22.37\mathrm{mi/h}\ . \tag{10}$$

where we just inserted the appropriate factors of unity and canceled out the redundant units.

Note the equality $1\,\mathrm{mi} = 1609.344\,\mathrm{m}$ is exact in the modern unit system (Wikipedia: Mile). Actually, one also has the following exact results: $1\,\mathrm{in} = 2.54\,\mathrm{cm}$, $1\,\mathrm{ft} = 0.3048\,\mathrm{m}$, and $1\,\mathrm{yd} = 0.9144\,\mathrm{m}$. The US is actually stealth metricated.

About how fast does a human walk in miles per hour?

A human walks about 1 m/s by casual observation, and thus just dividing our last conversion result by 10 gives 2.237 mi/h. I think this is a bit slow—more of as stroll than a walk. References typically say an average human pace is about 4 mi/h. Maybe that's a fastish pace.

Now for a tougher conversion example. The hogshead example of Appendix B.1. Let's go there.

And that's all there is to conversions.

Of course, if you stick to pure MKS units you never need to do conversions.

However, frequently in this class and in life you are given quantities not in MKS units and/or are asked for answers not in MKS units. So you either have to calculate in non-MKS units or do conversions and sometimes, of course, you have to do both.

6. DIMENSIONAL ANALYSIS

The word "dimension" in the context of dimensional analysis is rather tricky to define. At least it seems most textbooks and even Wikipedia do a rather poor job of it.

One can say a dimension is quantity where by quantity one means the physical nature of the quantity.

For example, 1 m, 2 m, and 3 m are all lengths using length to mean both nature and size. But one could say that they all have the nature of length. And, in fact, one says they have the dimension of length.

Now the mathematical laws of physics are relationship between quantities. It turns out the only meaningful (i.e., useful) way to produce or derive a new quantity from old quantities is by multiplication of powers of those old quantities: division occurs for inverse powers. For example, a length divided by the time it took to traverse that length is speed which is a physically meaningful quantity. The length plus or minus the traversal time has no meaning: i.e., we have no use for it in understanding the system. The dimension of speed is the dimension of length divided by the dimension of speed. Saying "dimension of something" is tedious and so when one knows what on means, one says "something". In the present case, speed is length divided by time.

It follows that the only way to produce or derive a new dimension from old ones is by multiplication of powers of those old dimensions: division occurs for inverse powers. But it's an odd kind multiplication. It's multiplication of "nature" rather than "number". Dimensional analysis is math without number. Well that has to be qualified since the number of factors of a given dimension is counted. For example acceleration is distance divided by time squared: i.e., the dimension of acceleration is the dimension of distance divided by the dimension of time squared.

A sort of anomaly is that it seems fine to say either the "dimension of a quantity" or the "dimensions of a quantity". Both locutions mean the same thing and both are used. Actually, the latter one seems favored—it just trips off the tongue.

Some physical quantities are dimensionless. For example, angles are dimensionless since they are, among other things, the ratio of circular arc length subtended by the angle to the radius to the arc length. So the dimension of angle must be length divided by length which in dimensional analysis is called dimensionless rather than 1. The transcendental functions and the logarithm functions yield dimensionless values. The transcendental functions can be, but not necessarily are, the ratios of side lengths of triangles. So they must be dimensionless. Exponents are dimensionless too. Also any quantity raised to the power zero is a dimensionless quantity.

Note that dimensionless quantities are sometimes given conventional units. Angles can be in radians or degrees, but angle is still a dimensionless quantity.

Now what does dimensional analysis mean in its usual meaning. It means the procedure for checking if an equation is dimensionally correct. If one has an equation which is dimensionally wrong, then the equation is wrong. By dimensionally wrong one means that the terms in the equation do not have the same dimension or the two sides of equal sign do not have the same dimension.

If an equation is dimensionally incorrect, it is physically meaningless which means it's just plain incorrect if it is supposed to mean something in physics. But dimensionally correct equations may be incorrect for other reasons and frequently are.

Being dimensionally correct is a necessary, but not a sufficient condition for an equation to be physically correct.

There is a second meaning for dimensional analysis which is discussed below in the optional § 6.1.

To carry out dimensional analysis, the dimensions of quantities with base units are given certain symbols. The conventional ones for length, mass, and time are, respectively L, M, and T. Note these symbols are roman letters. Conventional symbols exist for the other quantities which have base units and maybe for other quantities as well. But for one's own purposes, one can use any dimension symbols one likes actually.

There is a conventional function to evaluate the dimension of a variable x: the function function [ ]. One can write

$$[x] = \text{dimension symbol for the quantity of } x \ . \tag{11}$$

For example, say x is a length:

$$[x] = \mathrm{L} \ . \tag{12}$$

The dimension evaluation function distributes over all terms and factors in an equation. Acting on a dimensionless quantity [ ] yields 1 which is the dimensional analysis unit. One does not have to explicitly write the dimensional analysis unit, unless there are no dimensions to write down. What happens to the units of dimensionless quantities (e.g., radians) in formulae and calculations? Oh, they just appear or disappear as needed by the formulae.

The physical relationships between quantities dictate how to construct dimension symbols from other dimension symbols: one constructs them by multiplication of powers of the symbols. For example, an area is equal to the product of two perpendicular lengths. Thus, the dimension symbol for area is L^2. For another example, energy is a quantity that is calculated from formulae that multiply out to give quantities with the dimensions $\mathrm{ML}^2/\mathrm{T}^2$, and so $\mathrm{ML}^2/\mathrm{T}^2$ is the dimension of energy or $\mathrm{ML}^2/\mathrm{T}^2$ are the dimensions of energy.

As an example of dimensional analysis let's consider the horizontal range formula for projectile motion near the Earth's surface neglecting air resistance. The formula is

$$R = \frac{v^2}{g}\sin(2\theta) \ , \tag{13}$$

where R is the horizontal distance the projectile travels until it returns to its launch height, v is the launch speed, g is the acceleration due to gravity, and θ is the angle of launch from the horizontal (e.g., Serway & Jewett 2008, p. 79).

Applying the dimension evaluation function to both sides one gets

$$[R] = \mathrm{L} \qquad \text{and} \qquad \left[\frac{v^2}{g}\sin(2\theta)\right] = \frac{[v^2]}{[g]}[\sin(2\theta)] = \frac{\mathrm{L}^2/\mathrm{T}^2}{\mathrm{L}/\mathrm{T}^2} \times 1 = \mathrm{L} \ . \tag{14}$$

Thus, the horizontal range formula is dimensionally correct since both sides of it yields a quantity that has dimension length—or one can say that is a length.

As mentioned above, dimensional analysis is math without number. All dimension symbols have constant values. For example,

$$\mathrm{L} = \mathrm{L} + \mathrm{L} \ . \tag{15}$$

Also as mentioned above, dimensional analysis does have unit value which is the result of [ ] operating on a dimensionless quantity.

There seems no need to have an explicit zero in dimensional analysis.

Instead of using the conventional symbols for dimensional analysis, one can just use the units of quantities, usually the MKS units in intro physics. In fact, yours truly often does that. One remembers that the units of dimensionless quantities appear and disappear as needed by the formulae.

6.1. The Other Meaning of Dimensional Analysis

As well as meaning a procedure for checking formulae, dimensional analysis also means a procedure to create formulae.

Say you have a system for which you want to a formula for a particular quantity—but you are pretty clueless as to what the formula is or how to get it.

The dimensional analysis procedure is just to create a formulae that is dimensionally correct and which includes quantities from the system that are relevant. Of course, you could create a pretty lousy formula. But with some physical insight, you might get a formula that gives you order-of-magnitude accurate results or even better. (See § 8 for order-of-magnitude calculations.) The dimensional-analysis formula may help you to find an even more accurate formula.

An example of finding a formula by dimensional analysis is the Earth-rotational-kinetic energy example given in Appendix B.2. Let's go there.

7. SIGNIFICANT FIGURES

In the science context, significant means "means something" and insignificant means "means nothing".

Usually, one speaks of a result as being significant if has some level of accuracy, and so has some reliability and as being insignificant if it is completely unreliable. Of course, completely unreliable is an ideal of unreliability. At some point, some judgment based on some criterion must decide between reliable (significant) and unreliable (insignificant).

What are significant figures?

A measurement or a calculation frequently yields a number with some figures (i.e., digits) that are significant and some that are not. There are ways of deciding which are which that depend on case and which we don't have to specify here.

The significant figures are the **SIGNIFICANT FIGURES**; the others are the **IN-**

SIGNIFICANT FIGURES. Say x's stand for significant figures and y's for insignificant ones. Then for example one has

$$xxx.xxyyy\ , \qquad xxy.yyy\ , \qquad xxx.yyy\ , \tag{16}$$

where "." is the decimal point. Of course, all the figures could be significant (e.g., $xxx.xx$) and all-too-frequently all could be insignificant (e.g., $yy.yyy$).

Usually, though not always, the significant figures are the leading ones. Our examples show only the leading case. A case with non-leading significant figures would be for example $yyxx.xxy$. Hereafter, we assume that the significant figures are the leading ones to avoid dealing with weird cases that take special treatment.

When reporting a value in a detailed report, one usually should only report the significant figures. So if one calculates $xxx.xxyy$, one should report only $xxx.xx$. The insignificant figures are completely unreliable by some criterion, and therefore convey no information and give the misleading impression that they do if reported.

There are simple rules for determining significant (and insignificant) figures in output values from calculations given the significant figures in the input values.

But the rules are only approximate rules: i.e., they only give the approximately the right significant figure determinations. We specify these rules in the subsections below. For the sake of our discussion, we give these memorable rules **NAMES**, but they don't, in fact, have conventional names as far as yours truly knows.

Note that the criteria for judging significance of figures from observation and the rules for determining them in calculations are partially determined by the fact that we use a decimal or base-ten numeral system. Using a decimal numeral system means that our description of numbers is partially quantized in that in each decimal place (or numeral place to be generic) there are only ten possible values and in that the decimal place powers differ by a factor of ten. If we used a numeral system with a different base number, there would be at least small differences from the decimal numeral system in determinations of the degree of accuracy of values from both observation and calculation. But if one is doing an accurate job of determining accuracy, those small differences themselves will probably be insignificant.

The simple rules for determining significant figures in calculations are approximate as noted above. There are more accurate methods for determining the significant figures in the output values. Those methods are **NOT** very hard to understand or use, but they take a bit longer to use and are usually unneeded in an intro physics class where the calculations are for educational purposes—unless, of course, the educational purpose is to determine the output value significant figures accurately.

The more accurate methods of determining significant figures require specifying uncertainties (or errors as they are often called) with the input values to a calculation and using these uncertainties to calculate the output value uncertainties and significant figures. Some discussion of these methods is given in Appendix A.

In intro physics problems, uncertainties are not usually given with values. The values are simply given to so many figures and those are usually meant to be all significant figures. Although one doesn't have to do this, it is reasonable to assume that the uncertainty of the values given without uncertainty is numeral 1 in the decimal place of the last given figure. Intro physics problems are usually consistent with this assumption though not insistent on it. We can call this uncertainty the fiducial-numeral-1 uncertainty.

In courses of yours truly, you don't need to be too careful with significant figures, unless the problem specifically asks you to obey the significant figure rules. Your results should have about the number of significant figures for your input values. If you report a few insignificant figures in order to make sure you are not dropping significant figures, that's OK. But you shouldn't drop obviously significant figures, unless you are doing an order-of-magnitude calculation.

Now for the significant figure rules.

7.1. The Round-Off Rule

The round-off rule is explained as follows.

When dropping insignificant figures, one rounds the trailing significant figure down or up.

If the insignificant figures amount to number less than half of a 1 in the trailing significant figure place, round down. For example, say one has 6.44, but the last figure is insignificant, one rounds down to 6.4.

If the insignificant figures amount to number more than half of a 1 in the trailing significant figure place, round up. For example, say one has 6.46, but the last figure is insignificant, one rounds up to 6.5.

The insignificant figures are deemed meaningless, but one may wrong. They may have a tiny bit of meaning, and so the round-off rule exploits that possibility.

What if the insignificant figures are exactly half of a 1 trailing significant figure? Let's call this the half case

The procedure is round to the trailing significant figure to an even value.

For example if one has 6.45, but the last figure is insignificant, one rounds down to 6.4. For another example if one has 6.55, but the last figure is insignificant, one rounds up to 6.6.

The half case of round-off rule prevents a bias. If one is doing many calculations and assumes that trailing significant figures are even and odd with equal likelihood, the half case of round-off rule prevents the bias of rounding down more often than rounding up and vice versa. If one didn't prevent this bias, one might end up with results that were systematically too large or too small.

In short calculations, half cases probably happen rarely, and having a procedure for them is not so important. But it's good practice to use the half case of the round-off rule even when it is not important to do so.

People could have chosen to round to the odd trailing significant figure in the half case and had the same bias prevention. But having chosen to round to the even trailing significant figure, people must stick to it.

The round-off rule is used generally even when one is using a better method of significant figure determination than that following from the rest of the simple significant figure rules.

7.2. The Addition Rule

In addition (which includes subtraction as a special case of addition), the leading insignificant figure decimal place out of all the terms is the leading insignificant figure decimal place of the sum.

For example, add 3.15 and 2.1 with only significant figures reported. The leading insignificant figure decimal place in the terms is 2nd to the left of the decimal point. Therefore that is the leading insignificant figure decimal place in the sum. When one adds taking the values as exact one gets 5.25. We now round to significant figures (and we need the half case of the round-off rule) and get 5.2.

The reason of the addition rule is the unreliability of the leading insignificant figure of out of all the terms contaminates its decimal place in the sum with unreliability.

One should note that the subtraction of nearly equal values can lead to having single numeral zero as the only significant figure. For example,

$$10.31 - 10.315 = 0.00 \ , \tag{17}$$

where the final zero in the difference is the only significant figure: the other zeros are just placeholders.

Further discussion of the significant figures in addition calculations is given in Appendix A.1.

7.3. The Multiplication Rule

In multiplication and/or division of values, the significant figure number in the output value equals the minimum of the significant figure numbers among the input values.

For brevity, we call this rule the multiplication rule even though it treats division cases too.

For example, say we multiply 6.1 times 3.00. What is the product to significant figures?

You have 10 seconds. Go.

Right/wrong. The answer is 18 to significant figures. Treating the values as exact gives 18.3. Since the minimum input value number of significant figures is 2, one must round to 2 significant figures in the output value.

In Appendix A.2, we give a proof that the multiplication rule usually gives a good estimate of the significant figures of an output value in some cases. Those cases are when the relative uncertainties of the input values are much less than 1, but one of the relative uncertainties is much larger than the others (i.e., it is a dominant relative uncertainty). The input value with the largest relative uncertainty is usually the one with the fewest significant figures. This input value's large relative uncertainty largely determines the output value's significant figures. The multiplication rule gives the approximate way of making that determination.

What if the relatively uncertainties are relatively small, but there is no dominant one? In this case, the multiplication rule tends to overestimate the number of significant figures: i.e., the actual number of significant figures in the output value tends to be smaller than the smallest significant figure number of the input values because the relative uncertainties of the input values combine to increase the uncertainty over what it would be if only one input value had any uncertainty.

Note though that the multiplication rule can underestimate the number of significant figures too. For example, consider the multiplication of 5 ± 1 and $3.000\ldots$ (which is exactly 3). The actual result is 15 ± 3 where the uncertainty defines the decimal place of the last significant figure. There are actually two significant figures. But using the multiplication rule, the answer 2×10^1 with only 1 significant figure.

7.4. The Function Rule

The function rule is that the output value of a function of one argument has the same number of significant figures as that of the input value (i.e., the function argument).

For example what is the natural logarithm of 2.000 to significant figures?

You have 10 seconds. Go.

Right/wrong. It's 0.6931.

The function rule is generally justified by something is better than nothing.

Actually, it is clearly valid in two extreme limiting cases: 1) the argument has no significant figures; 2) the argument is exact.

We show that the function rule is justified in some other cases in § A.3.

What if the function has multiple arguments?

Just the assign to the output value the significant figure number of the input value with the smallest significant figure number value. This procedure is analogous to the multiplication rule (§ 7.3). It is also justified by being better than nothing and by being valid in the two extreme limiting cases: 1) the arguments have no significant figures; 2) the arguments are exact.

7.5. The Calculation Rule

It's usually best to carry some insignificant figures through intermediate calculations and only round off to significant figures as a last step. We call this rule the calculation rule.

There are four good reasons for doing rounding off only at the end.

First, the rules for significant figures are only approximately accurate. So every time you apply them you may be introducing error. To minimize the introduction of error, best to apply the rules only to the final output values

Second, worrying about the rules at every step is a bother. Just do it at the end if that is possible which it isn't always (see below). It's usually easier.

Third, keeping insignificant figure often allows for checks of your calculations. This is especially true when comparing to other people's calculations.

Fourth, if you are doing uncertainty analysis and your uncertainty estimates were too big, then rounding off for significant figures (as determined by the approximate rules or by an uncertainty analysis) could cause you to lose real accuracy in your final output values that you would not be able to notice at the end of the calculation. If you didn't round off, then your final output values might surprise you by their accuracy and alert you to your overestimates of the uncertainties.

The calculation rule is most useful if all the calculations are multiplications, divisions, and function evaluations. In such cases, it's usually easy to find the number of significant figures using the significant figure rules. The significant figure number in the output value is the smallest significant figure number of the input values.

But what if your calculations involve additions and subtractions? You don't have to round-off as the calculation proceeds and its good not too, but you do have to keep track of the significant figures at each step in order to know what to round-off when you get the final output value. This is particularly true in subtractions where a subtraction of nearly equal numbers can reduce the number of significant figures to 1: this figure being a zero in some decimal place you have to determine to be significant.

For example, say you had 3.23 and 3.215 where all the figures are significant. The difference $3.23 - 3.215 = 0.01$ which has only 1 significant figure using the significant figures rules, and not 3 significant figures as the input values might suggest misapplying the multiplication rule in an addition/subtraction case.

Here's an example multi-step calculation.

What is $\ln(3.11 - 3.06 + 1.1 \times 0.359)$ to significant figures calculated rounding off to significant figures at every step and calculated rounding off to significant figures only at the last step?

Go. You have 30 seconds.

Well

$$\ln(3.11 - 3.06 + 1.1 \times 0.359) = \ln(0.05 + 0.39) = \ln(0.44) = -0.82$$

and

$$\ln(3.11 - 3.06 + 1.1 \times 0.359) = \ln(0.05 + 0.3949) = \ln(0.4449) = -0.81 \ .$$

The two output values are different, but not significantly different.

But by the calculation rule, the second one is to be preferred. If your input values happened to be more exact than you thought, then second number would be more accurate. By using the calculation rule you take advantage of that possibility.

7.6. Final Word on the Significant Figure Rules

As mentioned above in § 7, the addition, multiplication, and function rules are all approximate rules. In some cases, they are accurate and in others they are not. In intro physics problems, we usually don't worry about when they fail since the problems are just for educational value.

But if you do worry, then you must do uncertainty analysis in order to determine the significant figures. We discuss uncertainty analysis in Appendix A.

8. ORDER-OF-MAGNITUDE VALUES AND CALCULATIONS

An order-of-magnitude value is one only known to within a factor of ten.

An order-of-magnitude calculation is one in which one only keeps track of the values to within an order of magnitude. Such calculations can be done with order-of-magnitude values written as powers of ten and doing ordinary addition/subtraction on the powers instead of multiplication/division on the values themselves.

There are two cases for doing an order-of-magnitude calculations.

First, one only knows one's values to order-of-magnitude. A sub-case of this case is the case of Fermi problems. We do an example Fermi problem below in § 8.1.

Second, one only want an order-of-magnitude accurate result. This may be the case for calculations that you want to do very quickly. For example, you are just trying to illustrate a problem to students on the board. Or maybe you are in the bar and just want to prove a bet. Or you done a high accuracy calculation, but want to check that you havn't made any simple math errors.

In the second case, you often have values that are known to better than order of magnitude. In this case, you have to round them to the nearest power of 10. The conventional way to do this is to use the significant figure round-off rule (see § 7.1) on the logarithm of the value treating the decimal fraction of the logarithm as insignificant.

But for quick calculations, one doesn't want to take the logarithm of a value since that involves calculational work that one is trying to avoid. One simply writes—if only in one's mind—the value in normalized scientific notation. If the coefficient of the value is less than $10^{1/2} = 3.162\ldots$, one rounds the coefficient down to 1 and if it is greater than that, one rounds it up to 10. If the coefficient is exactly $10^{1/2}$, then you have a half case in the logarithm of the value and round to the even power of 10 in accordance with the round-off rule (§ 7.1) applied to the decimal fraction of the logarithm of the value. If one is doing many calculations and assumes that even powers of 10 are as likely to be below as above, then the half case procedure prevents a bias of rounding down more often than rounding up. Of course, having a value of exactly $10^{1/2}$ is pretty rare unless you are dealing with values expressed as powers of ten.

As well as order-of-magnitude calculations, one can also do 1-digit calculations which are a lot more accurate and can be done sans calculator. Such calculations mean one just keeps about 1 significant figure in the calculations. There are no hard rules though. One can keep 2 significant figures sometimes and do compensations for dropping significant figures. These tricks help reduce round-off errors and may maintain real 1-digit accuracy throughout the calculation. Yours truly is very fond of doing 1-digit calculations for on-the-board examples.

Let's do some examples of order-of-magnitude calculations and 1-digit calculations.

8.1. Example: A Fermi Problem

In Fermi problems one uses order-of-magnitude estimates of input values allow you to find order-of-magnitude results that you couldn't find exactly by any easy means since you don't know the input values to better than order of magnitude.

The special feature of a Fermi problem that makes it a Fermi problem is that estimated input values are drawn from general knowledge not from special knowledge of what is needed to solve the problem.

Fermi problems are named for Enrico Fermi (1901–1954). He probably didn't invent them in any sense, but he was a famous practitioner of the art of solving them for both trivial problems (e.g., how many piano tuners are there in Chicago) and important ones (e.g., those needed solving along the way in building the first nuclear reactor—which was in Chicago too).

Here's a Fermi problem.

How many red cars are there in Idaho?

Well there are I'd say 10^6 people in Idaho to order of magnitude. I know there's more than 10^5, but 10^7 sounds way too big for a smallish-population state in a country of only 300×10^6 people.

How many cars does the average person own? Well many own none: three-year-olds, etc. But many own multiple cars—their pickup, their 1966 Mustang, their ordinary drive-to-work car. I estimate to order of magnitude the average person owns one car.

What fraction of cars are red? Well much less than 1, but probably higher than 10^{-2}. So I estimate 10^{-1}.

So the number of red cars in Idaho to order of magnitude is

$$10^6 \times 1 \times 10^{-1} = 10^5 \ . \tag{18}$$

There are a hundred thousand red cars in Idaho. It wouldn't surprise me if this number were too big by a factor of 10. I would be surprised if it were too small by a factor 10. I just can't believe there are a million red cars in Idaho.

I'll leave it as an exercise to the students to figure out how many woodpeckers there are in Latah County.

If there's time, let's try the chicken-fox problem in Appendix B.3.

8.2. Example: Order-of-Magnitude and 1-Digit Calculations Compared

Say I wanted to evaluate the Planck length which is a fundamental length scale constructed by dimensional analysis out of fundamental constants. Dimensional analysis here has its second meaning: the creation of formulae (see § 6.1). The exact physical significance of the Planck length is uncertain, but it is believed that it must play a role in the theory of everything.

The constants are $c = 2.99792458 \times 10^8$ m/s (the vacuum light speed), $G = 6.67428(67) \times$

$10^{-11}\,\mathrm{J\,m\,kg^{-2}}$ (the gravitational constant), and $\hbar = 1.054571628(53) \times 10^{-34}\,\mathrm{J\,s}$ (Dirac's constant or Planck's constant divided by 2π or, in common physics jargon, h-bar). The first thing we have to do is get rid of those joules and kilograms since a length has neither of those units. Now remember a joule is $\mathrm{kg\,m^2/s^2}$. So

$$\mathrm{unit}[\hbar G] = \mathrm{J\,s} \times \mathrm{J\,m\,kg^{-2}} = \mathrm{kg\,m^2\,s^{-1}} \times \mathrm{kg\,m^3\,s^{-2}\,kg^{-2}} = \mathrm{m^5\,s^{-3}} \ , \tag{19}$$

where unit[ ] is my own unit evaluator function. Now it's clear that

$$\mathrm{unit}\left[\sqrt{\frac{\hbar G}{c^3}}\right] = \mathrm{m} \ . \tag{20}$$

So the Planck length is

$$\sqrt{\frac{\hbar G}{c^3}} = \sqrt{\frac{1.054571628 \times 10^{-34} \times 6.67428 \times 10^{-11}}{(2.99792458 \times 10^8)^3}} = 1.61625 \times 10^{-35}\,\mathrm{m} \ , \tag{21}$$

where we report the number to correct significant figures according to the significant figure rules (which are only approximately right recall).

But say we didn't want to work so hard in the calculation. What is the Planck length to order of magnitude? Behold:

$$\begin{aligned} \sqrt{\frac{\hbar G}{c^3}} &= \sqrt{\frac{1.054571628 \times 10^{-34} \times 6.67428 \times 10^{-11}}{(2.99792458 \times 10^8)^3}} \\ &\approx \sqrt{\frac{10^{-34} \times 10^{-10}}{10^{24}}} = 10^{-34}\,\mathrm{m} \ . \end{aligned} \tag{22}$$

Note the straight order-of-magnitude calculation does **NOT** give the right order-of-magnitude result in this case. This sometimes happens because of the approximations made in dropping the coefficients. But the result is only off by 1 order of magnitude.

A 1-digit calculation does a lot better. Note we will keep 2 digits at times if we think the second digit is significant. Behold:

$$\sqrt{\frac{\hbar G}{c^3}} = \sqrt{\frac{1.054571628 \times 10^{-34} \times 6.67428 \times 10^{-11}}{(2.99792458 \times 10^8)^3}}$$

$$\approx \sqrt{\frac{10^{-34} \times 7 \times 10^{-11}}{30 \times 10^{24}}} \approx \sqrt{\frac{1}{4} \times 10^{-69}} \approx \sqrt{2.5 \times 10^{-70}}$$
$$\approx 1.6 \times 10^{-35}\,\mathrm{m} \ . \qquad (23)$$

The 1-digit-calculation result is correct to 2 significant figures. This not accidental. The significant figures we dropped only changed values by about 10 % or so, and so we achieve a higher accuracy for the result than 1 significant figure. Now 1-digit calculations are not always right to 1 or more significant figures because of round-off errors, but they are usually right to within a factor of 2 or so.

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A. SIGNIFICANT FIGURES

In this Appendix, we give some mathematical discussion of significant figures and the simple approximate rules used to treat them in cases where approximate nature of the rules is not a concern. The Appendix complements § 7 on significant figures.

A.1. A Discussion of Significant Figures in Addition Calculations

Here we discuss significant figures in addition calculations (which include subtraction calculations). The discussion includes a better treatment of significant figures in addition cases than given by the rules and a second proof the addition rule. (A first proof was given in § 7.2.)

Say you have two values $A \pm a$ and $B \pm b$ where a and b are the estimated uncertainties in

A and B, respectively. Note uncertainties are always positive values. Here we just consider the case where the uncertainties are meant to set the lower and upper limits on the values. With this assumption, the sum of the two values with uncertainty is $A + B \pm (a + b)$.

Given the significant figures of A, a, B, and b, what are significant figures of $A+B$ and $a + b$?

First, note that uncertainties are usually only estimates. They are not known exactly. Frequently, the uncertainties obtained from measurements and have only 1 significant figure. We make the simplifying uncertainty rule that the decimal place of the leading figure of the uncertainty defines the lowest decimal place for a significant figure in the value and in the uncertainty itself. In more advanced uncertainty analysis, one doesn't need to make use this uncertainty rule. But for simplicity and also because in a lot of practical applications it is adequate, we do make it. The uncertainty rule gives a simple practical test for finding the lowest significant figure. It may be that lower figures have some small significance, but in a lot of practical cases that significance is hard to assess adequately in any simple way.

With our uncertainty rule, the decimal place of the leading figure of $a + b$ is the last decimal place for significant figures for $A + B$ and $a + b$.

If that decimal place is the same as the largest decimal place a and b, then our lowest significant figure decimal place is the same as given by the addition rule. The decimal place can't be lower than the addition-rule decimal place, but it can be higher since the addition of $a + b$ can put a figure in a higher decimal place than the addition-rule decimal place.

We see that the addition rule is actually gives a lower limit on the lowest decimal place significant figure or an upper limit on the number of significant figures.

Thus, the addition rule can lead to reporting insignificant figures as significant figures. This is why the addition is rule is an approximate way to calculate significant figures, but it's quick and adequate for intro physics problems which are educational purposes.

As mentioned in § 7, in intro physics problems, uncertainties are not usually given with values. The values are simply given to so many figures and those are meant to be all significant figures. Although one doesn't have to do this, it is reasonable to assume that the uncertainty of these values is numeral 1 in the decimal place of the last given figure. Intro physics problems are usually consistent with this assumption thought not insistent on it. As mentioned in § 7, we can call this uncertainty the fiducial-numeral-1 uncertainty.

A.2. A Discussion of Significant Figures in Multiplication Calculations

Here we give a discussion of the significant figures in multiplication calculations. Division cases are included too, but we don't want to keep saying multiplication and division.

In § 7.3, we specified the multiplication rule. In this subsection, among other points of discussion, we give a proof. Also in this subsection, we assume familiarity with the mathematical discussion of the addition rule (§ A.1).

Say we have value $A \pm a$ where a is the uncertainty in A. The decimal place of the leading figure of a is the lowest decimal place for a significant figure by our uncertainty rule (see § A.1).

Of course, we can eye-ball $A \pm a$ to find the number of significant figures in $A \pm a$ if $A \pm a$ were a specified number. But how do we find the number of significant figures in general A by a mathematical formula that can be used in proofs?

Consider general number x in normalized scientific notation:

$$x = C \times 10^p \ , \tag{A1}$$

where C is the coefficient whose absolute value is in the range $[1, 10)$ and p is the power.

(Note square bracket means include the limit in the range and curly bracket means exclude the limit from the range.) The **LEADING NON-ZERO FIGURE** of the non-scientific-notation x is clearly in the decimal place p. Since x is general, we can't eye-ball it to find C and p. But note

$$\log(|x|) = \log\left(|C| \times 10^p\right) = \log(|C|) + p \ , \tag{A2}$$

where we are using the base 10 logarithm. Since $|C| \in \pm[1, 10)$, we have $\log(|C|) \in [0, 1)$. We find then that

$$p = \text{floor}[\log(|C|) + p] = \text{floor}[\log(|x|)] \ , \tag{A3}$$

where the floor function rounds its argument to the highest integer below it. So for general x, the leading non-zero figure is decimal place p given by the decimal-place formula

$$p = \text{floor}[\log(|x|)] \ . \tag{A4}$$

What if x is exactly zero? The formula gives $-\infty$ for p which is exactly right. There is no **LEADING NON-ZERO FIGURE**.

We can use the decimal-place formula equation (A4) to find the number of significant figures in general value with uncertainty $A \pm a$. Let this number be N. By inspection, we see that the significant-figure-number formula is

$$N = \text{floor}\{\log[\max(|A|, a)]\} - \text{floor}[\log(a)] + 1 \ , \tag{A5}$$

where recall that uncertainties are always positive values.

Note that if $|A| \leq a$, then the A's only significant figure is in the same decimal place as the leading figure of a and then

$$\text{floor}\left\{\log(\max(|A|, a)]\right\} - \text{floor}\left[\log(a)\right] = 0 \tag{A6}$$

and $N = 1$ according to the formula which is the correct result. If a isn't specified in a problem, then we can assume it is the fiducial-numeral-1 uncertainty (see § A.1). The value of a should always be greater than zero.

An approximate-significant-figure-number formula is obtained from equation (A5) by undistributing the floor function: i.e.,

$$\begin{aligned} N_{\text{approx}} &= \text{floor}\left\{\log[\max(|A|,a)] - \log(a)\right\} + 1 \\ &= \text{floor}\left\{\log\left[\frac{\max(|A|,a)}{a}\right]\right\} + 1 \\ &= \text{floor}\left\{\log\left[\max\left(\frac{|A|}{a},1\right)\right]\right\} + 1 \ , \end{aligned} \tag{A7}$$

or

$$N_{\text{approx}} = \text{floor}\left\{\log\left[\max\left(\frac{|A|}{a},1\right)\right]\right\} + 1 \ . \tag{A8}$$

Note that the argument for the floor function is always a positive number or zero. The approximate-significant-figure-number formula can give either less or greater value than the exact formula: i.e., it is not an lower bound nor upper bound formula.

Now let's consider how to find the significant figures of

$$\frac{(A \pm a)(B \pm b)}{C \pm c} \ , \tag{A9}$$

where $A \pm a$, $B \pm b$, and $C \pm c$ are general values with uncertainties. Equation (A9) is **NOT** a general product-division expression, but a general one can always be written as products of expressions like equation (A9), and so proofs with equation (A9) generalizes immediately to proofs for a general product-division expression.

The significant figures of the values in equation (A9) can be determined by the significant-figure-number formula equation (A5).

Now

$$\frac{(A \pm a)(B \pm b)}{C \pm c} = \frac{AB}{C}\left[1 \pm \left(\frac{a}{|A|} + \frac{b}{|B|} + \frac{c}{|C|}\right)\right] \tag{A10}$$

to 1st order in the relative uncertainties of A, B, and C. We have done Taylor expansions to 1st order to get equation (A10). We assume that 1st order Taylor expansions are adequate: i.e., that $|a|/A$, $|b|/B$, and $|c|/C$ are all much less than 1. Note that in order to do the Taylor expansions we have assumed that none of A, B, and C are zero to known significant figures. For the moment just accept Taylor expansions: we will introduce Taylor expansions in the lecture *NEWTONIAN PHYSICS II*. Of course, those of you will some calculus background may know all about Taylor expansions already.

The relative uncertainty in AB/C is, of course,

$$\frac{a}{|A|} + \frac{b}{|B|} + \frac{c}{|C|} \ . \tag{A11}$$

The exact number of significant figures in AB/C (assuming the 1st order Taylor expansions are adequate) is then

$$N \ = \ \text{floor}\left[\log\left(\left|\frac{AB}{C}\right|\right)\right] - \text{floor}\left\{\log\left[\left|\frac{AB}{C}\right|\left(\frac{a}{|A|} + \frac{b}{|B|} + \frac{c}{|C|}\right)\right]\right\} + 1 \ . \tag{A12}$$

Now equation (A12) is exact insofar as our assumptions are adequate. But it isn't a simple formula finding the number of significant figures in AB/C. For one thing, one has to have the uncertainties a, b, and c or assign them the value of the fiducial-numeral-1 uncertainty.

We can use the approximate-significant-figure-number formula equation (A8) to get a simpler result. We find

$$\begin{aligned} N_{\text{approx}} \ &= \ \text{floor}\left\{\log\left[\max\left(\frac{1}{a/|A| + b/|B| + c/|C|}, 1\right)\right]\right\} + 1 \\ &= \ \text{floor}\left[-\log\left(\frac{a}{|A|} + \frac{b}{|B|} + \frac{c}{|C|}\right)\right] + 1 \ , \end{aligned} \tag{A13}$$

where we have used the fact that $a/|A|$, $b/|B|$, and $c/|C|$ are all much less than 1 by assumption to eliminate the maximum function.

Can we get a still simpler result? Let's assume without loss of generality that relative uncertainty $a/|A|$ is greater than both of $b/|B|$ and $c/|C|$. Dropping $b/|B|$ and $c/|C|$, we get an approximate upper limit on N of

$$N_{\rm upper} = {\rm floor}\left[\log\left(\frac{|A|}{a}\right)\right] + 1\ . \tag{A14}$$

We see that $N_{\rm upper}$ for value AB/C is just the number of significant figures obtained from the approximate-significant-figure-number formula for A. Note that $N_{\rm approx}$ approaches $N_{\rm upper}$ in the limit that $a/|A|$ is much, much larger than $b/|B| + c/|C|$.

Does A necessarily have the least number of significant figures out of A, B, and C? No. It has the biggest relative uncertainty, but that does not imply it has the least significant figures. For example, 1.0 ± 0.6 has a bigger relative uncertainty than 9 ± 1, but still has more significant figures.

Nevertheless, there is a strong tendency for bigger relative uncertainty to lead to fewer significant figures. If relative uncertainty goes to 1, then the number of significant figures goes to 1. And if relative uncertainty goes to zero, then the number of significant figures goes to infinity.

Thus, our result that $N_{\rm upper}$ (which is approximately significant figure number of the input value with biggest relative uncertainty) is often going to equal the significant figure number of the input value with least significant figure number.

We conclude that we have proven the multiplication rule insofar as a crude approximate result can be proven: the output value significant figure number (which can be approximated by $N_{\rm upper}$ sometimes) should be chosen to the significant figure number of the input value with least significant figure number (which is often going to be $N_{\rm upper}$).

Of course, we've only looked at a special case where there are just three input values, but result generalizes immediately to cases with arbitrary number of input values by taking products of product-division expressions as mentioned above.

When all the input value relative uncertainties are small, the multiplication rule should be good when one input value has far fewer significant figures and therefore a much larger relative uncertainty than the other input values: in this case $N_{\rm upper} \approx N_{\rm approx} = N$. In the opposite case, the multiplication rule is likely to overestimate the number of significant figures in the output value since then $N_{\rm upper}$ is likely to be distinctly greater than $N_{\rm approx} = N$.

But the multiplication rule is not necessarily exactly right even when one input value has far fewer significant figures and therefore a much larger relative uncertainty than the other input values: For example, consider the multiplication of 5 ± 1 and $3.000\ldots$ (which is exactly 3). The actual result is 15 ± 3 where the uncertainty defines the decimal place of the last significant figure. There are actually two significant figures. But using the multiplication rule, the answer 2×10^1 with only 1 significant figure. In this case, the multiplication rule fails and underestimates the number of significant figures.

What if the relative uncertainties are not much less than 1? Well in this case, one has to do a fairly exact uncertainty treatment to find the number of significant figures in the output value. Only the uncertainty analysis can guarantee a good uncertainty. We won't go into all that now.

Is the multiplication rule still useful in such cases? Maybe in the sense that something is better than nothing I suppose. At the moment, yours truly is flummoxed for something better to say.

A.3. A Discussion of Significant Figures in Function Calculations

The function rule from § 7.4 is that the output value of a function of one argument has the same number of significant figures as that of the input value (i.e., the function argument).

As we said in § 7.4, the function rule is generally justified by something is better than nothing.

Actually, it is clearly valid in two extreme limiting cases: 1) the argument has no significant figures; 2) the argument is exact.

We can give an argument for rule that justifies it in some other cases. Say we have input value $x_0 \pm \Delta x$ where Δx is the uncertainty in x_0. The first order uncertainty propagation formula gives

$$\Delta f = \left| \left(\frac{df}{dx} \right)_{x_0} \right| \Delta x \ , \tag{A15}$$

where $(df/dx)_{x_0}$ the derivative of $f(x)$ evaluated at x_0.

The significant figures of $f(x_0)$ can be found from the significant-figure-number formula equation (A5) of § A.2. We find

$$N = \text{floor}\{\log[\max(|f(x_0)|, \Delta f)]\} - \text{floor}[\log(\Delta f)] + 1 \ . \tag{A16}$$

This result is accurate insofar as the 1st order uncertainty propagation formula is.

Now using the approximate-significant-figure-number formula equation (A8) of § A.2, we find

$$\begin{aligned} N_{\text{approx}} &= \text{floor}\left\{\log\left[\max\left(\frac{|f(x_0)|}{\Delta f}, 1\right)\right]\right\} + 1 \\ &= \text{floor}\left\{\log\left[\max\left(\frac{|f(x_0)|}{|(df/dx)_{x_0}|\Delta x}, 1\right)\right]\right\} + 1 \ . \end{aligned} \tag{A17}$$

The quantity

$$h = \frac{|f(x_0)|}{|(df/dx)_{x_0}|} \tag{A18}$$

is called the scale height of $f(x)$. It is the distance from x_0 along the x axis until the function $f(x)$ goes to zero if the $f(x)$ were linear in x. The scale height h follows from solving the first order Taylor expansion equation for the zero of $f(x)$: i.e.,

$$0 = f(x_0) + \delta x \left(\frac{df}{dx} \right)_{x_0} \ , \tag{A19}$$

where $h = |\delta x|$.

Now if the function $f(x)$ is roughly linear in x and intercept with the y axis is at least approximately zero, then $|x_0| \approx h$ and the approximate significant figure number for $f(x_0)$ will be the same as the approximate significant factor number x_0 itself. But usually these conditions on the function $f(x)$ will not hold.

But if h is of order x_0, then the significant figure number for $f(x_0)$ will be crudely approximately the significant factor number x_0 itself. Since the logarithm function varies relatively slowly with its argument, the approximation might not be so bad.

In the two cases, just discussed the function rule is justified since we have shown that the significant figure number for $f(x_0)$ will at least crudely approximate the significant factor number for x_0 itself.

But there are lots of other cases where the h will not be of order x_0. In those cases, the function rule is not justified and may give a very bad estimate of the significant figure number of the function. For intro physics problems, this defect of the function rule is usually not a worry since intro physics problems are for educational purposes. If one needs to do a better job than the function rule, one must use uncertainty analysis to determine the number of significant figures in the output of a function evaluation.

A.4. A Discussion of Significant Figures in Uncertainty Calculations

We will just say a brief word here.

If the simple significant figure rules are **NOT** adequate, then one must use an uncertainty calculation to find the significant figures in the output values of calculations as we have discussed in the preceding subsections.

Uncertainty calculations are often done using 1st order Taylor expansions to propagate uncertainties. This is fine if the uncertainties are relatively small and the propagation not complex.

If the uncertainties are relatively large and the propagation is not complex, then one can evaluate the output value uncertainties using the limits on possible input values.

But what if the propagation is complex whether the uncertainties are relatively small or no? Then one does a Monte Carlo calculation. One selects at random a large population of input values from the range or distribution of possible values and calculates the output values. The range or distribution of the output values allows one to find the uncertainties of the output values. From the output value uncertainties, one can then find significant figures of the output values. Doing a Monte Carlo calculation for uncertainties is a standard procedure for may realistic cases.

B. FURTHER EXAMPLE CALCULATIONS

In this appendix, we do an number of further example calculations. These examples are primarily intended for in-lecture practice, where the instructor starts the calculation and let's students individually or in groups race to solutions.

B.1. Conversion Example: Hogsheads per Hour to Cubic Meters per Second

You all know hogsheads—it's a non-SI unit of volume primarily used for alcohol beveridges. There are many kinds of hogsheads, but a US wine hogshead is 63 US liquid gallons and a US liquid gallon is defined to be exactly 3.785411784 liters and a liter is a cubic decimeter.

OK, convert 1 hogshead per hour to cubic meters per second. Go as individuals/groups. You've got 2 minutes. I'll work on it at the board.

My answer:

$$\begin{aligned} 1\,\text{hogshead/h} &= 1\,\text{hogshead/h} \times \frac{63\,\text{gallons}}{1\,\text{hogshead}} \times \frac{3.785411784\,\text{liters}}{1\,\text{gallon}} \times \left(\frac{1\,\text{m}}{10\,\text{dm}}\right)^3 \times \frac{1\,\text{h}}{3600\,\text{s}} \\ &= 63 \times 4 \times 10^{-3} \times 3 \times 10^{-4} \\ &= 7.5 \times 10^{-5}\text{m}^3/\text{s} \end{aligned} \tag{B1}$$

or to correct significant figures

$$1\,\text{hogshead/h} = 6.624470622 \times 10^{-5}\text{m}^3/\text{s}\ . \tag{B2}$$

The high accuracy result follows from the fortran 95 program fragment:

```
      print*
      con1=63.d0
      con2=3.785411784d0
      con3=1.d-3
      con4=1.d0/3600d0
      conv=con1*con2*con3*con4
      print*,'conv'
      print*,conv
!   6.624470622000000E-005
```

B.2. Dimensional Analysis Example: Rotational Kinetic Energy of the Earth

Energy has dimensions of ML^2/T^2.

Kinetic energy (KE) is the energy of motion.

So somehow some formula that has the dimensions of energy has to be developed for motion and the applied to the Earth to get its rotational kinetic energy.

We have to develop the formula and then estimate the Earth values to plug into it.

Let's take 5 minutes on this working in groups. Want you hints—as Yoda would say. The density of ordinary rock is of order $3\,\mathrm{g/cm^3}$ and the Earth's equatorial radius is about 6000 km. Use MKS units.

Go as individuals/groups. You have 30 seconds.

My answer:

A dimensionally correct formula for energy that is based on motion is

$$KE = mv^2 \ . \tag{B3}$$

Reasonably for the Earth m could be something of the order of magnitude of the Earth's mass and v something of order of the Earth's equatorial rotation speed.

Earth's mass? Well ordinary rock has density of order $3\,\mathrm{g/cm^3}$. So

$$m = \frac{4\pi}{3} r^3 \rho \approx 4 \times (6 \times 10^6)^3 \times 3 \times 10^{-3} \times 10^6 \approx 4 \times 200 \times 10^{18} \times 3 \times 10^3 \approx 2 \times 10^{24}\,\mathrm{kg} \tag{B4}$$

is probably of the right order of magnitude.

Earth's equatorial speed? Well

$$v = \frac{2\pi r}{p} \approx \frac{6 \times 6 \times 10^6}{86400} \approx 4 \times 10^2 = 400\,\mathrm{m/s} \ . \tag{B5}$$

So using our dimensional analysis formula and our order-of-magnitude values give

$$KE = mv^2 = 3 \times 10^{29}\,\mathrm{J} \ , \tag{B6}$$

where we recall that the MKS unit of is the joule (J).

What are the actual results?

The correct formula for rotational energy of a rigid rotator (which the Earth approximates) is

$$KE = \frac{1}{2} I\omega^2 = \frac{1}{2}\frac{I}{r^2}v^2 \ , \tag{B7}$$

where I is rotational inertia, I/r^2 has dimensions of mass and is of order the mass of the rigid rotator, r is the equatorial radius, ω is the angular velocity, and v is the equatorial speed. Long down the road in the lecture it ROTATIONAL DYNAMICS, we'll derive this formula.

Well $m = 5.9736 \times 10^{24}\,\mathrm{kg}$, and so our mass was off by a factor of 3.

Well again $v = 465.1\,\mathrm{m/s}$, and so our speed wasn't so bad.

From reasonable calculation Wikipedial (http://en.wikipedia.org/wiki/Rotational_energy) gives $2.14 \times 10^{29}\,\mathrm{J}$. The last digit place has some uncertainty.

So our value is within a factor of two of the correct value. That's not so bad for knowing almost nothing about anything involved in the problem.

B.3. Fermi Problem Example: Chickens Eaten Per Year Per Fox in the US

How many chickens are eaten per year per fox in the US?

Hm, tricky.

Do we count wild foxes only or foxes in fur farms too? Are there foxes in fur farms?

Let's just say wild foxes.

How many are there?

Well foxes are pretty common even in urban environments though you may not see them often. I've only see a fox twice. Once walking in from parking zone infinity in what was a semi-wilderness area and once just outside of St Stephen's Green in Dublin.

There's not 1 per human, but there may be 1 per 1000 humans.

So let's say there are 3×10^5 foxes in the US.

How many chickens do most get per year?

Well not a lot anymore—though in olden days probably a lot. Nowadays most chickens are in factory farms and the others are probably kept in tightly guarded coops. Probably there are relatively few chickens kept in open environments and these are likely pets. Chickens are the pets du jour you know.

At this point, the class can take over and work in groups. Let's give 3 minutes to find answer. Go.

My answer:

Well not everyone keeps chickens. At a guess only 1 person in 1000. I can't believe its 1 in 10^2 or 1 in 10^5.

So there are $300 \times 10^6/1000 = 3 \times 10^5$ chicken owners in the US.

So 3×10^5 foxes, 3×10^5 chicken owners—coincidence, I think not.

Say every chicken owner keeps 5 chickens. Now cats, dogs, hawks, and, snakes probably collectively get 1 per year from an owner I'd guess. Most people protect their chickens, but the predators are everywhere.

So 3×10^5 chickens predated per year. But given the competition, I'd guess foxes maybe only get 1 out of 3.

So 10^5 chickens get eaten by 3×10^5 foxes.

So the average fox—but **NO** fox is average—only gets about 1/3 of a chicken per year or to order of magnitude 1/10 of a chicken per year. It's not like the good old days.

This calculation could be wildly wrong. I doubt that true value could be much higher. It could be much lower, but pet chicken owners do have to worry about foxes.

Next we'll work on foxes and grapes . . .

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