**Intro Physics Semester I** **Name:**

**Homework 4: Multi-Dimensional Kinematics:** One or two full answer questions will be marked. There will also be a mark for completeness. Homeworks are due usually the day after the chapter they are for is finished. Solutions will be posted soon thereafter. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

## Answer Table

## Name:

| | a | b | c | d | e | | a | b | c | d | e |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. | O | O | O | O | O | 31. | O | O | O | O | O |
| 2. | O | O | O | O | O | 32. | O | O | O | O | O |
| 3. | O | O | O | O | O | 33. | O | O | O | O | O |
| 4. | O | O | O | O | O | 34. | O | O | O | O | O |
| 5. | O | O | O | O | O | 35. | O | O | O | O | O |
| 6. | O | O | O | O | O | 36. | O | O | O | O | O |
| 7. | O | O | O | O | O | 37. | O | O | O | O | O |
| 8. | O | O | O | O | O | 38. | O | O | O | O | O |
| 9. | O | O | O | O | O | 39. | O | O | O | O | O |
| 10. | O | O | O | O | O | 40. | O | O | O | O | O |
| 11. | O | O | O | O | O | 41. | O | O | O | O | O |
| 12. | O | O | O | O | O | 42. | O | O | O | O | O |
| 13. | O | O | O | O | O | 43. | O | O | O | O | O |
| 14. | O | O | O | O | O | 44. | O | O | O | O | O |
| 15. | O | O | O | O | O | 45. | O | O | O | O | O |
| 16. | O | O | O | O | O | 46. | O | O | O | O | O |
| 17. | O | O | O | O | O | 47. | O | O | O | O | O |
| 18. | O | O | O | O | O | 48. | O | O | O | O | O |
| 19. | O | O | O | O | O | 49. | O | O | O | O | O |
| 20. | O | O | O | O | O | 50. | O | O | O | O | O |
| 21. | O | O | O | O | O | 51. | O | O | O | O | O |
| 22. | O | O | O | O | O | 52. | O | O | O | O | O |
| 23. | O | O | O | O | O | 53. | O | O | O | O | O |
| 24. | O | O | O | O | O | 54. | O | O | O | O | O |
| 25. | O | O | O | O | O | 55. | O | O | O | O | O |
| 26. | O | O | O | O | O | 56. | O | O | O | O | O |
| 27. | O | O | O | O | O | 57. | O | O | O | O | O |
| 28. | O | O | O | O | O | 58. | O | O | O | O | O |
| 29. | O | O | O | O | O | 59. | O | O | O | O | O |
| 30. | O | O | O | O | O | 60. | O | O | O | O | O |

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004 qmult 00100 1 4 4 easy deducto-memory: kinematic quantities in multi-dimensions

1. "Let's play *Jeopardy*! For \$100, the answer is: The vectors

$$\vec{r}\ , \qquad \vec{v} = \frac{d\vec{r}}{dt}\ , \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}\ ."$$

What are the most obvious quantities of ____________, Alex?

a) time  b) light  c) dynamics  d) multi-dimensional kinematics  e) rest

**SUGGESTED ANSWER:** (d)

**Wrong answers:**
a) As Lurch would say AAAARGH.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00110 1 1 3 easy memory: unit vectors in Cartesian coordinates

2. In Cartesian coordinates, kinematics is very simple if $x(t)$, $y(t)$, and $z(t)$ are specified because the unit vectors of Cartesian coordinates are:

a) variables.  b) dependent on time.  c) constants.  d) dependent on position.
e) fewer than two.

**SUGGESTED ANSWER:** (c)

**Wrong answers:**
e) A nonsense answer.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00120 1 1 1 easy memory: unit vectors in polar/spherical coordinates

3. The unit vectors of polar and spherical polar coordinates are dependent on:

a) the angular coordinates of the displacement vector.
b) the magnitude of the displacement vector.  c) nothing.  d) time explicitly.  e) mass.

**SUGGESTED ANSWER:** (a)

**Wrong answers:**
d) Well no. They depend on the angular coordinates explicitly. If those coordinates depend on time, then there is an implicit time dependence. But there is no general time dependence.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00130 1 5 4 easy think: independent orthogonal motion

4. In Newtonian physics, the component motions of a particle in ____________ directions are independent of each other. This means that motion variables in one direction are **NOT** intrinsic functions of motion variables in ____________ directions. For example, a particular velocity in the $x$ direction does **NOT** intrinsically set the velocity in the $y$ direction. Now the initial or continuing conditions will set up relationships between motions in ____________ directions. But those conditions can be anything allowed by physics and relationships between the motions can be anything allowed by the conditions and physics. For example, the conditions could set the $x$ direction velocity to be a constant $v_x = 1\,\mathrm{m/s}$ and the $y$ direction velocity to a constant $v_y = 2\,\mathrm{m/s}$. Then one has, of course, that $v_y = 2v_x$, but the physically allowed conditions could have set the two velocity components to any constant values or anything else allowed by physics. There is **NO** intrinsic relationship between those velocity components. The independence of motions in ____________ directions is one of the amazing facts about the physical world and a vast simplification in dealing with it.

a) the same.  b) opposite.  c) both negative  d) orthogonal  e) dependent

**SUGGESTED ANSWER:** (d)

This is easy because it's a leading question. Deduction tells you some answers are nonsensical. Space itself (on our current level anyway) imposes no intrinsic relationships on motions in orthogonal directions.

**Wrong answers:**
a) A nonsense answer.

**Redaction:** Jeffery, 2001jan01

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004 qmult 00200 1 4 2 easy deducto-memory: projectile motion

5. "Let's play *Jeopardy*! For \$100, the answer is: Without qualifications, one usually means the non-powered flight of an object in the air or through space. The simplest in-air case is the one in which air drag is neglected. The science of such motions is ballistics.

   What is ____________, Alex?

   a) apparent motion b) projectile motion c) 1-dimensional motion
   d) trigonometric motion e) unstoppable motion

**SUGGESTED ANSWER:** (b)

**Wrong answers:**
d) A nonsense answer.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00210 2 1 1 mod. deducto-mem.: gravity is always downward on Earth
**Extra keywords:** physci

6. A ball is tossed into the air and falls to the ground some distance away. Consider its motion in the vertical direction only and neglect air drag.

   a) The ball has a constant acceleration downward.
   b) The ball first accelerates **UPWARD** on its rising path and then accelerates **DOWNWARD** on its falling path.
   c) The ball first accelerates **DOWNWARD** on its rising path and then accelerates **UPWARD** on its falling path.
   d) The ball does not accelerate at all.
   e) The ball is always accelerating in the upward direction.

**SUGGESTED ANSWER:** (a)

An easy memory question. Acceleration due to gravity alone is always downward and is a nearly constant near the Earth's surface. The magnitude of acceleration due to gravity alone is the magnitude of the gravitational field. Near the Earth's surface the gravitational field magnitude has fiducial value $g = 9.8\,\mathrm{m/s^2}$.

**Wrong answers:**

**Redaction:** Jeffery, 2001jan01

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004 qmult 00220 1 5 2 moderate thinking: projectile parabolic arc

7. Which best describes the path of a ball thrown on level ground at an angle 30° above the horizontal as seen from a side view.

   a) Two straight lines that meet at an apex: one for the rising phase; one the declining phase. The rising phase line is **TWICE** the length of the declining phase line.
   b) A smooth curve that rises and falls with distance. As far as the eye can tell, the curve could be parabolic.
   c) Two straight lines that meet at an apex: one for the rising phase; one the declining phase. The rising phase line is **HALF** the length of the declining phase line.
   d) A smooth curve that rises and falls with distance, but suddenly breaks off and descends vertically.
   e) A smooth curve that rises and falls with distance and then rises and falls again with distance. A Bactrian camel curve.

**SUGGESTED ANSWER:** (b)

An easy thinking question. People should be able to identify the only answer that corresponds to common observation. And some may have heard that projectile motion is parabolic aside from air drag effects. Neglecting air drag, the path is, in fact, a parabolic arc as function of the horizontal coordinate $x$. To understand this note that the vertical height is parabolic with time. The $x$ distance is linear in time. Therefore the vertical height is parabolic with time.

**Wrong answers:**
e) A two-hump camel curve?

**Redaction:** Jeffery, 2001jan01

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004 qmult 00230 1 1 4 easy memory: horizontal range formula

8. The horizontal range formula for projectile motion near the Earth's surface and neglecting air drag is:

a) $x_{\rm max} = \dfrac{8v_0^2}{g}\sin(2\theta)$ . b) $x_{\rm max} = \dfrac{v_0^2}{g}\cos(2\theta)$ . c) $x_{\rm max} = \dfrac{v_0^2}{g}$ . d) $x_{\rm max} = \dfrac{v_0^2}{g}\sin(2\theta)$ .
e) $x_{\rm max} = \dfrac{4v_0^2}{g}\sin(2\theta)$ .

**SUGGESTED ANSWER:** (d)

Here's the quick derivation. In the two independent directions one has

$$x = (v_0\cos\theta)t \qquad \text{and} \qquad y = -\frac{1}{2}gt^2 + (v_0\sin\theta)t \ ,$$

where we've taken the inital launch position to be the origin to avoid fruitless generality and the meaning of the symbols is obvious. As a function of $x$, we find that

$$y = -\frac{1}{2}\frac{gx^2}{v_0^2\cos^2\theta} + x\tan\theta \ .$$

The equation is a quadratic for $x$. There two solutions for $y = 0$. One is the launch solution $x = 0$. The other follows from

$$0 = -\frac{1}{2}\frac{gx}{v_0^2\cos^2\theta} + \tan\theta \ .$$

We find

$$x = \frac{2v_0^2}{g}\sin\theta\cos\theta = \frac{v_0^2}{g}\sin(2\theta) \ ,$$

where we have used a trigonometric identity. That completes the derivation.

**Wrong answers:**
c) This is the maximum horizontal range formula.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00240 1 1 2 easy memory: horizontal range maximum angle

9. The maximum horizontal range value for projectile motion near the Earth's surface for a given launch speed $v_0$ and neglecting air drag is obtained for launch angle:

a) 30°. b) 45°. c) 60°. d) 82.245°. e) 90°.

**SUGGESTED ANSWER:** (b)

**Wrong answers:**
e) You are shooting the object straight up.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00260 1 1 1 easy memory: ratio of height to range formulae

10. The ratio of the maximum height formula (for projectile motion near the Earth's surface, neglecting air drag, and measuring from the launch height) to the horizontal range formula (for projectile motion near the Earth's surface and neglecting air drag) is ____________ and its value for the maximum range is ________.

a) $\dfrac{y_{\rm max}}{x_{\rm range}} = \dfrac{1}{4}\tan\theta$; $\frac{1}{4}$ b) $\dfrac{y_{\rm max}}{x_{\rm range}} = \dfrac{1}{4}\tan\theta$; 1 c) $\dfrac{y_{\rm max}}{x_{\rm range}} = \tan\theta$; 1 d) $\dfrac{y_{\rm max}}{x_{\rm range}} = \tan\theta$; $\frac{1}{4}$
e) $\dfrac{y_{\rm max}}{x_{\rm range}} = \dfrac{1}{4}\tan\theta$; $\infty$

**SUGGESTED ANSWER:** (a)

The horizontal range and maximum height formulae are, respectively,

$$x_{\rm range} = \frac{2v_0^2}{g}\sin\theta\cos\theta = \frac{v_0^2}{g}\sin(2\theta) \qquad \text{and} \qquad y_{\rm max} = \frac{v_0^2}{2g}\sin^2\theta \ .$$

Taking the ratio gives

$$\frac{y_{\rm max}}{x_{\rm range}} = \frac{1}{4}\tan\theta \ .$$

For the maximum range for which $\theta = 45^\circ$, one finds

$$\frac{y_{\rm max}}{x_{\rm range}} = \frac{1}{4} \ .$$

**Wrong answers:**

e) The value is what you'd get for $\theta = 90^\circ$. In this case, the height is finite, but the range has gone to zero.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00300 1 4 5 easy deducto-memory: relative motion explained

11. "Let's play *Jeopardy*! For \$100, the answer is: Motion of something with respect to something else. To be a bit more explicit, say you have two objects. The relative displacement of object 2 from object 1 is defined to be

$$\vec r = \vec r_2 - \vec r_1 \ .$$

From this definition, the relative velocity and acceleration follow from differentiation:

$$\vec v = \vec v_2 - \vec v_1 \qquad \text{and} \qquad \vec a = \vec a_2 - \vec a_1 \ .$$

The description of the motion in these terms clarifies the initial answer statement."

What is ____________, Alex?

a) variable motion b) relativistic motion c) no motion d) abosulte motion
e) relative motion

**SUGGESTED ANSWER:** (e)

**Wrong answers:**

d) Exactly wrong.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00350 2 5 5 moderate thinking: inertial frame motion is relative

12. You are in a featureless narrow room playing catch with a friend. How can you tell if the room is in a building or is a sealed compartment on super-smoothly running, non-accelerating train (or plane)? **HINT:** Review your whole life experience; try an experiment (but not while you are driving).

a) When you throw the ball **ALONG** the long axis of the room, it would have different speeds (relative to the room) in the two possible directions if you were on a train.
b) When you thow the ball **PERPENDICULAR** to the long axis of the room it would curve off a straight line (relative to the room) if you were on a train.
c) On a train the thrown ball would zigzag wildly in flight.
d) On a train the thrown ball would do loops in flight.
e) There is no way to tell as long as the train motion is very smooth.

**SUGGESTED ANSWER:** (e)

A moderate thinking question. People have been on trains and planes and cars and they know nothing weird happens as long as there is no acceleration. Fundamentally, non-accelerating frames are inertial frames and the laws of physics are the same relative to all of them. But in the context of a 5-minute quiz, it could be tough.

**Wrong answers:**

**Redaction:** Jeffery, 2001jan01

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004 qmult 00360 2 3 1 moderate math: ground speed from Pyth. theorem
**Extra keywords:** physci

13. You are flying a plane. Air velocity (i.e., plane velocity relative to the air) is 40 mi/h due north. Wind velocity is 30 mi/h due west. What is the magnitude of ground velocity (i.e., the ground speed)?

a) 50 mi/h. b) −50 mi/h. c) 40 mi/h. d) 10 mi/h. e) 2500 mi/h.

**SUGGESTED ANSWER:** (a)

This question was a natural for my pilot students at Middle Tennessee State University (MTSU). The magnitude of the ground velocity can be found by the Pythagorean theorem in this case because the two velocity vectors to be added are at right angles to each other. Deduction should also give the answer. But more formally:

$$\vec{v}_{\rm air} = (0,40) \qquad \text{and} \qquad \vec{v}_{\rm wind} = (30,0)$$

Thus,

$$\vec{v}_{\rm ground} = (30,40) \qquad \text{and} \qquad v_{\rm ground} = \sqrt{30^2+40^2} = 50\,\text{mi/h}\ .$$

**Wrong answers:**
b) Magnitude or speed is never negative.
d) No. This speed is only possible if the plane direction were opposite the wind direction.
e) You forgot to take the square root. But no subsonic aircraft velocity and wind velocity can give you this speed.

**Redaction:** Jeffery, 2001jan01

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004 qmult 00400 1 1 2 easy memory: dividing a circle

14. A circle can be divided into:

a) 360 divisions only. b) any number of divisions you like. c) $2\pi$ divisions only.
d) $\pi$ divisions only. e) 360 or $2\pi$ divisions only.

**SUGGESTED ANSWER:** (b)

**Wrong answers:**
a) A nonsense answer. **Redaction:** Jeffery, 2008jan01

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004 qmult 00410 1 1 2 easy memory: radians in a circle 1

15. How many radians are there in a circle?

a) $\pi$. b) $2\pi$. c) $3\pi$. d) $360°$. e) 360.

**SUGGESTED ANSWER:** (b)

**Wrong answers:**
e) The trick answer.

**Redaction:** Jeffery, 2001jan01

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004 qmult 00420 1 5 1 easy thinking: 24 factors in 360

16. The division of the circle into 360° was an arbitrary choice—and we don't know why. We just know the ancient Mesopotamian mathematicians and astronomers did it this way—you know Mesopotamia—ancient Iraq: "the cradle of civilization". Their choice was just adopted by the ancient Greeks and got passed on to us. In the French Revolutionary epoch, the decimal system was adopted for most measures, but the revolutionaries didn't get around (you might say) to the circle. We can guess that one reasons is that the ancient Mesopotamians had a preference for whole number arithmetic particularly in division and 360 has a lot of whole number factors. How many whole number (i.e., integer) factors does 360 have counting 1 and 360 itself?

a) 24. b) 360. c) 6. d) 7. e) 12.

**SUGGESTED ANSWER:** (a)

Below are the whole number factors of 360 table format:

| count | factor | complement factor |
|---|---|---|
| 2 | 1 | 360 |
| 4 | 2 | 180 |
| 6 | 3 | 120 |
| 8 | 4 | 90 |
| 10 | 5 | 72 |
| 12 | 6 | 60 |
| 14 | 8 | 45 |
| 16 | 9 | 40 |
| 18 | 10 | 36 |
| 20 | 12 | 30 |
| 22 | 15 | 24 |
| 24 | 18 | 20 |

**Wrong answers:**
b) A specious guess.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00430 1 3 4 easy math: radian to degree conversion

17. What is the approximate conversion factor from radians to degrees?

a) 1/60 degrees/radian. b) $\pi$ degrees/radian. c) $2\pi$ degrees/radian.
d) 60 degrees/radian. e) 360 degrees/radian.

**SUGGESTED ANSWER:** (d)

Behold

$$180^\circ = \pi \ , \qquad \text{and so} \qquad \frac{180^\circ}{\pi} \approx 57.2958 \approx 60\,\text{degrees/radian} \ .$$

**Wrong answers:**
a) Wrong conversion factor: this is for degrees to radians.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00440 1 5 5 easy thinking: the 2 pi unit ti

18. There are $2\pi$ radians in a circle. Its rather inconvenient that this means that there are $2\pi = 6.2831853\ldots$ radians in a circle which is an irrational number. For convenience we might invent a new base unit: the $2\pi$ with symbol "ti" (for **T**wo p**I** and pronounced tie). One hundredth of a ti would be a:

a) exati. b) megati. c) kiloti. d) deciti. e) centiti.

**SUGGESTED ANSWER:** (e)

I think the idea of the ti makes sense. We could then drop this non-metric degree unit and use centitis ($3.6^\circ$) and millitis ($0.36^\circ$) for most purposes. But no one ever listens to me.

**Wrong answers:**
a) Eek, $10^{18}$ ti.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00450 1 3 1 easy math: hand angular measure

19. Approximately, at arm's length a finger subtends $1^\circ$, a fist $10^\circ$, and a spread hand $18^\circ$. These numbers, of course, vary a bit depending on person and exactly how the operation is done. What are these angles approximately in radians?

a) 1/60, 1/6, and 1/3 radians. b) 60, 600, and 1800 radians.
c) $\pi/12$, $\pi/3$, and $\pi/2$ radians. d) $\pi/12$, $\pi/3$, and $\pi$ radians. e) $\pi/12$, $\pi/3$, and $2\pi$ radians.

**SUGGESTED ANSWER:** (a)

I've used the conversion factor ($\pi$ radians)/(180° approximated to (1 radian/60°.

**Wrong answers:**

b) This looks like a conversion from radians to degrees where one uses the approximate conversion factor 60 degrees/radian.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00460 1 5 5 easy thinking: covering the Moon

20. Can you cover the Moon with your finger held at arm's length? **HINT:** You could try for yourself if you are not in a a test *mise en scène*.

a) No. The Moon is much larger in angle than a finger. Just think how huge the Moon looks on the horizon sometimes.
b) It depends critically on the size of one's finger and arm. People with huge hands can to it and those without can't.
c) Yes. A finger at arm's length typically subtends about 10° and the Moon subtends 0.01°.
d) No. The Moon's diameter is about 3470 km and a finger is about a centimeter or so in width.
e) Usually yes. A finger at arm's length typically subtends about 1° and the Moon subtends 0.5°.

**SUGGESTED ANSWER:** (e)

**Wrong answers:**

d) Yes, this makes sense.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00470 1 1 5 easy memory: small angle approximations

21. For small angles $\theta$ measured in radians and with increasing accuracy as $\theta$ goes to zero (where the formulas are in fact exact), one has the small angle approximations:

a) $\sin\theta \approx \cos\theta \approx 1 - \dfrac{1}{2}\theta^2$. b) $\cos\theta \approx \tan\theta \approx 1 - \dfrac{1}{2}\theta^2$. c) $\sin\theta \approx \cos\theta \approx \theta$.
d) $\cos\theta \approx \tan\theta \approx \theta$. e) $\sin\theta \approx \tan\theta \approx \theta$.

**SUGGESTED ANSWER:** (e)

The proof of these approximations follows from the Taylor expansions of sine and tangent about $\theta = 0$: i.e.,

$$\sin\theta = \sum_{n=0} \frac{1}{(2n+1)!}\theta^{2n+1} = \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 - \frac{1}{5040}\theta^7 + \ldots \ ,$$
$$\tan\theta = \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5 + \frac{17}{315}\theta^7 + \ldots \ ,$$

where one can get these from

`http://en.wikipedia.org/wiki/Tangent_function#Series_definitions` .

**Wrong answers:**

a) One has $\cos\theta \approx 1 - (1/2)\theta^2$ for small angles, in fact.
b) One has $\cos\theta \approx 1 - (1/2)\theta^2$ for small angles, in fact.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00500 1 1 5 easy memory: polar coordinates

22. In 2-dimensional Cartesian coordinates, a displacement vector $\vec{r}$ is given by

$$\vec{r} = (x, y) = x\hat{x} + y\hat{y} \ ,$$

where the unit vectors $\hat{x}$ and $\hat{y}$ are constants. In polar coordinates,

$$\vec{r} = (r, \theta) = r\hat{r} \ ,$$

where the unit vector

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \ .$$

The polar coordinates are obtained from the Cartesian ones by the formulae

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \ .$$

In calculational work one must be aware that a negative argument of $\tan^{-1}$ is treated by calculators and computers as implying that $y > 0$ and $x < 0$. If the reverse is true, one must explicitly add or subtract $180°$ from the calculated result. The Cartesian components are obtained from the polar coordinates by the formulae

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta \ .$$

As well as $\hat{r}$, one needs another unit vector for polar coordinates that is perpendicular to the $\hat{r}$ and that is used for velocity and acceleration vectors and changes in the displacement vector. This is the unit vector ____________ given by

$$\underline{\qquad\qquad} = \vec{r}(\theta + 90°) = -\sin\theta\hat{x} + \cos\theta\hat{y} \ ,$$

a) $\hat{\alpha}$ b) $\hat{\omega}$ c) $\hat{n}$ d) $\hat{z}$ e) $\hat{\theta}$

**SUGGESTED ANSWER:** (e)

**Wrong answers:**
c) This is the usual symbol for the normal unit vector.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00530 1 4 1 easy deducto-memory: acceleration in polar coordinates

23. "Let's play *Jeopardy*! For $100, the answer is: The formulae

$$\vec{r} = r\hat{r} \ ,$$
$$\vec{v} = \frac{dr}{dt}\hat{r} + \frac{d\theta}{dt}\hat{\theta} \ ,$$
$$\vec{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\hat{\theta} \ ,$$

where $r$ is magnitude of displacement from the origin (or radial component of the displacement $\vec{r}$), $\hat{r}$ is the unit vector of the radial component, $\theta$ the angular component of $\vec{r}$, $\hat{\theta}$ is the unit vector of the angular coordinate, and one often writes $d\theta/dt$ is as $\omega$ which is called the angular velocity. The unit vectors are functions of the angular component:

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \qquad \text{and} \qquad \hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y} \ .$$

What are displacement, velocity, acceleration in ____________ coordinates, Alex?

a) polar b) Cartesian c) spherical polar d) elliptical e) hyperbolical

**SUGGESTED ANSWER:** (a) See Fr-557 and

`http://en.wikipedia.org/wiki/Kinematics#Cylindrical_coordinates` .

**Wrong answers:**
c) You really don't want to know.
d) I really don't want to know.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00550 1 4 3 easy deducto-memory: centripetal acceleration defined sort of
**Extra keywords:** physci

24. "Let's play *Jeopardy*! For $100, the answer is: It is the acceleration in a case of circular motion."

What is ____________, Alex?

a) net acceleration b) centrifugal (center-fleeing) acceleration
c) centripetal (center-pointing) acceleration d) deceleration e) zero

**SUGGESTED ANSWER:** (c)

**Wrong answers:**
e) The magnitude of the velocity is unchanging, but the direction changes continually. Hence there is an acceleration.

**Redaction:** Jeffery, 2001jan01

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004 qmult 00552 1 1 3 easy memory: centripetal acceleration formula 1
25. The radial component of acceleration in polar coordinates

$$a_r = \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2 r}{dt^2} - r\omega^2$$

specializes to ____________ if the motion is circular and centered on the origin. In this case, the radial component of acceleration is called centripetal (meaning center pointing) since, in fact, it is always negative (i.e., the radial component of acceleration always points toward the origin). The radial component of velocity for circular motion is zero naturally and the angular component, often called the tangential velocity is given by

$$v_\theta = r\omega\ .$$

Usually, one drops the subscripts $r$ and $\theta$ on $a_r$ and $v_\theta$ if the quantities are identified by context.

a) $a_r = -r\omega = -\frac{v_\theta}{r}$ b) $a_r = r\omega^2 = \frac{v_\theta^2}{r}$ c) $a_r = -r\omega^2 = -\frac{v_\theta^2}{r}$ d) $a_r = \omega^2 = v_\theta^2$
e) $a_r = -\omega^2 = -v_\theta^2$

**SUGGESTED ANSWER:** (c)

**Wrong answers:**
a) Not dimensionally correct.

**Redaction:** Jeffery, 2008jan01

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004 qmult 00556 2 5 2 moderate thinking memory: cent. accel. behavior
26. The formula for centripetal acceleration magnitude for circular motion is

$$a_{\rm cen} = \frac{v^2}{r}\ ,$$

where $v$ is the tangential velocity of the motion and $r$ is the radius of the circle. The centripetal acceleration ____________ with $v$ and ____________ with $r$.

a) increases linearly; decreases inverse linearly b) increases quadratically; decreases inverse linearly c) decreases inverse linearly; increases quadratically d) increases quadratically; decreases inverse quadratically e) constant; decreases inverse linearly

**SUGGESTED ANSWER:** (b)

**Wrong answers:**
e) As Lurch would say: "Aaaarh."

**Redaction:** Jeffery, 2001jan01

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004 qmult 00560 1 1 5 easy memory: uniform circular motion
27. In uniform circular motion, position, velocity, and acceleration are continually changing. The velocity and acceleration magnitudes are:

a) continually changing too. b) continually changing and constant, respectively.
c) constant and continually changing, respectively. d) constant and undefined, respectively.
e) both constant.

**SUGGESTED ANSWER:** (e)

**Wrong answers:**
d) A nonsense answer.

**Redaction:** Jeffery, 2008jan01

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004 qfull 00200 2 3 0 moderate math: quadratic parabolic I

28. The formula for a general quadratic is

$$f(x) = ax^2 + bx + c \ .$$

The shape of a quadratic is actually a simple parabola with the vertical symmetry axis offset from the origin.

a) Show this by completing the square in the function

$$f(x) = ax^2 + bx + c$$

and find the $x$ coordinate of the symmetry axis. The completed square form is called the vertex form.

b) What is the formula for a transformed coordinate $x'$ that explicitly turns the quadratic function into a simple parabola?

**SUGGESTED ANSWER:**

a) Behold:

$$\begin{aligned} f(x) &= ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right] \\ &= a\left\{\left[x - \left(-\frac{b}{2a}\right)\right]^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right\} \ . \end{aligned}$$

Now it is clear that the function gives a simple parabola with symmetry axis at $x = -b/(2a)$.

b) The required formula is

$$x' = x + \frac{b}{2a} \ .$$

**Redaction:** Jeffery, 2008jan01

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004 qfull 00210 3 5 0 tough thinking: horizontal range formula algebra

**Extra keywords:** Suitable for the algebra-based course.

29. A projectile is launched from $x$-$y$ origin which is on the ground level of large level plain. The $x$ direction is the horizontal and the $y$ direction is the vertical: upward is the positive $y$ direction. The projectile is launched in the positive $x$ direction. The initial launch speed is $v_0$ at angle $\theta$ above the horizontal. Air drag is neglected.

a) Find the expressions for $x$ and $y$ position as functions of time $t$ entirely in symbols and with any dependences on $\theta$ shown explicitly. Drop any symbols that stand for known zeros.

b) Now find $y$ as a function of $x$ by eliminating $t$. Now find the horizontal range formula: i.e., the expression for $x$ when the projectile returns to $y = 0$. Simplify the formula as much as is reasonably possible. **HINT:** The trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$ helps simplifying formula.

c) Using the horizontal range formula, find by any means the angle $\theta$ a maximum range holding all the other variables constant. Briefly explain how you arrived at your answer. **HINT:** The sine function has only one maximum in the domain of its argument $[0°, 180°]$.

**SUGGESTED ANSWER:**

a) From the kinematic equations for constant acceleration, one immediately finds:

$$x = (v_0 \cos\theta)\, t$$

and

$$y = -\frac{1}{2}gt^2 + (v_0 \sin\theta)\, t \ .$$

b) Well

$$t = \frac{x}{v_0 \cos\theta} \ .$$

Substituting this into the expression for $y$ gives

$$y = -\frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2 + x\tan\theta \ .$$

Well by setting the last expression for $y$ equal to zero we obtain

$$0 = -\frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2 + x\tan\theta \ .$$

One solution for $x$ is $x = 0$ which is just the launch solution. The other solution is the horizontal range solution which is nonzero. Since it is non-zero, we can divide through by this solution $x$ and obtain a linear equation for $x$:

$$0 = -\frac{1}{2}g\left(\frac{x}{v_0^2\cos^2\theta}\right) + \tan\theta \ .$$

Solving for $x$ gives

$$x = \frac{2v_0^2}{g}\sin\theta\cos\theta = \frac{v_0^2}{g}\sin(2\theta)$$

(See also HRW-57). This solution is the horizontal range formula.

c) After an exhausting search that took hours and hours, I found that $\theta = 45^\circ$ gave $\sin(2\theta)$ a maximum value of about 1 and therefore a maximum range for the projectile.

All right I lied, I didn't do that. I know that the sine function in the domain for its argument $(0^\circ, 180^\circ)$ has a maximum of 1 for $90^\circ$. Thus, the range is maximum for $2\theta = 90^\circ$ or $\theta = 45^\circ$. Of course, if I didn't know where the maximum was I could have used calculus to find it, but calculus is *hors de combat* as *les Français* might say in algebra-based introductory physics.

**Redaction:** Jeffery, 2008jan01

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004 qfull 00230 3 3 0 tough math: projectile motion in volleyball

30. The women's volleyball court has a net height of 2.24 m and extends 9.0 m on either side of the net. On a jump serve, a player spikes (I think that's the word) the ball at 3.00 m above the court in a direction perpendicular to the net. The initial velocity is **HORIZONTAL** and the net is 8.0 m away from the server. Neglect air drag.

This is a problem in which you want to find the conditions that lead to the desired result. It's really a pretty common kind of problem—in life as well as in physics.

a) First sketch the system: launch, court, net. Then sketch a general trajectory that lands before the net and one that lands on the other side of then (i.e., that clears the net).

b) First, solve in **SYMBOLS** for initial velocity $v_0$ as a function of **ONLY** the variables $x$ (horizontal position measured from the launch point) and $y$ (height), and the constants $y_0$ (launch height) and $g$. The time variable $t$ should be eliminated. Essentially, for a general point $(x, y)$ on a general trajectory (that had a horizontal launch velocity recall) you get the initial velocity $v_0$ which will get you there.

Sketch a graph of $v_0$ as a function of $y$ for constant $x$: indicate on sketch the $v_0$ for $y = 0$ in symbols and the location of the maximum of $v_0$. What is the significance of this maximum? Then sketch $v_0$ as a function of $x$ for constant $y$.

c) What is the minimum velocity $v_0$ needed for the ball to clear the net? Assume the ball is a point mass for this part.

d) What is the maximum velocity $v_0$ allowed if the ball is to stay in court? Assume no one touches the ball and assume the ball is a point mass for this part.

**SUGGESTED ANSWER:**

a) You will have to imagine the sketch.

b) The kinematic equations for position as a function of time are for the $x$ and $y$ directions, respectively,

$$x = v_0 t \qquad \text{and} \qquad y = -\frac{1}{2}gt^2 + y_0 \ .$$

Now we want to find $v_0$ as a function of $x$, $y_0$, $y$, and $g$. So clearly we must combine the above formulae and eliminate $t$. There are more than one way to do this. The one that comes to mind first is to find $t$ as a function of $y$. Behold:

$$t = \sqrt{\frac{y - y_0}{-(1/2)g}} = \sqrt{\frac{2(y_0 - y)}{g}} \ .$$

The negative solution for time was for the mythical extrapolation of the kinematic equations to negative times before the spiking event. Using our expression for time and the $x$ position expression, we find

$$v_0 = \frac{x}{t} = x\sqrt{\frac{g}{2(y_0 - y)}} \ .$$

You will have imagine the plot of $v_0$ versus $y$. For $x$ fixed, one one can see that $v_0$ rises from a minimum $x\sqrt{g/(2y_0)}$ for $y = 0$ (which is when the ball lands on the ground at $x$) and goes to maximum of infinity for $y = y_0$. This latter case shows that you need infinite initial horizontal velocity to keep the ball from falling downward when $x > 0$.

You will have imagine the plot of $v_0$ versus $x$. But you can see that is a simple linear plot. The slope is $\sqrt{g/[2(y_0 - y)]}$. This shows that for fixed $y$, one requires requires a higher initial magnitude of velocity as $x$ increases. Another way of looking at it is that to put the arrival at height $y$ farther away (i.e., at increased $x$) requires increasing $v_0$.

This problem illustrates how initial conditions can be determined so that the outcome is what one desires.

c) The final $x$ position in this case is fixed and is $x = 8\,\text{m}$ measured relative to the launch position. Thus the $v_0$ can be viewed as a function of final $y$. Since $v_0$ increases with $y$, the minimum launch speed to clear the net will be for $y$ equal to the net height. Thus,

$$v_0 \approx 8 \times \sqrt{\frac{10}{2 \times (3.00 - 2.24)}} \approx 8 \times \sqrt{\frac{10}{1.5}} \approx 20\,\text{m/s}$$

to about 1-digit accuracy. To better accuracy

$$v_0 = 20.3\,\text{m/s} \ .$$

d) In this case, the final height is fixed at $y = 0$ which is the minimum landing height of the ball. Now $v_0$ increases with $x$ and the maximum in-court $x$ is the distance from the spiker to the far end of the court (i.e., $x = 17\,\text{m}$), and so the maximum velocity for the ball staying in-court is given by

$$v_0 \approx 17.0 \times \sqrt{\frac{10}{2 \times (3.00 - 0.00)}} \approx 17.0 \times \sqrt{\frac{10}{6.00}} \approx 22\,\text{m/s}$$

to about 1-digit accuracy. To better accuracy

$$v_0 = 21.7\,\text{m/s} \ .$$

The speed range for a good serve is very narrow under the given conditions. I suspect that in actual play there is probably a broader range. Mediocre servers probably just shoot somewhat upward at less-than-killer speed to make sure the ball gets over and stays in play. Expert servers may be able hit the horizontal launch speed rather closely, but through practice not calculation. By the way the spiking height of 3.00 m (9.84 ft) is really high it seems to me: maybe the spiker is an Olympian.

Fortran Code

```
        print*
        conv=3.28  ! conversion factor for meters to feet
        xx1=8.
        g=9.8
        y1=2.24
        y0=3.
        y0feet=y0*conv
        vv1=xx1*sqrt( g/ (2.*(y0-y1)) )
        xx2=17.
        y2=0.
        vv2=xx2*sqrt( g/ (2.*(y0-y2)) )
        print*,'vv1,vv2,y0feet'
        print*,vv1,vv2,y0feet
*             20.3133354  21.7263279  9.84
```

**Redaction:** Jeffery, 2001jan01

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004 qfull 00240 3 3 0 tough math: orient express

31. You are on the last run of the historic Orient Express train in 1939 traveling from Paris to Istanbul. Somewhere between Belgrade and Sofia, the train ominously starts accelerating in the reverse direction—which we will call the negative $x$ direction. The **MAGNITUDE** of this acceleration is $a_{\rm tr}$. The train is on a **STRAIGHT**, level line of track. Neglect **AIR DRAG**.

   a) Inside the train car a projectile is launched in the positive $x$ direction (i.e., in the forward direction). What is the horizontal acceleration **RELATIVE** to the train car in terms of knowns? What is the vertical acceleration **RELATIVE** to the train car in terms of knowns? Take the **UP** direction as the positive $y$ direction. **HINT:** No elaborate calculations are needed. We are just looking for simple, short, symbol answers. Remember the formula for **RELATIVE ACCELERATION** of an object 2 with respect to object 1:

   $$\vec{a}_{\rm rel21} = \vec{a}_2 - \vec{a}_1 \ .$$

   b) You and your mysterious *compagnon de voyage* M. Achille find yourselves locked in your car—and attempt some two-dimensional kinematics. What are the $x$ and $y$ positions relative to the car as functions of time $t$ for the projectile launched from the origin (which is fixed to the car) at time zero with launch speed $v_0$ (relative to the car) and at an angle $\theta$ (in the range $0°$ to $90°$ and also relative to the car) to the positive $x$ axis? Express these positions using the accelerations found in part (a), time $t$, $v_0$ and $\theta$.

   c) For reasons known to himself alone, M. Achille insists that you find the horizontal range formula for the projectile in the reference frame of the car: i.e., a formula giving the horizontal range (the $x$ displacement from launch height to launch height) in terms of variables **NOT** including time $t$. Do so and simplify it as much as reasonably possible.

   d) What is the horizontal range formula for the car in the case that $a_{\rm tr}$ goes to zero?

**SUGGESTED ANSWER:**

   a) Since the car is accelerating in the negative direction with acceleration of magnitude $a_{\rm tr}$ relative to the ground, any object without $x$ direction forces acting on it must accelerate in the positive $x$ direction with accelerartion $a_{\rm tr}$ relative to the car. It is a positive acceleration. In fact, relative to the ground there is no $x$ acceleration for the projectile. As a formula, the $x$ acceleration relative to the car is given by

   $$a_x = 0 - (-a_{\rm tr}) = a_{\rm tr} \ .$$

The $y$ direction motion is, of course, independent of the $x$ direction motion. In the $y$ direction, the acceleration relative to the car (and also to the ground) is $-g$ or as a formula

$$a_y = -g \ .$$

b) Applying the constant-acceleration kinematic equations to the two independent directions, we find

$$x = \frac{1}{2}a_{\rm tr}t^2 + (v_0\cos\theta)t$$

and

$$y = -\frac{1}{2}gt^2 + (v_0\sin\theta)t \ .$$

c) The horizontal range formula is obtained by setting $y = 0$ in the equation for the $y$ position. When that is done, there are two solutions for time $t$. One is just the launch time $t = 0$. The other solution gives the time when the projectile has returned to the height $y = 0$: this location is at the horizontal range in the $x$ coordinate. We set $y$ equal to zero in the $y$ position equation and divide through by $t$ for the non-zero time solution to get

$$0 = -\frac{1}{2}gt + v_0\sin\theta \ .$$

Solving for time gives

$$t = \frac{2v_0\sin\theta}{g} \ .$$

Substituting this time into the $x$ position equation gives

$$x = \frac{2a_{\rm tr}v_0^2}{g^2}\sin^2\theta + \frac{2v_0^2}{g}\cos\theta\sin\theta$$

or

$$x = \frac{2a_{\rm tr}v_0^2}{g^2}\sin^2\theta + \frac{v_0^2}{g}\sin(2\theta) \ .$$

The last formula is the horizontal range formula for the accelerated frame of the train car.

d) If $a_{\rm tr} \to 0$, then the horizontal range formula reduces to

$$x = \frac{v_0^2}{g}\sin(2\theta)$$

which is just the ordinary horizontal range formula for projectile motion relative to the ground or frames unaccelerated with respect to the ground.

**NOTE:** M. Achille may be the brother of Hercule Poirot, but then again Hercule Poirot may never have had a brother—except temporarily. But if M. Achille is not Poirot's brother, then he could be ...

**Redaction:** Jeffery, 2008jan01

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004 qfull 00310 2 3 0 moderate math: swimming English channel

32. You have decided to swim the English channel from Gris-Nez to Folkestone: a distance of about 40 km roughly to the north-west. In this problem, treat the distances and directions as exact: e.g., 40 km is the exact distance from Gris-Nez to Folkestone north-west is exactly 45° west of north, and south-west is exactly 135° west of north. Also treat the Earth as flat. **HINT:** Smear your body all over with fats to prevent hypothermia.

a) If you swim at 4.0 km/h, how long will it take you?

b) As it turns out you land at New Romney about 20 km south-west of Folkestone. What is the average channel current velocity during your swim?

c) What was your average velocity (note velocity, not just speed) during the swim relative to the fixed Earth? As noted in the preamble, assume that Folkestone is exactly north-west of Gris-Nez and New Romney is exactly south-west of Folkestone. Give the velocity in magnitude-direction format. **HINT:** Velocity requires a direction specification too.

**SUGGESTED ANSWER:**

a) Behold:

$$t = \frac{d}{v} = \frac{40}{4.0} = 10\,\mathrm{h} \ .$$

b) First note that since the unexpected displacment to the southwest was exactly perpendicular to the displacement that your velocity relative to the water would have given, the water must have been moving exactly perpendicular to velocity relative to the water. Therefore, the displacement to the southwest was entirely due to the water's motion.

With that established, one finds

$$v_{\mathrm{current}} = \frac{d}{t} = \frac{20}{10} = 2\,\mathrm{km/h} \ .$$

The direction is south-west: i.e., 135° west of north.

c) Well the two velocities, swimming and current, are exactly perpendicular (in the assumptions of this problem), and so the magnitude of the net velocity is

$$\sqrt{4^2 + 2^2} = \sqrt{20} \approx 4.5\,\mathrm{km/h} \ .$$

The several ways of finding the direction. Probably, the simplest is just to the inverse tangent function to find the angle $\theta$ that you actually traveled counterclockwise from due north-west. Behold:

$$\theta = \tan^{-1}\left(\frac{20}{40}\right) \approx 27^\circ \ .$$

Thus net velocity was at about 72° degrees west of north. A map actually suggests the direction would be more like 75° west of north. Given the crudity of the calculation anything in the vicinity of 70°–80° is OK.

**Redaction:** Jeffery, 2001jan01

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004 qfull 00320 3 5 0 tough thinking: Amazon crossing 1

33. Deep in the Amazonian jungle you wish to cross a river livid with piranha and crocodiles. Let $x$ be the coordinate **ALONG** the river and $y$ the coordinate **PERPENDICULAR** to the river. The river width is $y_{\mathrm{max}} = 20\,\mathrm{m}$. You are going to paddle across (on an unstable rotting log—using your bare hands) and your paddling speed is $v_{\mathrm{paddle}} = 0.50\,\mathrm{m/s}$.

a) How long does it take you to cross assuming that you (wisely) aim straight in the $y$-direction?

b) Assuming the river flows at a steady velocity $v_x = 0.20\,\mathrm{m/s}$ how far downstream (i.e., in the $x$ direction) have you gone while crossing?

c) What is your velocity relative to the shore during the crossing? Give this velocity in magnitude-direction format. **HINT:** Remember to specify direction.

d) Now let's make the not-too-likely assumption that the river velocity in the $x$-direction is a linear function of $y$, the distance in the $y$-direction. Let

$$v_x(y) = v_{x\ \mathrm{max}} \frac{y}{y_{\mathrm{max}}} \ ,$$

where $y = 0$ at the starting shore and $v_{x\ \mathrm{max}} = 0.8\,\mathrm{m/s}$. Given that everything, except the river velocity, is the same as in the earlier parts of the question, how far downstream (i.e., in the $x$ direction) do you move?

**SUGGESTED ANSWER:**

a) Behold

$$t_{\rm cross} = \frac{y_{\rm max}}{v_{\rm paddle}} = \frac{20}{0.5} = 40\,{\rm s}\ .$$

b) Behold

$$x_{\rm cross} = v_x t_{\rm cross} = 8\,{\rm m}\ .$$

c) The magnitude of the net velocity is

$$v = \sqrt{v_{\rm paddle}^2 + v_x^2} = 0.54\,{\rm m/s}$$

and the direction measured from the $x$-axis is

$$\theta = \tan^{-1}\left(\frac{v_{\rm paddle}}{v_x}\right) = 68^\circ\ .$$

d) Well your $y$ location is a function of time $y = v_{\rm paddle}t$, and thus

$$v_x(y) = v_{x\ \rm max}\frac{y}{y_{\rm max}} = v_{x\ \rm max}\frac{v_{\rm paddle}}{y_{\rm max}}t\ .$$

Thus the problem is a constant acceleration problem in the $x$ direction with the acceleration being

$$a = v_{x\ \rm max}\frac{v_{\rm paddle}}{y_{\rm max}}\ .$$

The initial $x$-direction velocity is zero and the initial $x$ position can be taken to be zero. The downstream distance in this case is

$$x_{\rm cross} = \frac{1}{2}at^2 = \frac{1}{2}v_{x\ \rm max}\frac{v_{\rm paddle}}{y_{\rm max}}t^2 = 16\,{\rm m}\ .$$

Another approach is to use the midstream velocity as the average velocity. Then

$$x_{\rm cross} = v_{\rm mid}t = v_{x\ \rm max}\frac{(y_{\rm max}/2)}{y_{\rm max}}t = 0.4\times 40 = 16\,{\rm m}\ .$$

This actually takes some justification. From the standard kinematic equations in one dimension,

$$v = v_0 + a\Delta t\ ,$$

and so

$$v_{\rm mid} = v_0 + a\frac{\Delta t}{2} = v_0 + \frac{1}{2}(v - v_0) = \frac{1}{2}(v_0 + v)\ .$$

Now $v_{\rm mid}$ is the average velocity in the sense of being the average of the two endpoint velocities. But it is also the average velocity in the ordinary sense. Recall the other standard 1-dimensional kinematic equation

$$x = x_0 + \frac{1}{2}(v_0 + v)\Delta t\ .$$

Now the average velocity is

$$v_{\rm average} = \frac{x - x_0}{\Delta t} = \frac{1}{2}(v_0 + v) = v_{\rm mid}\ .$$

Thus for constant acceleration cases, $v_{\rm average} = v_{\rm mid}$ and our use of the midstream velocity as the average velocity is justified.

Fortran Code

```
      print*
      pi=acos(-1.)
      raddeg=180./pi
      vpad=.5
```

```
         vx=.2
         ymax=20.
         tcross=ymax/vpad
         xcross=vx*tcross
         v=sqrt(vpad**2+vx**2)
         theta=atan(vpad/vx)*raddeg
         vxmax=.8
         xcross2=.5*vxmax*(vpad/ymax)*tcross**2
         print*,'tcross,xcross,v,theta,xcross2'
         print*,tcross,xcross,v,theta,xcross2
*   40.  8.  0.538516462  68.1985855  16.
```

**Redaction:** Jeffery, 2001jan01

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004 qfull 00500 2 3 0 moderate math: v and a in polar coordinates 1

34. Let's do a general treatment—like it or not—of circular motion in polar coordinates.

a) The radial and angular unit vectors of polar coordinates are, respectively,

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \qquad \text{and} \qquad \hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y} \ ,$$

where the Cartesian unit vectors $\hat{x}$ and $\hat{y}$ are constants and $\theta$ is angular position. Note that $\hat{r}$ is the unit vector that points in the $\theta$ direction. Both polar coordinate unit vectors are position dependent, unlike the Cartesian unit vectors $\hat{x}$ and $\hat{y}$. Show $\hat{\theta}$ points $\pi/2$ counterclockwise from $\hat{r}$. **HINT:** Evaluate $\hat{r}$ for $\theta + \pi/2$ (i.e., evaluate the unit vector that points in the $\theta + \pi/2$ direction) and make use of standard trig identities

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \qquad \text{and} \qquad \sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \ .$$

b) Differentiate $\hat{r}$ and $\hat{\theta}$ with respect to time and write the results in terms of $\hat{r}$ and $\hat{\theta}$ and the angular velocity $\omega = d\theta/dt$. **HINT:** Use the chain rule.

c) The position vector for circular motion in polar coordinates is

$$\vec{r} = r\hat{r} \ ,$$

where $r$ is a constant (since the motion is cirular). Derive the velocity and acceleration formulae making use of the results of the part (b) answer. Note that angular acceleration is given the symbol $\alpha$: thus,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \ .$$

d) Write down the formulae for the tangential and radial components of velocity and acceleration. The radial component of the acceleration is called the centripetal acceleration which means center seeking acceleration since for circular motion it always points toward the center (i.e., is always negative). Write the centripetal acceleration in terms of $\omega$ and tangential velocity component.

e) The position vector for in polar coordinates in general (i.e., for varying direction and magnitude) is

$$\vec{r} = r\hat{r} \ ,$$

Derive the general velocity and acceleration formulae making use of the results of the part (b) answer and using the symbols for angular velocity ($\omega$) and angular acceleration ($\alpha$) for simplification.

**SUGGESTED ANSWER:**

a) Behold:

$$\hat{r}(\theta + \pi/2) = \cos(\theta + \pi/2)\hat{x} + \sin(\theta + \pi/2)\hat{y} = -\sin\theta\hat{x} + \cos\theta\hat{y} = \hat{\theta} \ ,$$

where have used

$$\cos(\theta + \pi/2) = \cos(\theta)\cos(\pi/2) - \sin(\theta)\sin(\pi/2) = -\sin(\theta)$$

and

$$\sin(\theta + \pi/2) = \sin(\theta)\cos(\pi/2) + \cos(\theta)\sin(\pi/2) = \cos(\theta) \ .$$

Thus, we see that the unit vector rotated $\pi/2$ counterclockwise from $\hat{r}$ is $\hat{\theta}$.

b) Behold:

$$\frac{d\hat{r}}{dt} = [-\sin\theta\hat{x} + \cos\theta\hat{y}]\omega = \omega\hat{\theta}$$

and

$$\frac{d\hat{\theta}}{dt} = [-\cos\theta\hat{x} - \sin\theta\hat{y}]\omega = -\omega\hat{r} \ .$$

Compactly, the results are

$$\frac{d\hat{r}}{dt} = \omega\hat{\theta} \qquad \text{and} \qquad \frac{d\hat{\theta}}{dt} = -\omega\hat{r} \ .$$

c) Behold:

$$\vec{v} = r\frac{d\hat{r}}{dt} = r\omega\hat{\theta} \ ,$$

and

$$\vec{a} = r\alpha\hat{\theta} - r\omega^2\hat{r} \ .$$

d) Behold:

$$v_{\text{tan}} = r\omega \ , \qquad a_{\text{tan}} = r\alpha \ , \qquad a_{\text{cen}} = -r\omega^2 = -v_{\text{tan}}\omega = -\frac{v_{\text{tan}}^2}{r} \ ,$$

where the subscripts tan and cen stand for, respectively, tangential and centripetal. The magnitude of the centripetal acceleration is often written

$$a = \frac{v^2}{r}$$

which is fine as long as one knows what one means.

e) Behold:

$$\vec{v} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \ ,$$

and

$$\begin{aligned} \vec{a} &= \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\omega\hat{\theta} + \frac{dr}{dt}\omega\hat{\theta} + r\alpha\hat{\theta} - r\omega^2\hat{r} \\ &\left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(2\frac{dr}{dt}\omega + r\alpha\right)\hat{\theta} \ . \end{aligned}$$

We can somewhat simplify the results using Newton's dot notation where one dot over a quantity means first derivative with respect to time and two means second derivative with respect to time. Newton's dot notation is conventional in the case of angular motion. We find

$$\begin{aligned} \vec{r} &= r\hat{r} \\ \vec{v} &= \dot{r}\hat{r} + r\omega\hat{\theta} \\ \vec{a} &= \left(\ddot{r} - r\omega^2\right)\hat{r} + (2\dot{r}\omega + r\alpha)\,\hat{\theta} \ . \end{aligned}$$

If $r$ is constant (i.e., we have circular motion), then these formulae reduce to We find

$$\begin{aligned} \vec{r} &= r\hat{r} \\ \vec{v} &= r\omega\hat{\theta} \\ \vec{a} &= -r\omega^2\hat{r} + r\alpha\hat{\theta} \end{aligned}$$

as we found in the part(a) answer.

**Redaction:** Jeffery, 2008jan01

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004 qfull 00530 2 3 0 moderate math: uniform circular motion and TGV

35. Consider a TGV (Train à Grande Vitesse).

a) Say it has a speed of 216 km/h and is moving in a circle. What is the minimum radius of the circle if the centripetal acceleration is to be kept less than $0.05g$? Explain why it must be the minimum radius. **HINT:** Schematically plot centripetal acceleration versus $r$.

b) If the radius is 1.00 km, what is the maximum speed (in m/s and km/hr) before the acceleration exceeds $0.05g$? Explain why it is the maximum speed. **HINT:** Schematically plot centripetal acceleration versus $v$.

**NOTE:** Here we use $g = 9.8\,\text{m/s}^2$ as a unit of acceleration. An accelerated frame is exactly like being gravitational field so far a mechanical experiments internal to the frame are concerned. This is actually a profound observation that started Einstein on his way to his general theory of relativity. The pseudo-gravitational force points opposite to the acceleration and has magnitude $ma$. This pseudo or inertial force is sometimes called the g-force and is specified as a force per unit mass in units of $g$. Thus circling with acceleration $0.05g$ is just like being in gravitational field of $0.05g$ where the gravitational force points radially outward.

**SUGGESTED ANSWER:**

TGVs are Trains a Grand Vitesse that run in France. Top commercial speed is 300 km/h, but the record speed is 515.3 km/h. I rode one in 1990 September from Geneve to Paris on my way from a joyous month-long stay (studying physics) at a lodge in the French Alps near Chamonix and Mont Blanc. For more on the TGV see:
http://en.wikipedia.org/wiki/TGV .

a) For the result we first need to convert the speed to MKS units:

$$216\,\text{km/h} = 216\,\text{km/h} \times \left(\frac{1000\,\text{m}}{1\,\text{km}}\right)\left(\frac{1\,\text{h}}{3600\,\text{s}}\right) = 60\,\text{m/s} .$$

Now recall the magnitude of centripetal acceleration is given by

$$a = \frac{v^2}{r} ,$$

where $v$ is the constant or uniform tangential speed and $r$ is the radius. We see that $a$ increases as $r$ decreases. Thus the smallest $r$ before the acceleration exceeds $0.05g$ is given by:

$$r = \frac{v^2}{a} = \frac{v^2}{0.05g} \approx \frac{3600}{0.5} = 7200\,\text{m} = 7.2\,\text{km}$$

to 1-digit accuracy. To better accuracy the radius is 7.35 km.

b) Since $a$ increases with $v$, we see that $v$ must be upper-bounded if $a$ is not to exceed $0.05g$. The maximum speed is given by

$$v = \sqrt{ar}\sqrt{0.05g \times 1000} \approx \sqrt{0.5 \times 1000} \approx 22.5\,\text{m/s} \approx 80\,\text{km/h} .$$

to 1-digit accuracy. To better accuracy the $22.2\,\text{m/s} = 79.7\,\text{km/h}$.

Fortran Code

```
print*
g=9.8
acel=g*.05
conv=1000./3600.
v=216.*conv
rmin=v**2/acel
rad=1000.
vmax=sqrt(acel*rad)
```

```
        vmaxkmhr=vmax/conv
        print*,'v,rmin,vmax,vmaxkmhr'
        print*,v,rmin,vmax,vmaxkmhr
*    60.0000038  7346.93945  22.1359444  79.6893997
```

**Redaction:** Jeffery, 2001jan01

# Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

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## 1 Constants

$$c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 2.998 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \qquad \text{exact by definition}$$

$$e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$G = 6.67428(67) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(2006, CODATA)}$$

$$g = 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 8.987551787\ldots \times 10^9 \approx 8.99 \times 10^9 \approx 10^{10}\,\mathrm{N\,m^2/C^2}\text{exact by definition}$$

$$k_{\mathrm{Boltzmann}} = 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} = 0.8617343(15) \times 10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,\mathrm{eV/K}$$

$$m_e = 9.10938215(45) \times 10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV}$$

$$m_p = 1.672621637(83) \times 10^{-27}\,\mathrm{kg} = 938.272013(23), \mathrm{MeV}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = 8.8541878176\ldots \times 10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11} \qquad \text{vacuum permittivity (exact by definition)}$$

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad \text{exact by definition}$$

---

## 2 Geometrical Formulae

$$C_{\mathrm{cir}} = 2\pi r \qquad A_{\mathrm{cir}} = \pi r^2 \qquad A_{\mathrm{sph}} = 4\pi r^2 \qquad V_{\mathrm{sph}} = \frac{4}{3}\pi r^3$$

$$\Omega_{\mathrm{sphere}} = 4\pi \qquad d\Omega = \sin\theta\,d\theta\,d\phi$$

---

## 3 Trigonometry Formulae

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}$$

$$f(\theta) = f(\theta + 360^\circ)$$

$$\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin(\theta + 90^\circ) = \cos(\theta) \qquad \cos(\theta + 90^\circ) = -\sin(\theta) \qquad \tan(\theta + 90^\circ) = -\tan(\theta)$$

$$\sin(180^\circ - \theta) = \sin(\theta) \qquad \cos(180^\circ - \theta) = -\cos(\theta) \qquad \tan(180^\circ - \theta) = -\tan(\theta)$$

$$\sin(90^\circ - \theta) = \cos(\theta) \qquad \cos(90^\circ - \theta) = \sin(\theta) \qquad \tan(90^\circ - \theta) = \frac{1}{\tan(\theta)} = \cot(\theta)$$

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a)$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a - b) - \cos(a + b)\right] \qquad \cos(a)\cos(b) = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a - b) + \sin(a + b)\right]$$

$$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)] \qquad \cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)] \qquad \sin(a)\cos(a) = \frac{1}{2}\sin(2a)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

---

## 4 Approximation Formulae

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx} \qquad \frac{1}{1 - x} \approx 1 + x \;:\; (x << 1)$$

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2 \qquad \text{all for } \theta << 1$$

---

## 5 Quadratic Formula

$$\text{If} \qquad 0 = ax^2 + bx + c\,, \qquad \text{then} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

## 6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

## 7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

Taylor's series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) \\ &= f(x_0) + (x - x_0)f^{(1)}(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x - x_0)^3}{3!} f^{(3)}(x_0) + \ldots \end{aligned}$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

## 8 One-Dimensional Kinematics

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$v = at + v_0 \qquad x = \frac{1}{2}at^2 + v_0 t + x_0 \qquad v^2 = v_0^2 + 2a(x - x_0)$$

$$x = \frac{1}{2}(v_0 + v)t + x_0 \qquad x = -\frac{1}{2}at^2 + vt + x_0 \qquad g = 9.8\,\mathrm{m/s^2}$$

$$x_{\text{rel}} = x_2 - x_1 \qquad v_{\text{rel}} = v_2 - v_1 \qquad a_{\text{rel}} = a_2 - a_1$$

$$x' = x - v_{\text{frame}} t \qquad v' = v - v_{\text{frame}} \qquad a' = a$$

## 9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\text{avg}} = \frac{\Delta \vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\text{avg}} = \frac{\Delta \vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

## 10 Projectile Motion

$$x = v_{x,0} t \qquad y = -\frac{1}{2} g t^2 + v_{y,0} t + y_0 \qquad v_{x,0} = v_0 \cos\theta \qquad v_{y,0} = v_0 \sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0 \cos\theta} \qquad y = y_0 + x \tan\theta - \frac{x^2 g}{2 v_0^2 \cos^2\theta}$$

$$x_{\text{for } y \text{ max}} = \frac{v_0^2 \sin\theta \cos\theta}{g} \qquad y_{\text{max}} = y_0 + \frac{v_0^2 \sin^2\theta}{2g}$$

$$x(y = y_0) = \frac{2 v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin(2\theta)}{g} \qquad \theta_{\text{for max}} = \frac{\pi}{4} \qquad x_{\text{max}}(y = y_0) = \frac{v_0^2}{g}$$

$$x(\theta = 0) = \pm v_0 \sqrt{\frac{2(y_0 - y)}{g}} \qquad t(\theta = 0) = \sqrt{\frac{2(y_0 - y)}{g}}$$

## 11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

## 12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2 r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad \vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2 \qquad a_{\text{tan}} = r\alpha$$