Intro Physics I EXAM 3
2012 March 28, Wednesday NAME:

Instructions: There are 20 multiple-choice problems each worth 1 mark for a total of 20 marks altogether. Choose the **BEST** answer, completion, etc., and **DARKEN** fully the appropriate circle on the table provided below. Read all responses carefully. **NOTE** long detailed responses won't depend on hidden keywords: keywords in such responses are bold-faced capitalized. Harder/longer multiple-choice problems (which are sometimes based on full-answer problems) are marked by asterisks: $*$ for easy, $**$ for moderate, $***$ for hard: all in the judgment of the instructor.

There are **THREE** full-answer problems worth 10 marks for a total of 30 marks altogether. Answer them all. It is important that you **SHOW** (**SHOW, SHOW, SHOW**) how you got the answer for the full-answer problem. Don't give up on problems where you can't do the first part: sometimes later parts can be done independently. Some full-answer problems may be multiple-pagers: make sure you have answered everything. And **BOX-IN** your final answers.

This is a **CLOSED-BOOK** exam. **NO** cheat sheets allowed. An **EQUATION SHEET** is provided. Calculators are permitted—but **ONLY** for calculations. There are **SCRATCH PAGES** for auxiliary calculations. Remember your name (and write it down on the exam too).

The exam is out of 50 marks altogether and is a 50-minute exam.

Answer Table for the Multiple-Choice Questions

	a	b	c	d	e		a	b	c	d	e
1.	O	O	O	O	O	11.	O	O	O	O	O
2.	O	O	O	O	O	12.	O	O	O	O	O
3.	O	O	O	O	O	13.	O	O	O	O	O
4.	O	O	O	O	O	14.	O	O	O	O	O
5.	O	O	O	O	O	15.	O	O	O	O	O
6.	O	O	O	O	O	16.	O	O	O	O	O
7.	O	O	O	O	O	17.	O	O	O	O	O
8.	O	O	O	O	O	18.	O	O	O	O	O
9.	O	O	O	O	O	19.	O	O	O	O	O
10.	O	O	O	O	O	20.	O	O	O	O	O

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005 qmult 00670 1 5 5 easy thinking: hanger center of mass

1. Where, roughly speaking, is the center of mass of a coat hanger? **HINT:** Imagine letting it hang from two different free pivot points: this is called a Gedanken (thought) experiment in physics speak. If you arn't in a test *mise en scène*, you could actually do the experiment.

   a) At the end of the hook.
   b) At the top of the hook.
   c) At the left end of the triangular loop.
   d) Nowhere since a center of mass must be inside the material of an object to be a center of mass.
   e) Oh, somewhere not so far from the middle region of the triangular loop.

   SUGGESTED ANSWER: (e)

   Wrong answers:
   c) What if the left side of a hanger? The one to which the hook grabs or the other? Maybe Charles Dodgson would know.

   Redaction: Jeffery, 2001jan01

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009 qmult 00160 2 5 1 moderate thinking: KE change and momentum change
Extra keywords: physci KB-94-13

2. If the kinetic energy of an object is doubled, the momentum magnitude changes by a factor of:

   a) $\sqrt{2}$. b) 2. c) 1/2. d) $1/\sqrt{2}$. e) 1.

   SUGGESTED ANSWER: (a)

   Recall $\vec{p} = m\vec{v}$, and thus $\vec{v} = \vec{p}/m$. Thus, $KE = mv^2/2 = p^2/(2m)$, and thus $p = \sqrt{2mKE}$. Thus, momentum magnitude increases as the square root of KE. Thus, if KE increases by 2, momentum magnitude increases by $\sqrt{2}$.

   Wrong answers:
   b) Not a good guess, but better than some others anyway.

   Redaction: Jeffery, 2001jan01

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009 qmult 00270 1 1 2 easy memory: conservation of momentum, Thor
Extra keywords: physci

3. The mighty Thor is trapped in the eternal vacuum of gravity-free space with nothing to push on. But he sees Asgard glittering **YONDER** (i.e., over there). Having taken introductory physics in his young Viking days, he realizes that he will soar straight to Asgard if, with awesome strength, he throws his hammer:

   a) yonder. b) anti-yonder. c) any which way. d) left.
   e) in a parabolic arc.

   SUGGESTED ANSWER: (b)

   Wrong answers:
   e) Not in gravity-free space.

Redaction: Jeffery, 2001jan01

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010 qmult 00710 1 5 1 easy thinking: identical masses collide

4. Two identical point masses elastically collide in a 1-dimensional setup. Mass 1 had velocity v before the collision and velocity zero after. Mass 2's velocity was initially zero and after the collision it was:

   a) v. b) $2v$. c) $-v$. d) $-2v$. e) still zero.

SUGGESTED ANSWER: (a)

Conservation of momentum alone allows only answer (a). Actually, you did not need the information that mass 1's velocity was zero after the collision. In 1-dimensional elastic collisions, conservation of momentum and kinetic energy completely determine the outcome of a collision. The outcome of mass 1 at rest and mass 2 having velocity v is consistent with conservation of momentum and kinetic energy, and so it is the only possible outcome. (We are neglecting the ghost solution where mass 1 goes through mass 2 without an interaction at all.)

The above result can be obtained from the full formulae for post-collision velocities of a 1-dimensional elastic collision of point masses 1 and 2:

$$v_1' = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2} ,$$

$$v_2' = \frac{(m_2 - m_1)v_2 + 2m_1v_1}{m_2 + m_1} ,$$

where the prime indicates post-collision and the unprime pre-collision. If $m_1 = m_2$, then $v_1' = v_2$ and $v_2' = v_1$ always.

Wrong answers:

e) Where did all the momentum go.

Redaction: Jeffery, 2001jan01

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010 qmult 10200 2 5 2 moderate math: goats, completely inelastic coll. fullmult

5.** Richthofen and Mannock are two goats with masses 80 kg and 50 kg, respectively. They charge each other head-on on a icy pond—amazingly they stay upright, but goats are sure-hooféd. Just before impact Richthofen is moving at 10.0 m/s—he's way out of control—and Mannock's moving at −4.0 m/s. On impact they lock horns literally (i.e., stick together). Treat the goats as point masses and their collision as an ideal collision (i.e., one in which only the collision forces are non-negligible).

   i) What is the center-of-mass velocity of the Richthofen-Mannock system during the collision?

   ii) Given that the hoof-ice coefficient of kinetic friction is 0.05 (which is the same as steel on ice) how far does the Richthofen-Mannock system slide after the collision before coming to rest?

   a) (i) 10 m/s (ii) 1.1 m. b) (i) 4.6 m/s (ii) 22 m. c) (i) 1.7 m/s (ii) 1.1 m.
   d) (i) 3.6 m/s (ii) 17 m. e) (i) 3.6 m/s (ii) 14 m.

SUGGESTED ANSWER: (b)

i) Since the collision is an ideal collision momentum is conserved through the collision event. Thus, the center-of-mass velocity is conserved through the collision event. This center-of-mass velocity is

$$v = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = 4.6\,\mathrm{m/s}\ ,$$

where Richthofen is object 1 and Mannock is object 2.

ii) After the collision, the Richthofen-Mannock system slides and the only external force acting on it is kinetic friction. From the work-kinetic-energy theorem, we have

$$\begin{aligned} \Delta KE &= W \\ 0 - \frac{1}{2}mv^2 &= -\mu_{\mathrm{ki}} mg\Delta s \\ \Delta s &= \frac{v^2}{2\mu_{\mathrm{ki}} g} = 21.7\,\mathrm{m}\ . \end{aligned}$$

Thus, the Richthofen-Mannock system slides about 22 m.

Wrong answers:

c) These were answers to the full-problem version of this problem.

Fortran-95 Code

```
        xm1=80.d0
        v1=10.d0
        xm2=50.d0
        v2=-4.d0
        xmu=0.05
        gg=9.8d0
        xm=xm1+xm2
        vcen=(xm1*v1+xm2*v2)/xm
        xkecen=.5d0*xm*vcen**2
        dels=vcen**2/(2.d0*xmu*g)
        print*,'vcen,xkecen,dels'
        print*,vcen,xkecen,dels
!   4.615384615384615 1384.6153846153843 21.73650450603764
```

Redaction: Jeffery, 2008jan01

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004 qmult 00420 1 5 1 easy thinking: 24 factors in 360

6. The division of the circle into 360° was an arbitrary choice—and we don't know why. We just know the ancient Mesopotamian mathematicians and astronomers did it this way—you know Mesopotamia—ancient Iraq: "the cradle of civilization". Their choice was just adopted by the ancient Greeks and got passed on to us. In the French Revolutionary epoch, the decimal system was adopted for most measures, but the revolutionaries didn't get around (you might say) to the circle. We can guess that one reasons is that the ancient Mesopotamians had a preference for whole number arithmetic particularly in division and 360 has a lot of whole number factors. How many whole number (i.e., integer) factors does 360 have counting 1 and 360 itself?

a) 24. b) 360. c) 6. d) 7. e) 12.

SUGGESTED ANSWER: (a)

Below are the whole number factors of 360 table format:

count	factor	complement factor
2	1	360
4	2	180
6	3	120
8	4	90
10	5	72
12	6	60
14	8	45
16	9	40
18	10	36
20	12	30
22	15	24
24	18	20

Wrong answers:

b) A specious guess.

Redaction: Jeffery, 2008jan01

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011 qmult 00200 1 4 2 easy deducto-memory: rotational kinematic equations

7. The rotational constant-angular-acceleration kinematic equations:

a) have no resemblance to the linear kinematic equations.
b) are exactly the same as the linear kinematic equations, except that the angular kinematic equations relate **ANGULAR** rather than linear variables.
c) are exactly the same as the linear kinematic equations, except that the angular kinematic equations relate **LINEAR** rather than angular variables.
d) do not allow for angular acceleration.
e) include torque terms.

SUGGESTED ANSWER: (b)

Wrong answers:

c) Oh, c'mon.

Redaction: Jeffery, 2001jan01

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011 qmult 00232 1 1 5 easy memory: timeless equation use

8. The rotational constant-angular-acceleration equation

$$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

by itself alone does **NEVER** allows you to solve for:

a) α. b) $\Delta\theta$. c) ω_0. d) ω. e) t.

SUGGESTED ANSWER: (e)

The equation is what I call the timeless equation.

Wrong answers:

a) Given the other 3 variables that appear in the equation you can solve for this.

Redaction: Jeffery, 2008jan01

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011 qmult 00310 1 3 3 easy math: find the initial omega 1

9. A wheel spins π radians in 10.0 s with an angular acceleration of 4.00 radians/s^2. What is its final angular velocity?

a) 80.1 radians/s. b) 203 radians/s. c) 20.3 radians/s.
d) 3.14 radians/s. e) 6.28 radians/s.

SUGGESTED ANSWER: (c)

Of the 5 standard variables (α, ω_0, ω, $\Delta\theta$, t) of the (constant-angular-acceleration) rotational kinematic equations, we don't know ω nor ω_0. We don't want to know ω_0. This looks like a job for the rarely used 5th rotational constant-acceleration kinematic equation

$$\Delta\theta = -\frac{1}{2}\alpha t^2 + \omega t$$

since it doesn't contain the unwanted variable ω_0, and so allows a solution for the unknown ω from one equation. Behold:

$$\omega = \frac{\Delta\theta + (1/2)\alpha t^2}{t} = \frac{\pi + 200}{10.0} = 20.3\,\text{radians/s} .$$

Wrong answers:

b) Forgot to divide by time.

Fortran-95 Code

```
          print*
          pi_con=acos(-1.d0)
          theta=pi_con
          t=10.d0
          alpha=4.d0
    ! theta=-(1/2)*alpha*t**2+omega*t ! Rarely used 5th kinematic
equation.
    ! omega=(theta+(1/2)*alpha*t**2)/t
          omega=(theta+.5d0*alpha*t**2)/t
          print*,'omega'
          print*,omega
    !  20.3141592653590
```

Redaction: Jeffery, 2008jan01

011 qmult 10330 1 3 3 easy math: Waldo centrifuge, Waldo orbit fullmult

Extra keywords: just rotational kinematics

10.** Waldo's back—you know, Waldo Pepper, the Playful Pig—just accept it. This time, the Bold Boar has decided to become an astronaut and is training on NASA's giant centrifuge—the one in the film *The Right Stuff*. Let's guess it has a radius of 10.0 m. The centrifuge spins in the horizontal: i.e., the centrifuge axis is perpendicular to the ground.

i) Starting from rest the centrifuge goes into a constant angular acceleration phase for 10.0 s. At this point Waldo—who does indeed have a mass of 150 kg—notes that the vertical weighing scale he is nauseatingly pressed on reads 2000 N. What is the centripetal force on Waldo?

ii) The Sentient Swine now does some math—some correct math. What does Waldo find for his angular velocity ω at the 10 s mark?

iii) Doing a little more correct math, the Heck-of-a-Hog now finds his angular acceleration α from time zero to the 10 s mark. What is this angular acceleration?

a) (i) 2000 N (ii) 1.33 radians/s (iii) 0.133 radians/s^2.
b) (i) 200 N (ii) 0.36 radians/s (iii) 0.036 radians/s^2.
c) (i) 2000 N (ii) 1.15 radians/s (iii) 0.115 radians/s^2.
d) (i) 2000 N (ii) 0.36 radians/s (iii) 0.036 radians/s^2.
e) (i) 200 N (ii) 0.36 radians/s (iii) 0.00 radians/s^2.

SUGGESTED ANSWER: (c)

i) Waldo's pressing on the weighing scale causes it to read 2000 N means that the weighting scale is forcing Waldo into circular motion with a force of 2000 N. The weighing scale force is a normal force. At least it can be called that viewing it from the outside. Internally, the force is probably a spring force of some kind. Note that there must be some normal force by the floor to hold Waldo up against gravity too, but that doesn't come into the problem.

ii) Well from

$$F_{\text{centripetal}} = \frac{mv^2}{r} = m\omega^2 r ,$$

Waldo finds his his angular velocity is

$$\omega = \sqrt{\frac{F}{mr}} = \sqrt{\frac{2000}{150 \times 10}} = \sqrt{\frac{4}{3}} \approx 1.15\,\text{radians/s} .$$

iii) Since the angular acceleration is constant, we know that

$$\alpha = \frac{\omega - \omega_{\text{initial}}}{t} \approx 0.115\,\text{radians/s}^2 ,$$

where ω_{initial} is the angular velocity and t is time.

Fortran-95 Code

```
print*
```

```
          t=10.d0
          f=2000.d0
          r=10.d0
          xm=150.d0
          omega=sqrt(f/(r*xm))
          alpha=omega/t
          print*,omega,alpha
          print*,'omega,alpha'
!   1.1547005383792514 0.11547005383792515
```

Redaction: Jeffery, 2008jan01

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012 qmult 00100 1 1 1 easy memory: rotational principles

11. Classical rotational dynamics principles are:

a) secondary principles derived from the fundamental principles of Newtonian physics (i.e., Newton's three laws, force laws, energy, etc.).
b) independent postulates completely unrelated to the fundamental principles of Newtonian physics (i.e., Newton's three laws, force laws, the energy concept, etc.).
c) secondary principles derived from quantum mechanics.
d) all gross approximations derived from the fundamental principles of Newtonian physics (i.e., Newton's three laws, force laws, the energy concept, etc.).
e) independent, but very approximate, postulates completely unrelated to the fundamental principles of Newtonian physics (i.e., Newton's three laws, force laws, the energy concept, etc.).

SUGGESTED ANSWER: (a)

This is, of course, the right answer for classical physics. But in quantum mechanics it seems to me that intrinsic angular momentum (electron spin, etc.) comes in as its own postulate.

Wrong answers:

b) Nope.

Redaction: Jeffery, 2001jan01

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012 qmult 00220 1 1 2 easy memory: cross product standard values

12. Behold:

$$\vec{a}\times\vec{b} = \begin{cases} ab\sin\theta\hat{n} & \text{in general;} \\ 0 & \text{for } \theta = 0^\circ \text{ or } 180^\circ; \\ ab\hat{n} & \text{for } \theta = 90^\circ; \\ \underline{\hspace{3cm}} & \text{in general.} \end{cases}$$

a) $\vec{b}\times\vec{a}$. b) $-\vec{b}\times\vec{a}$. c) $-\vec{a}\times\vec{b}$. d) $-\vec{b}\cdot\vec{a}$. e) $\vec{a}\cdot\vec{b}$.

SUGGESTED ANSWER: (b)

It is also called the vector product since the product is a vector. But "vector" has two syllables and is less easy to say.

Wrong answers:
c) Not in general. In the special case of $\vec{a} \times \vec{b} = 0$, this is true.

Redaction: Jeffery, 2008jan01

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012 qmult 00720 1 4 5 easy deducto-memory: gravitational torque

13. "Let's play *Jeopardy*! For \$100, the answer is: Its torque about any origin can be calculated as if all the object's mass were located at the center of mass."

What is ____________, Alex?

a) a contact force b) friction c) a tension force d) a normal force
e) gravity near the Earth's surface

SUGGESTED ANSWER: (e)

For any system of particles of mass m_i and position r_i relative to any origin and with $\hat{y}$ specifying the upward direction, one has

$$\tau = \sum_i \vec{r}_i \times (-m_i g\hat{y}) = \left(\sum_i m_i \vec{r}_i\right) \times (-g\hat{y}) = \vec{r} \times (-mg\hat{y}) ,$$

where we have used the center of mass definition

$$\vec{r} = \frac{\sum_i m_i \vec{r}_i}{m}$$

with m being total mass.

Wrong answers:
a) Contact force: no way.

Redaction: Jeffery, 2001jan01

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012 qmult 01210 1 4 3 easy deducto-memory: no-slip condition

14. "Let's play *Jeopardy*! For \$100, the answer is: The condition that is required for wheels in most ordinary circumstances: e.g., for car wheels."

What is the ____________ condition, Alex?

a) no-trip b) no-rip c) no-slip d) no-grip e) no-blip

SUGGESTED ANSWER: (c)

There are other useful conditions: no-tip (poor service), no-crip (no gangs), no-gyp (no swindling), no-kip (no salted herrings), no-flip (and no-flop), no-split (infinitives), no-spit (speaks for itself), ...

Wrong answers:
b) Well this one too actually.

Redaction: Jeffery, 2008jan01

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012 qmult 11250 2 3 4 moderate math: rolling ball on incline fullmult

15.** A uniform solid ball of mass 10.0 kg starting from **REST** rolls down an incline of angle $\theta = 30°$. Note the ball is a **ROLLER**, not a **SLIDER**. There is no slipping between the ball and incline: i.e., the no-slip condition is imposed.

i) Find the ball's center-of-mass acceleration. **HINT:** You could check an equation table.

ii) What is its center-of-mass velocity after 10 s?

iii) How far has it traveled down the incline in 10 s?

a) (i) $4.9\,\mathrm{m/s^2}$ (ii) $49\,\mathrm{m/s}$ (iii) $490\,\mathrm{m}$.
b) (i) $4.9\,\mathrm{m/s^2}$ (ii) $49\,\mathrm{m/s}$ (iii) $245\,\mathrm{m}$.
c) (i) $3.5\,\mathrm{m/s^2}$ (ii) $35\,\mathrm{m/s}$ (iii) $350\,\mathrm{m}$.
d) (i) $3.5\,\mathrm{m/s^2}$ (ii) $35\,\mathrm{m/s}$ (iii) $175\,\mathrm{m}$.
e) (i) $4.9\,\mathrm{m/s^2}$ (ii) $4.9\,\mathrm{m/s}$ (iii) $49\,\mathrm{m}$.

SUGGESTED ANSWER: (d)

i) Somewhere along the line, we found the result for a roller on an incline with the no-slip condition imposed:

$$a = \frac{g\sin\theta}{1 + I/(mr^2)} \ .$$

Applying this result, we find:

$$a = \frac{g\sin\theta}{1 + I/(mr^2)} = \frac{9.8 \times 1/2}{1 + 2/5} = 3.5\,\mathrm{m/s^2} \ .$$

ii) From one of the constant-acceleration kinematic equation, we find

$$v = at = 35\,\mathrm{m/s} \ .$$

iii) From one of the constant-acceleration kinematic equation, we find

$$x = \frac{1}{2}at^2 = 175\,\mathrm{m} \ .$$

Fortran-95 Code

```
          print*
          t=10.d0
          theta=30.d0
          gg=9.8d0
          xrotinert=.4d0
          a=gg*sin(theta/raddeg)/(1.d0+xrotinert)
          v=a*t
          x=.5d0*a*t**2
          print*,'a,v,x'
          print*,a,v,x
!       3.5 35.0 175.0
```

Redaction: Jeffery, 2008jan01

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013 qmult 00100 1 1 3 easy memory: rotational equilibrium

16. To be in rotational equilibrium relative to some origin in an inertial frame, an object must have (relative to that origin):

    a) zero angular momentum. b) non-zero angular momentum.
    c) constant angular momentum. d) non-constant angular momentum.
    e) no hair.

    SUGGESTED ANSWER: (c)

    Wrong answers:
    a) No. It can have non-zero angular momentum.
    b) No. It can have zero angular momentum.
    d) Exactly wrong.
    e) Black holes have no hair.

    Redaction: Jeffery, 2001jan01

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013 qmult 00200 2 5 3 moderate thinking: static equilibrium

17. In **STATIC** equilibrium for a rigid body:

    a) there is no center-of-mass or rotational acceleration, but there can be **NONZERO** center-of-mass velocity and angular velocity.
    b) there is no center-of-mass or rotational acceleration, and **NO** center-of-mass or rotational velocity. If static equilibrium exists in a specific reference frame, it exists in **ALL** reference frames no matter how those reference frames may be moving.
    c) there is no center-of-mass or rotational acceleration, and **NO** center-of-mass or rotational velocity. If static equilibrium exists in a specific reference frame, it exists **ONLY** in reference frames **NOT** moving with respect to the specific reference frame.
    d) there are no forces at all.
    e) there are no torques at all.

    SUGGESTED ANSWER: (c) The students have to absorb the idea of moving frames of reference.

    Wrong answers:
    b) A table in static equilibrium on a train, is not in static equilibrium relative to the ground.
    d) There can be no net force.
    e) There can be no net torque.

    Redaction: Jeffery, 2001jan01

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013 qmult 00400 1 3 5 easy math: simple beam torque calculation

18. An object of mass 1 kg sits on a horizontal beam at 1 m from a point fulcrum. What is the torque about the fulcrum that the weight of the mass causes?

    a) 1 N m. b) 2 N m. c) 3 N m. d) 4 N m. e) 9.8 N m.

    SUGGESTED ANSWER: (e)

But the student does really have to know how to calculate a torque. Note that the units of torque are dimensionally the same as energy. But despite this dimensional likeness, torque and energy are different things.

Wrong answers:

Redaction: Jeffery, 2001jan01

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013 qmult 00500 2 3 1 moderate math: torque calculation with a beam

19. Two objects are sitting on a horizontal beam. The beam rests on a point fulrum at its center of mass. The beam is free to rotate about the fulcrum. Object 1 sits on the left-hand side of the pivot at a distance ℓ_1 from the fulcrum. Object 2 sits on the right-hand side at a distance ℓ_2. Given $m_1 = Nm_2$, what is ℓ_2 in terms of ℓ_1? **HINT:** Draw a diagram.

    a) $\ell_2 = N\ell_1$. b) $\ell_2 = \ell_1/N$. c) $\ell_2 = \ell_1$. d) $\ell_2 = 2\ell_1$. e) $\ell_2 = 0$.

SUGGESTED ANSWER: (a)

Sheer common sense and deductions should lead to the right answer. For equilibrium, $\tau_{\text{net}} = -m_1 g\ell_1 + m_2 g\ell_2 = 0$ taking the fulcrum as the origin. Now g cancels out. Masses determined by balancing are independent of g. Balances really measure mass, not weight. Thus, $\ell_2 = \ell_1 m_1/m_2 = \ell_1 N$.

Note that the beam gravity force and fulcrum normal force exert no torques since they both effectively act at the fulcrum point. Also note that the beam does not have to be uniform for it to have zero torque: it just have to have its center of mass at the fulcrum point. If the masses are removed the beam stays balance since the torques about the fulcrum point are still zero.

Wrong answers:

e) Not unless $N = 0$.

Redaction: Jeffery, 2001jan01

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013 qmult 00600 1 1 3 easy memory: indeterminate equilibrium cases

20. In a planar or 2-dimensional case of static equilibrium with no special rules relating forces, can you solve for four unknown forces assuming perfectly rigid objects?

    a) No. The system is **INDETERMINATE**: you only have **FOUR** equilibrium equations.
    b) Yes. The system is **DETERMINATE** since you have **FOUR** equilibrium equations.
    c) No. The system is **INDETERMINATE**: you only have **THREE** equilibrium equations.
    d) Yes. The system is **DETERMINATE**: you have **THREE** equilibrium equations.
    e) No. The system is **INDETERMINATE**: you only have **TWO** equilibrium equations.

SUGGESTED ANSWER: (c)

Wrong answers:

e) Nah you have three: the x and y force equations and the z torque equation.

Redaction: Jeffery, 2001jan01

010 qfull 00500 2 5 0 moderate thinking: bullet colliding with plank

21. A 10 g bullet is shot vertically upward at 1000 m/s. Just after firing it passes straight through a 10 kg plank resting across a couple of rafters—the roof hasn't been added yet. The bullet rises to a **MAXIMUM HEIGHT** of 500 m above the rafters after going through the plank. Assume **NO** air drag and exact 1-dimensionality. **HINT:** The collision with the plank is **NOT** elastic.

a) What is the bullet's velocity immediately after the collision with the plank (i.e., immediately after it has exited the plank)?

b) What is the plank's velocity immediately after the collision (i.e., immediately after the bullet has exited the plank)? **HINT:** Make the ideal collision approximation.

c) How high does the plank rise above its initial position?

d) Approximately calculate the bullet's speed when it returns approximately to the ground level.

e) Is it dangerous to fire bullets straight up into the air? First assume no air drag and show by **SYMBOLIC CALCULATION** whether the return-to-Earth bullet velocity is dangerous or not.

Next assume there is air drag and argue whether the return-to-Earth velocity is dangerous or not. **HINT:** Raindrops reach a terminal speed of 7 m/s after a fall from rest of about 6 m. Also think of everyday experiences.

SUGGESTED ANSWER:

a) By conservation of energy,

$$\frac{1}{2}m_1 v_0^2 = m_1 g \Delta y \ ,$$

where m_1 is the bullet mass, v_0 is the bullet's immediate post-collision speed, and Δy is maximum bullet height. Thus the bullet's immediate post-collision speed is given by

$$v_0 = \sqrt{2gy} = 99\,\mathrm{m/s}$$

to 2-digit accuracy.

One can, of course, use the timeless equation too:

$$v^2 = v_0^2 + 2a\Delta y$$

where in this case $v = 0$, v_0 is the unknown to be solved for, $a = -g$, and $\Delta y = 500\,\mathrm{m}$. The solution is

$$v_0 = \sqrt{2g\Delta y}$$

as before.

b) In the ideal collision approximation, momentum is conserved through the collision. By conservation of momentum,

$$m_1 v_1 = m_1 v_{1'} + m_2 v_{2'} \ ,$$

m_1 is the bullet mass, where v_1 is the pre-collision bullet velocity, m_2 is the plank mass, and $v_{2'}$ is the post-collision plank velocity. It follows that

$$v_{2'} = \frac{m_1}{m_2}(v_1 - v_{1'}) = 0.90\,\mathrm{m/s}$$

to 2-digit accuracy.

c) By conservation of energy,

$$\frac{1}{2}m_2 v_{2'}^2 = m_2 g \Delta y \ ,$$

where Δy is maximum plank height and the other symbols are as in the part (b) answer. It follows that

$$\Delta y = \frac{v_{2'}^2}{2g} = 0.041\,\mathrm{m} = 4.1\,\mathrm{cm}$$

to 2-digit accuracy.

d) By conservation of energy, the bullet's final speed when it gets back to Earth is roughly equal to its immediate post collision speed $v_{\mathrm{post}} = 99\,\mathrm{m/s}$.

e) In the absence of air drag, bullets would come down at the same speed they went up. This follows from the 1-dimensional constant-acceleration kinematic equation

$$v^2 = v_0^2 + 2a\Delta y \ ,$$

where v_0 is initial velocity and Δy is change position from the initial position. An upward fired bullet is only accelerated downward by gravity. So it rises, stops, falls, and eventually gets back to $\Delta y = 0$ where its speed is the same as the muzzle speed and its velocity is given by

$$v = -v_0 \ .$$

Obviously, the bullet is deadly coming down.

Now the no-air-drag case is unrealistic, except in vacuum cases such as on the Moon. Air drag changes things considerably. The bullet is significantly slowed by air drag going up and won't reach the same height as in vacuum. Thus, it will have a shorter distance to fall and falling there will also be air drag. Remember that at the top of the trajectory, it's velocity is zero and then gravity starts accelerating it downward.

Now air drag is speed dependent and increases from zero as speed increases. It always opposes the direction of motion. As the bullet accelerates downward air drag increases slowing its fall. If the fall is long enough, the bullet will reach terminal speed where air drag and gravity cancel: i.e.

$$f(v_{\mathrm{term}}) = mg$$

where m is the bullet mass, v_{term} is the terminal speed, and $f(v)$ is the air drag force's magnitude. Since air drag increases with speed, terminal speed increases with mg.

Air drag is a pretty complex force in general. It must depend on the shape, orientation, and texture of the body. One thing it does depend linearly to a high degree is cross-sectional area of the object perpendicular to the direction of motion. If we divide through by cross-sectional area A, we get

$$\frac{f(v_{\mathrm{term}})}{A} = \frac{mg}{A} = \sigma g$$

where σ is the area density of the object. For many kinds of objects, area density and ordinary volume density are proportional: e.g., spheres, cubes, any standardized shape. Thus, often terminal speed increases with density if $f(v_{\mathrm{term}})$ increases with v_{term}. Since, in fact, we know from everyday life that terminal speeds do usually increase with density in a non-wild manner, we can guess that $f(v_{\mathrm{term}})$ goes as some power of v_{term}.

Now water drops and bullets have very roughly speaking the same shape. Water drops have a terminal speed of about 7 m/s. A steel bullet has a density of about 8 times that of water, and so we can guess that bullet terminal speeds will of order of magnitude 50 m/s assuming. Everyday experience suggests such bullet speeds are dangerous if not necessarily lethal. So if a falling bullet reaches terminal speed, it is probably dangerous. One guesses that it should reach terminal speed or a fair fraction of it, since everyday experience suggests bullets will rise tens of meters before they start to fall. It takes longer falling distances for bullets than raindrops to reach terminal speed, but tens of meters is probably of order enough. Even if the bullet doesn't reach terminal velocity it will still probably moving pretty fast as just ordinary experience with falling objects tells us. I'd say that a falling bullet would be dangerous, but I couldn't offhand argue that it would be necessarily deadly. A unfortunate impact site on a body could be very bad though.

Consulting the web (which isn't possible in a test *mise en scène* yet) gives some help. Bullet terminal velocity is general not readily predictable even for bullets of the same type because it depends on bullet orientation and whether the bullet is tumbling or not. Web sources suggest that typically terminal velocities are of order 100 m/s: this is subsonic, but still pretty dangerous. Depending on where it hits, 100 m/s bullet could be deadly. See

http://www.loadammo.com/Topics/March01.htm .

Obviously firing into the air is criminally negligent.

Fortran-95 Code

```
print*
gg=9.8d0
xm1=.01d0
v1=1000.d0
y1=500.d0
xm2=10.d0
v2=0.d0
v1p=sqrt(2.d0*gg*y1)
v2p=xm1*(v1-v1p)/xm2
```

```
      y2=v2p**2/(2.d0*gg)
      print*,'v1p,v2p,y2'
      print*,v1p,v2p,y2
!  98.99494936611666 0.9010050506338834 0.0414188827177432
```

Redaction: Jeffery, 2001jan01

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012 qfull 00640 3 3 0 tough math: Pippa on merry-go-round

22. Wee Pippa Passing runs up to a playground merry-go-round, initially at rest, and jumps **RADIALLY** onto the rim (i.e., all her horizongal impulse is pointed to the center of the merrry-go-round).

    a) What is the torque she exerts about the rotational axis of the merry-go-round? Does the merry-go-round start to rotate? Why or why not? Does the merry-go-round move at all? Why or why not?

    b) Pippa and the merry-go-round both can have angular momentum about the merry-go-round axis. She and the merry-go-round are coupled together by the static frictional force between her feet and the surface. But Pippa can directly control the relative velocity between herself and the merry-go-round by walking or running: thus she can change the coupling condition. When she is at rest on the merry-go-round, she and the merry-go-round constitute one rigid rotator. But when she moves they constitute two rigid rotators about the merry-go-round axis.

       Say Pippa starts running just on the rim of the merry-go-round just after jumping on. The merry-go-round axis is frictionless: thus the total angular momentum of the system about the axis cannot change. Using conservation of angular momentum for an isolated system find an expression for the merry-go-round angular frequency $\omega_{\rm m}$ in terms of Pippa's relative angular frequency $\omega_{\rm p\ rel}$ and the rotational inertias about the axis of Pippa $I_{\rm p}$ and the merry-go-round $I_{\rm m}$. Note that $\omega_{\rm p\ rel} = \omega_{\rm p} - \omega_{\rm m}$. Show your derivation.

    c) Give the expression for Pippa's final angular velocity to the ground (i.e., $\omega_{\rm p}$) using the part (b) result. What would Pippa's final angular velocity be in the limits that $I_{\rm m} \to \infty$ and $I_{\rm m} \to 0$? Show your derivation.

    d) Pippa runs on the rim at 3.0 m/s relative to the rim. The radius of the merry-go-round is 3 m. The tangential rim velocity of the merry-go-round is -2.0 m/s when Pippa is running. Pippa has a mass of 40 kg. Assuming the merry-go-round is a uniform disk, what is its mass. Show your calculation.

    e) Is it at all possible with Pippa and merry-go-round starting from rest relative to the ground that both Pippa and the merry-go-round could be made to spin in the same direction relative to the ground without external torques about the merry-go-round axis? Why or why not?

SUGGESTED ANSWER:

a) Since she jumps on radially, she exerts zero torque about the axis. Therefore, merry-go-round doesn't move. Her linear momentum becomes the linear momentum of the Pippa-merry-go-round-Earth system since the merry-go-round is rigidly attached to the Earth Thus, effectively the linear momentum disappears into the great Earth momentum sink and the merry-go-round doesn't noticeably move. Actually, there might be some flexing that is not apparent to the eye.

b) For the system, the total angular momentum L is given by

$$\begin{aligned} L &= I_{\rm m}\omega_{\rm m} + I_{\rm p}\omega_{\rm p} \\ &= I_{\rm m}\omega_{\rm m} + I_{\rm p}(\omega_{\rm p\ rel} + \omega_{\rm m}) \\ &= (I_{\rm m} + I_{\rm p})\omega_{\rm m} + I_{\rm p}\omega_{\rm p\ rel} \ . \end{aligned}$$

Sans external torques, L is conserved. Thus we can write the equation for both before and after Pippa starts running and solve for the final $\omega_{\rm m}$. Doing so gives

$$\begin{aligned} \omega_{\rm m} &= \frac{L - I_{\rm p}\omega_{\rm p\ rel}}{I_{\rm m} + I_{\rm p}} \\ &= \frac{(I_{\rm m} + I_{\rm p})\omega_{\rm m,0} + I_{\rm p}\omega_{\rm p\ rel,0} - I_{\rm p}\omega_{\rm p\ rel}}{I_{\rm m} + I_{\rm p}} \\ &= \omega_{\rm m,0} + \left(\frac{I_{\rm p}}{I_{\rm m} + I_{\rm p}}\right)(\omega_{\rm p\ rel,0} - \omega_{\rm p\ rel}) \ , \end{aligned}$$

where we denote the inital values with subscript 0. Note the initial values in equation are for after Pippa has jumped on merry-go-round, but before she starts running. In this case, of course, $\omega_{\rm m,0} = \omega_{\rm p\ rel,0} = 0$: thus

$$\omega_{\rm m} = -\frac{I_{\rm p}}{I_{\rm m} + I_{\rm p}}\omega_{\rm p\ rel} \ .$$

c) Pippa's final angular velocity relative to the ground is

$$\omega_{\rm p} = \omega_{\rm p\ rel} + \omega_{\rm m} = \omega_{\rm p\ rel}\left(1 - \frac{I_{\rm p}}{I_{\rm m} + I_{\rm p}}\right) \ .$$

If $I_{\rm m} \to \infty$, $\omega_{\rm p} \to \omega_{\rm p\ rel}$. Thus, if the merry-go-round has infinite rotational inertia, it will not move relative to the ground. If $I_{\rm m} \to 0$, then $\omega_{\rm p} \to 0$. Thus, if the merry-go-round rotational inertia is vanishingly small, Pippa cannot move relative to the ground by running around the rim. Weird isn't it.

d) Multiplying the part (b) answer by r gives

$$v_{\rm m} = -\frac{I_{\rm p}}{I_{\rm m} + I_{\rm p}}v_{\rm p\ rel}$$

for the merry-go-round rim velocity. Rearranging gives

$$I_{\rm m} = -I_{\rm p}\left(1 + \frac{v_{\rm p\ rel}}{v_{\rm m}}\right) \ .$$

Now the rotational inertias about the merry-go-round axis are $I_{\rm m} = (1/2)m_{\rm m}r^2$ and $I_{\rm p} = m_{\rm p}r^2$ where the radii are the same since Pippa is running on the rim. Thus one finds

$$m_{\rm m} = -2m_{\rm p}\left(1 + \frac{v_{\rm p\ rel}}{v_{\rm m}}\right) = 40\,{\rm kg}\ .$$

It is a very light merry-go-round.

e) No. The system has zero total angular momentum at the start and sans external torques must continue to have zero total angular momentum. If both Pippa and the merry-go-round spun in the same sense, the total z angular momentum would not be zero.

Redaction: Jeffery, 2001jan01

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013 qfull 00310 2 3 0 moderate math: equilibrium ladder

23. A ladder leans against a wall in static equilibrium. Ladder, wall, and ground are perfectly rigid. The ladder has mass m, length ℓ, and center of located at $\ell_{\rm cm}$ along its length measuring from its base. The problem is 2-dimensional: the ladder and wall are seen in the xy plane and a z axis is the only rotation axis.

a) Draw a good diagram marking on all possible forces: gravity, ground normal force $F_{\rm N1}$ ground friction force $F_{\rm f1}$ wall normal force $F_{\rm N2}$ and. wall friction force $F_{\rm f2}$. Mark the forces where they act; in the case of gravity, the center of mass is the appropriate place. Draw the ladder leaning to the **RIGHT** so that we are all consistent. The angle between the ladder and the **VERTICAL** is θ. Make the diagram large enough to be easily read.

b) Write out all the equations of equilibrium including all possible forces. Just so we are all on the same wavelength, take the origin for the torque equation to be the contact point between ladder and ground. Why is this a good choice? **HINT:** Using moment arms is a convenient way to determine the torques, but write them out in terms of ℓ, $\ell_{\rm cm}$, and trigonometric functions of θ. Also, in setting up the equations you must adopt some conventions about which directions are positive for which forces and what is the positive torque direction. As long as you are consistent everything works out the same physically no matter what conventions you adopt.

c) In our idealized system, we have have no general formulae for normal forces or friction forces. We must must solve for them from the laws of motion or rotational motion. Given only m, g, θ as knowns, can we solve for all of the for the normal and wall forces? Explain your answer?

d) Assuming the wall is frictionless, derive the formulae for $F_{\rm N1}$, $F_{\rm f1}$ and $F_{\rm N2}$. Are these general formulae for these forces? Explain your answer.

e) Given ordinary static friction between the ladder and ground and still zero friction for the wall, what must happen as θ increases, but before it reaches 90°? Explain your answer.

f) Again assume the wall is frictionless. Say the ladder is just on the verge of slipping at $\theta_{\rm slip}$. Derive the formula for the static friction coefficient of the ground.

g) Assuming the floor is frictionless, derive the formulae for $F_{\rm N1}$, $F_{\rm N2}$, and $F_{\rm f2}$. What ordinary friction rule is violated in this case?

SUGGESTED ANSWER:

a) You will have to imagine the diagram.

b) The three equations of equilibrium are

$$\begin{aligned} 0 &= F_{\rm f1} - F_{\rm N2} \ , \\ 0 &= F_{\rm N1} + F_{\rm f2} - mg \ , \\ 0 &= -mg\ell_{\rm cm}\sin\theta + F_{\rm f2}\ell\sin\theta + F_{\rm N2}\ell\cos\theta \ , \end{aligned}$$

where we have taken the ladder base on the ground as the origin for the torque equation. The signed moment arms were easily determined geometrically. One could, of course, use the torque definition and the trigonometric identities

$$\sin(\pi-\theta) = \sin\theta \qquad \text{and} \qquad \sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$$

to find the torques. In writing down the torque equation, we have taken counterclockwise as positive.

In principle, we could choose any point in the plane of the problem as the origin for the torque equation. For equilibrium, clearly the net torque about any origin at all must be zero: individual torques remain origin dependent, but forces friction and normal forces are, of course, origin independent. The choice of origin is a good one because the torque of the forces F_1 and $F_{\rm f1}$ are zero for this choice. Thus, the choice of origin simplifies the equations to be solved.

We cannot solve for normal and frictional forces since we only three equations of equilibrium and four unknowns. The problem is indeterminate using the idealized perfectly rigid objects that we have invoked. Nature has no problem though giving those forces definite values. This is because in reality all objects show some deformation under applied force (unless the applied force is uniform field force as gravity is for most small objects) and the deformation causes an equal and opposite restoring force. The force laws governing those deformations and restoring forces provide enough constraints that the realistic problem is always determinate. It is well beyond our scope, however, to go into elasticity theory.

c) No. There are four unknown forces and only three equations of equilibrium. We cannot solve for the unknowns without more information.

d) Given $F_{\rm f2} = 0$, we can solve the equations for the three unknown forces by inspection. We obtain

$$\begin{aligned} F_{\rm N1} &= mg \ , \\ F_{\rm f1} &= mg\frac{\ell_{\rm cm}}{\ell}\tan\theta \\ F_{\rm N2} = F_{\rm f1} &= mg\frac{\ell_{\rm cm}}{\ell}\tan\theta \ . \end{aligned}$$

These are **NOT** general formulae for these forces. They are formulae for what the forces must be given our static system.

e) Given $F_{\rm f2} = 0$, we can solve the equations for the ground friction $F_{\rm f1}$ must be to maintain static equilibrium. We obtained this formula in the part (d) answer. However, static friction has an upper limit $\mu_{\rm st}|F_{\rm N}|$. If θ grows sufficiently large, then this limit will be exceeded since according to our formula for $F_{\rm f1}$ goes to infinity as $\theta \to 90^\circ$ while the $F_{\rm N1}$ stays constant. So static equilibrium will fail and the ladder will slide to the ground. I once saw this happen with a worker—who was not hurt to save suspense—on a ladder in the computer room of the astronomy department of the University of Barcelona.

f) Using the approximate law $|F_{\rm st,max}| = \mu_{\rm st}|F_{\rm N}|$ and the part (d) answer, we find

$$\mu_{\rm st} = \left|\frac{F_{\rm f1}}{F_{\rm N1}}\right| = \frac{\ell_{\rm cm}}{\ell}\tan\theta_{\rm slip} \ .$$

g) Given $F_{\rm f1} = 0$, it follows from the equations in the part (d) answer that

$$\begin{aligned} F_{\rm N1} &= 0 \ , \\ F_{\rm f2} &= mg\frac{\ell_{\rm cm}}{\ell} \ , \\ F_{\rm N2} &= m_{\rm ladder}g\left(1 - \frac{\ell_{\rm cm}}{\ell}\right) \ . \end{aligned}$$

It certainly violates our ordinary friction rules to have a wall frictional force without a wall normal force. But those rules don't actually account for all cases. Here the wall must be sticky—maybe with fresh paint.

Redaction: Jeffery, 2001jan01

SCRATCH PAGE

SCRATCH PAGE

Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

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1 Constants

$$c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 2.998 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \qquad \text{exact by definition}$$

$$e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$G = 6.67384(80) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(2012, CODATA)}$$

$$g = 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 8.987551787\ldots \times 10^9 \approx 8.99 \times 10^9 \approx 10^{10}\,\mathrm{N\,m^2/C^2}\text{exact by definition}$$

$$k_{\text{Boltzmann}} = 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} = 0.8617343(15) \times 10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,\mathrm{eV/K}$$

$$m_e = 9.10938215(45) \times 10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV}$$

$$m_p = 1.672621637(83) \times 10^{-27}\,\mathrm{kg} = 938.272013(23), \mathrm{MeV}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = 8.8541878176\ldots \times 10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11} \qquad \text{vacuum permittivity (exact by definition)}$$

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad \text{exact by definition}$$

---

2 Geometrical Formulae

$$C_{\text{cir}} = 2\pi r \qquad A_{\text{cir}} = \pi r^2 \qquad A_{\text{sph}} = 4\pi r^2 \qquad V_{\text{sph}} = \frac{4}{3}\pi r^3$$

$$\Omega_{\text{sphere}} = 4\pi \qquad d\Omega = \sin\theta\,d\theta\,d\phi$$

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3 Trigonometry Formulae

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}$$

$$f(\theta) = f(\theta + 360^\circ)$$

$$\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin(\theta + 90^\circ) = \cos(\theta) \qquad \cos(\theta + 90^\circ) = -\sin(\theta) \qquad \tan(\theta + 90^\circ) = -\tan(\theta)$$

$$\sin(180^\circ - \theta) = \sin(\theta) \qquad \cos(180^\circ - \theta) = -\cos(\theta) \qquad \tan(180^\circ - \theta) = -\tan(\theta)$$

$$\sin(90^\circ - \theta) = \cos(\theta) \qquad \cos(90^\circ - \theta) = \sin(\theta) \qquad \tan(90^\circ - \theta) = \frac{1}{\tan(\theta)} = \cot(\theta)$$

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a)$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a - b) - \cos(a + b)\right] \qquad \cos(a)\cos(b) = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a - b) + \sin(a + b)\right]$$

$$\sin^2\theta = \frac{1}{2}[1-\cos(2\theta)] \qquad \cos^2\theta = \frac{1}{2}[1+\cos(2\theta)] \qquad \sin(a)\cos(a) = \frac{1}{2}\sin(2a)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

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4 Approximation Formulae

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx} \qquad \frac{1}{1-x} \approx 1 + x \ : \ (x << 1)$$

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2 \qquad \text{all for } \theta << 1$$

---

5 Quadratic Formula

If $\quad 0 = ax^2 + bx + c\ ,\quad$ then $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\dfrac{b}{2a} \pm \sqrt{\left(\dfrac{b}{2a}\right)^2 - \dfrac{c}{a}}$

---

6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

---

7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

Taylor's series

$$f(x) = \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0)$$
$$= f(x_0) + (x - x_0) f^{(1)}(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x - x_0)^3}{3!} f^{(3)}(x_0) + \ldots$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

---

8 One-Dimensional Kinematics

$$v_{\rm avg} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\rm avg} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2 x}{dt^2}$$

$$v = at + v_0 \qquad x = \frac{1}{2}at^2 + v_0 t + x_0 \qquad v^2 = v_0^2 + 2a(x - x_0)$$

$$x = \frac{1}{2}(v_0 + v)t + x_0 \qquad x = -\frac{1}{2}at^2 + vt + x_0 \qquad g = 9.8\,\mathrm{m/s^2}$$

$$x_{\rm rel} = x_2 - x_1 \qquad v_{\rm rel} = v_2 - v_1 \qquad a_{\rm rel} = a_2 - a_1$$

$$x' = x - v_{\text{frame}} t \qquad v' = v - v_{\text{frame}} \qquad a' = a$$

9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\text{avg}} = \frac{\Delta \vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\text{avg}} = \frac{\Delta \vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

10 Projectile Motion

$$x = v_{x,0} t \qquad y = -\frac{1}{2} g t^2 + v_{y,0} t + y_0 \qquad v_{x,0} = v_0 \cos\theta \qquad v_{y,0} = v_0 \sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0 \cos\theta} \qquad y = y_0 + x \tan\theta - \frac{x^2 g}{2 v_0^2 \cos^2\theta}$$

$$x_{\text{for } y \text{ max}} = \frac{v_0^2 \sin\theta \cos\theta}{g} \qquad y_{\text{max}} = y_0 + \frac{v_0^2 \sin^2\theta}{2g}$$

$$x(y = y_0) = \frac{2 v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin(2\theta)}{g} \qquad \theta_{\text{for max}} = \frac{\pi}{4} \qquad x_{\text{max}}(y = y_0) = \frac{v_0^2}{g}$$

$$x(\theta = 0) = \pm v_0 \sqrt{\frac{2(y_0 - y)}{g}} \qquad t(\theta = 0) = \sqrt{\frac{2(y_0 - y)}{g}}$$

11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad a_{\tan} = r\alpha$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2 = v\omega$$

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13 Very Basic Newtonian Physics

$$\vec{r}_{\text{cm}} = \frac{\sum_i m_i\vec{r}_i}{m_{\text{total}}} = \frac{\sum_{\text{sub}} m_{\text{sub}}\vec{r}_{\text{cm sub}}}{m_{\text{total}}} \qquad \vec{v}_{\text{cm}} = \frac{\sum_i m_i\vec{v}_i}{m_{\text{total}}} \qquad \vec{a}_{\text{cm}} = \frac{\sum_i m_i\vec{a}_i}{m_{\text{total}}}$$

$$\vec{r}_{\text{cm}} = \frac{\int_V \rho(\vec{r})\vec{r}\,dV}{m_{\text{total}}}$$

$$\vec{F}_{\text{net}} = m\vec{a} \qquad \vec{F}_{21} = -\vec{F}_{12} \qquad F_g = mg \qquad g = 9.8\,\text{m/s}^2$$

$$\vec{F}_{\text{normal}} = -\vec{F}_{\text{applied}} \qquad F_{\text{linear}} = -kx$$

$$f_{\text{normal}} = \frac{T}{r} \qquad T = T_0 - F_{\text{parallel}}(s) \qquad T = T_0$$

$$F_{\text{f static}} = \min(F_{\text{applied}}, F_{\text{f static max}}) \qquad F_{\text{f static max}} = \mu_{\text{static}}F_{\text{N}} \qquad F_{\text{f kinetic}} = \mu_{\text{kinetic}}F_{\text{N}}$$

$$v_{\text{tangential}} = r\omega = r\frac{d\theta}{dt} \qquad a_{\text{tangential}} = r\alpha = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} \qquad \vec{F}_{\text{centripetal}} = -m\frac{v^2}{r}\hat{r}$$

$$F_{\text{drag,lin}} = bv \qquad v_{\text{T}} = \frac{mg}{b} \qquad \tau = \frac{v_{\text{T}}}{g} = \frac{m}{b} \qquad v = v_{\text{T}}(1 - e^{-t/\tau})$$

$$F_{\text{drag,quad}} = bv^2 = \frac{1}{2}C\rho A v^2 \qquad v_{\text{T}} = \sqrt{\frac{mg}{b}}$$

14 Energy and Work

$$dW = \vec{F} \cdot d\vec{s} \qquad W = \int \vec{F} \cdot d\vec{s} \qquad KE = \frac{1}{2}mv^2 \qquad E_{\text{mechanical}} = KE + PE$$

$$P_{\text{avg}} = \frac{\Delta W}{\Delta t} \qquad P = \frac{dW}{dt} \qquad P = \vec{F} \cdot \vec{v}$$

$$\Delta KE = W_{\text{net}} \quad \Delta PE_{\text{of a conservative force}} = -W_{\text{by a conservative force}} \quad \Delta E = W_{\text{nonconservative}}$$

$$F = -\frac{dPE}{dx} \qquad \vec{F} = -\nabla PE \qquad PE = \frac{1}{2}kx^2 \qquad PE = mgy$$

15 Momentum

$$\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}} \qquad \Delta KE_{\text{cm}} = W_{\text{net,external}} \qquad \Delta E_{\text{cm}} = W_{\text{not}}$$

$$\vec{p} = m\vec{v} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}_{\text{total}}}{dt}$$

$$m\vec{a}_{\text{cm}} = \vec{F}_{\text{net non-flux}} + (\vec{v}_{\text{flux}} - \vec{v}_{\text{cm}})\frac{dm}{dt} = \vec{F}_{\text{net non-flux}} + \vec{v}_{\text{rel}}\frac{dm}{dt}$$

$$v = v_0 + v_{\text{ex}} \ln\left(\frac{m_0}{m}\right) \qquad \text{rocket in free space}$$

16 Collisions

$$\vec{I} = \int_{\Delta t} \vec{F}(t)\,dt \qquad \vec{F}_{\text{avg}} = \frac{\vec{I}}{\Delta t} \qquad \Delta p = \vec{I}_{\text{net}}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \qquad \vec{v}_{\text{cm}} = \frac{\vec{p}_1 + \vec{p}_2}{m_{\text{total}}}$$

$$KE_{\text{total } f} = KE_{\text{total } i} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{1'} = \frac{(m_1 - m_2)v_1 + 2m_2 v_2}{m_1 + m_2} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{2'} - v_{1'} = -(v_2 - v_1) \qquad v_{\text{rel}'} = -v_{\text{rel}} \qquad \text{1-d Elastic Collision Expressions}$$

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17 Rotational Kinematics

$$2\pi = 6.2831853\ldots \qquad \frac{1}{2\pi} = 0.15915494\ldots$$

$$\frac{180^\circ}{\pi} = 57.295779\ldots \approx 60^\circ \qquad \frac{\pi}{180^\circ} = 0.017453292\ldots \approx \frac{1}{60^\circ}$$

$$\theta = \frac{s}{r} \qquad \omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{a}{r} \qquad f = \frac{\omega}{2\pi} \qquad P = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$\omega = \alpha t + \omega_0 \qquad \Delta\theta = \frac{1}{2}\alpha t^2 + \omega_0 t \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \frac{1}{2}(\omega_0 + \omega)t \qquad \Delta\theta = -\frac{1}{2}\alpha t^2 + \omega t$$

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18 Rotational Dynamics

$$\vec{L} = \vec{r} \times \vec{p} \qquad \vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}$$

$$L_z = RP_{xy}\sin\gamma_L \qquad \tau_z = RF_{xy}\sin\gamma_\tau \qquad L_z = I\omega \qquad \tau_{z,\text{net}} = I\alpha$$

$$I = \sum_i m_i R_i^2 \qquad I = \int R^2 \rho\, dV \qquad I_{\text{parallel axis}} = I_{\text{cm}} + mR_{\text{cm}}^2 \qquad I_z = I_x + I_y$$

$$I_{\text{cyl,shell,thin}} = MR^2 \qquad I_{\text{cyl}} = \frac{1}{2}MR^2 \qquad I_{\text{cyl,shell,thick}} = \frac{1}{2}M(R_1^2 + R_2^2)$$

$$I_{\text{rod,thin,cm}} = \frac{1}{12}ML^2 \qquad I_{\text{sph,solid}} = \frac{2}{5}MR^2 \qquad I_{\text{sph,shell,thin}} = \frac{2}{3}MR^2$$

$$a = \frac{g\sin\theta}{1 + I/(mr^2)}$$

$$KE_{\text{rot}} = \frac{1}{2}I\omega^2 \qquad dW = \tau_z\, d\theta \qquad P = \frac{dW}{dt} = \tau_z\omega$$

$$\Delta KE_{\text{rot}} = W_{\text{net}} = \int \tau_{z,\text{net}}\, d\theta \qquad \Delta PE_{\text{rot}} = -W = -\int \tau_{z,\text{con}}\, d\theta$$

$$\Delta E_{\text{rot}} = KE_{\text{rot}} + \Delta PE_{\text{rot}} = W_{\text{non,rot}} \qquad \Delta E = \Delta KE + KE_{\text{rot}} + \Delta PE = W_{\text{non}} + W_{\text{rot}}$$

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19 Static Equilibrium

$$\vec{F}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = \tau'_{\text{ext,net}} \quad \text{if } F_{\text{ext,net}} = 0$$

$$0 = F_{\text{net }x} = \sum F_x \qquad 0 = F_{\text{net }y} = \sum F_y \qquad 0 = \tau_{\text{net}} = \sum \tau$$

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20 Gravity

$$\vec{F}_{1\text{ on }2} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} \qquad \vec{g} = -\frac{GM}{r^2}\hat{r} \qquad \oint \vec{g}\cdot d\vec{A} = -4\pi GM$$

$$PE = -\frac{Gm_1m_2}{r_{12}} \qquad V = -\frac{GM}{r} \qquad v_{\text{escape}} = \sqrt{\frac{2GM}{r}} \qquad v_{\text{orbit}} = \sqrt{\frac{GM}{r}}$$

$$P^2 = \left(\frac{4\pi^2}{GM}\right) r^3 \qquad P = \left(\frac{2\pi}{\sqrt{GM}}\right) r^{3/2} \qquad \frac{dA}{dt} = \frac{1}{2} r^2 \omega = \frac{L}{2m} = \text{Constant}$$

$$R_{\text{Earth,mean}} = 6371.0\,\text{km} \qquad R_{\text{Earth,equatorial}} = 6378.1\,\text{km} \qquad M_{\text{Earth}} = 5.9736 \times 10^{24}\,\text{kg}$$

$$R_{\text{Earth mean orbital radius}} = 1.495978875 \times 10^{11}\,\text{m} = 1.0000001124\,\text{AU} \approx 1.5 \times 10^{11}\,\text{m} \approx 1\,\text{AU}$$

$$R_{\text{Sun,equatorial}} = 6.955 \times 10^{8} \approx 109 \times R_{\text{Earth,equatorial}} \qquad M_{\text{Sun}} = 1.9891 \times 10^{30}\,\text{kg}$$