Intro Physics Semester I **Name:**

Homework 4: Multi-Dimensional Kinematics: One or two full answer questions will be marked. There will also be a mark for completeness. Homeworks are due usually the day after the chapter they are for is finished. Solutions will be posted soon thereafter. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

Answer Table

Name:

	a	b	c	d	e		a	b	c	d	e
1.	O	O	O	O	O	31.	O	O	O	O	O
2.	O	O	O	O	O	32.	O	O	O	O	O
3.	O	O	O	O	O	33.	O	O	O	O	O
4.	O	O	O	O	O	34.	O	O	O	O	O
5.	O	O	O	O	O	35.	O	O	O	O	O
6.	O	O	O	O	O	36.	O	O	O	O	O
7.	O	O	O	O	O	37.	O	O	O	O	O
8.	O	O	O	O	O	38.	O	O	O	O	O
9.	O	O	O	O	O	39.	O	O	O	O	O
10.	O	O	O	O	O	40.	O	O	O	O	O
11.	O	O	O	O	O	41.	O	O	O	O	O
12.	O	O	O	O	O	42.	O	O	O	O	O
13.	O	O	O	O	O	43.	O	O	O	O	O
14.	O	O	O	O	O	44.	O	O	O	O	O
15.	O	O	O	O	O	45.	O	O	O	O	O
16.	O	O	O	O	O	46.	O	O	O	O	O
17.	O	O	O	O	O	47.	O	O	O	O	O
18.	O	O	O	O	O	48.	O	O	O	O	O
19.	O	O	O	O	O	49.	O	O	O	O	O
20.	O	O	O	O	O	50.	O	O	O	O	O
21.	O	O	O	O	O	51.	O	O	O	O	O
22.	O	O	O	O	O	52.	O	O	O	O	O
23.	O	O	O	O	O	53.	O	O	O	O	O
24.	O	O	O	O	O	54.	O	O	O	O	O
25.	O	O	O	O	O	55.	O	O	O	O	O
26.	O	O	O	O	O	56.	O	O	O	O	O
27.	O	O	O	O	O	57.	O	O	O	O	O
28.	O	O	O	O	O	58.	O	O	O	O	O
29.	O	O	O	O	O	59.	O	O	O	O	O
30.	O	O	O	O	O	60.	O	O	O	O	O

1. "Let's play *Jeopardy*! For \$100, the answer is: The vectors

$$\vec{r}\ , \qquad \vec{v} = \frac{d\vec{r}}{dt}\ , \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}\ ."$$

   What are the most obvious quantities of __________, Alex?

   a) time b) light c) dynamics d) multi-dimensional kinematics e) rest

2. In Cartesian coordinates, kinematics is very simple if $x(t)$, $y(t)$, and $z(t)$ are specified because the unit vectors of Cartesian coordinates are:

   a) variables. b) dependent on time. c) constants. d) dependent on position.
   e) fewer than two.

3. The unit vectors of polar and spherical polar coordinates are dependent on:

   a) the angular coordinates of the displacement vector.
   b) the magnitude of the displacement vector. c) nothing. d) time explicitly. e) mass.

4. In Newtonian physics, the component motions of a particle in __________ directions are independent of each other. This means that motion variables in one direction are **NOT** intrinsic functions of motion variables in __________ directions. For example, a particular velocity in the x direction does **NOT** intrinsically set the velocity in the y direction. Now the initial or continuing conditions will set up relationships between motions in __________ directions. But those conditions can be anything allowed by physics and relationships between the motions can be anything allowed by the conditions and physics. For example, the conditions could set the x direction velocity to be a constant $v_x = 1\,\mathrm{m/s}$ and the y direction velocity to a constant $v_y = 2\,\mathrm{m/s}$. Then one has, of course, that $v_y = 2v_x$, but the physically allowed conditions could have set the two velocity components to any constant values or anything else allowed by physics. There is **NO** intrinsic relationship between those velocity components. The independence of motions in __________ directions is one of the amazing facts about the physical world and a vast simplification in dealing with it.

   a) the same. b) opposite. c) both negative d) orthogonal e) dependent

5. "Let's play *Jeopardy*! For \$100, the answer is: Without qualifications, one usually means the non-powered flight of an object in the air or through space. The simplest in-air case is the one in which air drag is neglected. The science of such motions is ballistics.

   What is __________, Alex?

   a) apparent motion b) projectile motion c) 1-dimensional motion
   d) trigonometric motion e) unstoppable motion

6. A ball is tossed into the air and falls to the ground some distance away. Consider its motion in the vertical direction only and neglect air drag.

   a) The ball has a constant acceleration downward.
   b) The ball first accelerates **UPWARD** on its rising path and then accelerates **DOWNWARD** on its falling path.
   c) The ball first accelerates **DOWNWARD** on its rising path and then accelerates **UPWARD** on its falling path.
   d) The ball does not accelerate at all.
   e) The ball is always accelerating in the upward direction.

7. Which best describes the path of a ball thrown on level ground at an angle 30° above the horizontal as seen from a side view.

   a) Two straight lines that meet at an apex: one for the rising phase; one the declining phase. The rising phase line is **TWICE** the length of the declining phase line.
   b) A smooth curve that rises and falls with distance. As far as the eye can tell, the curve could be parabolic.
   c) Two straight lines that meet at an apex: one for the rising phase; one the declining phase. The rising phase line is **HALF** the length of the declining phase line.
   d) A smooth curve that rises and falls with distance, but suddenly breaks off and descends vertically.

e) A smooth curve that rises and falls with distance and then rises and falls again with distance. A Bactrian camel curve.

8. The horizontal range formula for projectile motion near the Earth's surface and neglecting air drag is:

a) $x_{\text{max}} = \dfrac{8v_0^2}{g}\sin(2\theta)$. b) $x_{\text{max}} = \dfrac{v_0^2}{g}\cos(2\theta)$. c) $x_{\text{max}} = \dfrac{v_0^2}{g}$. d) $x_{\text{max}} = \dfrac{v_0^2}{g}\sin(2\theta)$.
e) $x_{\text{max}} = \dfrac{4v_0^2}{g}\sin(2\theta)$.

9. The maximum horizontal range value for projectile motion near the Earth's surface for a given launch speed v_0 and neglecting air drag is obtained for launch angle:

a) 30°. b) 45°. c) 60°. d) 82.245°. e) 90°.

10. The ratio of the maximum height formula (for projectile motion near the Earth's surface, neglecting air drag, and measuring from the launch height) to the horizontal range formula (for projectile motion near the Earth's surface and neglecting air drag) is ____________ and its value for the maximum range is ________.

a) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \dfrac{1}{4}\tan\theta$; $\frac{1}{4}$ b) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \dfrac{1}{4}\tan\theta$; 1 c) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \tan\theta$; 1 d) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \tan\theta$; $\frac{1}{4}$
e) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \dfrac{1}{4}\tan\theta$; ∞

11. "Let's play *Jeopardy*! For \$100, the answer is: Motion of something with respect to something else. To be a bit more explicit, say you have two objects. The relative displacement of object 2 from object 1 is defined to be

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \ .$$

From this definition, the relative velocity and acceleration follow from differentiation:

$$\vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \text{and} \qquad \vec{a} = \vec{a}_2 - \vec{a}_1 \ .$$

The description of the motion in these terms clarifies the initial answer statement."

What is ____________, Alex?

a) variable motion b) relativistic motion c) no motion d) abosulte motion
e) relative motion

12. You are in a featureless narrow room playing catch with a friend. How can you tell if the room is in a building or is a sealed compartment on super-smoothly running, non-accelerating train (or plane)? **HINT:** Review your whole life experience; try an experiment (but not while you are driving).

a) When you throw the ball **ALONG** the long axis of the room, it would have different speeds (relative to the room) in the two possible directions if you were on a train.
b) When you thow the ball **PERPENDICULAR** to the long axis of the room it would curve off a straight line (relative to the room) if you were on a train.
c) On a train the thrown ball would zigzag wildly in flight.
d) On a train the thrown ball would do loops in flight.
e) There is no way to tell as long as the train motion is very smooth.

13. You are flying a plane. Air velocity (i.e., plane velocity relative to the air) is 40 mi/h due north. Wind velocity is 30 mi/h due west. What is the magnitude of ground velocity (i.e., the ground speed)?

a) 50 mi/h. b) −50 mi/h. c) 40 mi/h. d) 10 mi/h. e) 2500 mi/h.

14. A circle can be divided into:

a) 360 divisions only. b) any number of divisions you like. c) 2π divisions only.
d) π divisions only. e) 360 or 2π divisions only.

15. How many radians are there in a circle?

a) π. b) 2π. c) 3π. d) 360°. e) 360.

16. The division of the circle into 360° was an arbitrary choice—and we don't know why. We just know the ancient Mesopotamian mathematicians and astronomers did it this way—you know Mesopotamia—ancient Iraq: "the cradle of civilization". Their choice was just adopted by the ancient Greeks and got passed on to us. In the French Revolutionary epoch, the decimal system was adopted for most measures, but the revolutionaries didn't get around (you might say) to the circle. We can guess that one reasons is that the ancient Mesopotamians had a preference for whole number arithmetic particularly in division and 360 has a lot of whole number factors. How many whole number (i.e., integer) factors does 360 have counting 1 and 360 itself?

    a) 24. b) 360. c) 6. d) 7. e) 12.

17. What is the approximate conversion factor from radians to degrees?

    a) 1/60 degrees/radian. b) π degrees/radian. c) 2π degrees/radian.
    d) 60 degrees/radian. e) 360 degrees/radian.

18. There are 2π radians in a circle. Its rather inconvenient that this means that there are $2\pi = 6.2831853\ldots$ radians in a circle which is an irrational number. For convenience we might invent a new base unit: the 2π with symbol "ti" (for **T**wo p**I** and pronounced tie). One hundredth of a ti would be a:

    a) exati. b) megati. c) kiloti. d) deciti. e) centiti.

19. Approximately, at arm's length a finger subtends 1°, a fist 10°, and a spread hand 18°. These numbers, of course, vary a bit depending on person and exactly how the operation is done. What are these angles approximately in radians?

    a) 1/60, 1/6, and 1/3 radians. b) 60, 600, and 1800 radians.
    c) $\pi/12$, $\pi/3$, and $\pi/2$ radians. d) $\pi/12$, $\pi/3$, and π radians. e) $\pi/12$, $\pi/3$, and 2π radians.

20. Can you cover the Moon with your finger held at arm's length? **HINT:** You could try for yourself if you are not in a a test *mise en scène*.

    a) No. The Moon is much larger in angle than a finger. Just think how huge the Moon looks on the horizon sometimes.
    b) It depends critically on the size of one's finger and arm. People with huge hands can to it and those without can't.
    c) Yes. A finger at arm's length typically subtends about 10° and the Moon subtends 0.01°.
    d) No. The Moon's diameter is about 3470 km and a finger is about a centimeter or so in width.
    e) Usually yes. A finger at arm's length typically subtends about 1° and the Moon subtends 0.5°.

21. For small angles θ measured in radians and with increasing accuracy as θ goes to zero (where the formulas are in fact exact), one has the small angle approximations:

    a) $\sin\theta \approx \cos\theta \approx 1 - \frac{1}{2}\theta^2$. b) $\cos\theta \approx \tan\theta \approx 1 - \frac{1}{2}\theta^2$. c) $\sin\theta \approx \cos\theta \approx \theta$.
    d) $\cos\theta \approx \tan\theta \approx \theta$. e) $\sin\theta \approx \tan\theta \approx \theta$.

22. In 2-dimensional Cartesian coordinates, a displacement vector $\vec{r}$ is given by

$$\vec{r} = (x, y) = x\hat{x} + y\hat{y} \ ,$$

where the unit vectors $\hat{x}$ and $\hat{y}$ are constants. In polar coordinates,

$$\vec{r} = (r, \theta) = r\hat{r} \ ,$$

where the unit vector

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \ .$$

The polar coordinates are obtained from the Cartesian ones by the formulae

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \ .$$

In calculational work one must be aware that a negative argument of $\tan^{-1}$ is treated by calculators and computers as implying that $y > 0$ and $x < 0$. If the reverse is true, one must explicitly add or subtract

180° from the calculated result. The Cartesian components are obtained from the polar coordinates by the formulae

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta \ .$$

As well as $\hat{r}$, one needs another unit vector for polar coordinates that is perpendicular to the $\hat{r}$ and that is used for velocity and acceleration vectors and changes in the displacement vector. This is the unit vector ____________ given by

$$\underline{\hspace{4em}} = \vec{r}(\theta + 90^\circ) = -\sin\theta\hat{x} + \cos\theta\hat{y} \ ,$$

a) $\hat{\alpha}$ b) $\hat{\omega}$ c) $\hat{n}$ d) $\hat{z}$ e) $\hat{\theta}$

23. "Let's play *Jeopardy*! For \$100, the answer is: The formulae

$$\vec{r} = r\hat{r} \ ,$$
$$\vec{v} = \frac{dr}{dt}\hat{r} + \frac{d\theta}{dt}\hat{\theta} \ ,$$
$$\vec{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\hat{\theta} \ ,$$

where r is magnitude of displacement from the origin (or radial component of the displacement $\vec{r}$), $\hat{r}$ is the unit vector of the radial component, θ the angular component of $\vec{r}$, $\hat{\theta}$ is the unit vector of the angular coordinate, and one often writes $d\theta/dt$ is as ω which is called the angular velocity. The unit vectors are functions of the angular component:

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \qquad \text{and} \qquad \hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y} \ .$$

What are displacement, velocity, acceleration in ____________ coordinates, Alex?

a) polar b) Cartesian c) spherical polar d) elliptical e) hyperbolical

24. "Let's play *Jeopardy*! For \$100, the answer is: It is the acceleration in a case of circular motion."

What is ____________, Alex?

a) net acceleration b) centrifugal (center-fleeing) acceleration
c) centripetal (center-pointing) acceleration d) deceleration e) zero

25. The radial component of acceleration in polar coordinates

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2r}{dt^2} - r\omega^2$$

specializes to ____________ if the motion is circular and centered on the origin. In this case, the radial component of acceleration is called centripetal (meaning center pointing) since, in fact, it is always negative (i.e., the radial component of acceleration always points toward the origin). The radial component of velocity for circular motion is zero naturally and the angular component, often called the tangential velocity is given by

$$v_\theta = r\omega \ .$$

Usually, one drops the subscripts r and θ on a_r and v_θ if the quantities are identified by context.

a) $a_r = -r\omega = -\frac{v_\theta}{r}$ b) $a_r = r\omega^2 = \frac{v_\theta^2}{r}$ c) $a_r = -r\omega^2 = -\frac{v_\theta^2}{r}$ d) $a_r = \omega^2 = v_\theta^2$
e) $a_r = -\omega^2 = -v_\theta^2$

26. The formula for centripetal acceleration magnitude for circular motion is

$$a_{\text{cen}} = \frac{v^2}{r} \ ,$$

where v is the tangential velocity of the motion and r is the radius of the circle. The centripetal acceleration ____________ with v and ____________ with r.

a) increases linearly; decreases inverse linearly b) increases quadratically; decreases inverse linearly c) decreases inverse linearly; increases quadratically d) increases quadratically; decreases inverse quadratically e) constant; decreases inverse linearly

27. In uniform circular motion, position, velocity, and acceleration are continually changing. The velocity and acceleration magnitudes are:

    a) continually changing too. b) continually changing and constant, respectively.
    c) constant and continually changing, respectively. d) constant and undefined, respectively.
    e) both constant.

28. The formula for a general quadratic is

$$f(x) = ax^2 + bx + c \ .$$

The shape of a quadratic is actually a simple parabola with the vertical symmetry axis offset from the origin.

a) Show this by completing the square in the function

$$f(x) = ax^2 + bx + c$$

and find the x coordinate of the symmetry axis. The completed square form is called the vertex form.

b) What is the formula for a transformed coordinate x' that explicitly turns the quadratic function into a simple parabola?

29. A projectile is launched from x-y origin which is on the ground level of large level plain. The x direction is the horizontal and the y direction is the vertical: upward is the positive y direction. The projectile is launched in the positive x direction. The initial launch speed is v_0 at angle θ above the horizontal. Air drag is neglected.

a) Find the expressions for x and y position as functions of time t entirely in symbols and with any dependences on θ shown explicitly. Drop any symbols that stand for known zeros.

b) Now find y as a function of x by eliminating t. Now find the horizontal range formula: i.e., the expression for x when the projectile returns to $y = 0$. Simplify the formula as much as is reasonably possible. **HINT:** The trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$ helps simplifying formula.

c) Using the horizontal range formula, find by any means the angle θ a maximum range holding all the other variables constant. Briefly explain how you arrived at your answer. **HINT:** The sine function has only one maximum in the domain of its argument $[0°, 180°]$.

30. The women's volleyball court has a net height of 2.24 m and extends 9.0 m on either side of the net. On a jump serve, a player spikes (I think that's the word) the ball at 3.00 m above the court in a direction perpendicular to the net. The initial velocity is **HORIZONTAL** and the net is 8.0 m away from the server. Neglect air drag.

This is a problem in which you want to find the conditions that lead to the desired result. It's really a pretty common kind of problem—in life as well as in physics.

a) First sketch the system: launch, court, net. Then sketch a general trajectory that lands before the net and one that lands on the other side of then (i.e., that clears the net).

b) First, solve in **SYMBOLS** for initial velocity v_0 as a function of **ONLY** the variables x (horizontal position measured from the launch point) and y (height), and the constants y_0 (launch height) and g. The time variable t should be eliminated. Essentially, for a general point (x, y) on a general trajectory (that had a horizontal launch velocity recall) you get the initial velocity v_0 which will get you there.

Sketch a graph of v_0 as a function of y for constant x: indicate on sketch the v_0 for $y = 0$ in symbols and the location of the maximum of v_0. What is the significance of this maximum? Then sketch v_0 as a function of x for constant y.

c) What is the minimum velocity v_0 needed for the ball to clear the net? Assume the ball is a point mass for this part.

d) What is the maximum velocity v_0 allowed if the ball is to stay in court? Assume no one touches the ball and assume the ball is a point mass for this part.

31. You are on the last run of the historic Orient Express train in 1939 traveling from Paris to Istanbul. Somewhere between Belgrade and Sofia, the train ominously starts accelerating in the reverse direction—which we will call the negative x direction. The **MAGNITUDE** of this acceleration is a_{tr}. The train is on a **STRAIGHT**, level line of track. Neglect **AIR DRAG**.

    a) Inside the train car a projectile is launched in the positive x direction (i.e., in the forward direction). What is the horizontal acceleration **RELATIVE** to the train car in terms of knowns? What is the vertical acceleration **RELATIVE** to the train car in terms of knowns? Take the **UP** direction as the positive y direction. **HINT:** No elaborate calculations are needed. We are just looking for simple, short, symbol answers. Remember the formula for **RELATIVE ACCELERATION** of an object 2 with respect to object 1:

    $$\vec{a}_{\text{rel21}} = \vec{a}_2 - \vec{a}_1 \ .$$

    b) You and your mysterious *compagnon de voyage* M. Achille find yourselves locked in your car—and attempt some two-dimensional kinematics. What are the x and y positions relative to the car as functions of time t for the projectile launched from the origin (which is fixed to the car) at time zero with launch speed v_0 (relative to the car) and at an angle θ (in the range 0° to 90° and also relative to the car) to the positive x axis? Express these positions using the accelerations found in part (a), time t, v_0 and θ.

    c) For reasons known to himself alone, M. Achille insists that you find the horizontal range formula for the projectile in the reference frame of the car: i.e., a formula giving the horizontal range (the x displacement from launch height to launch height) in terms of variables **NOT** including time t. Do so and simplify it as much as reasonably possible.

    d) What is the horizontal range formula for the car in the case that a_{tr} goes to zero?

32. You have decided to swim the English channel from Gris-Nez to Folkestone: a distance of about 40 km roughly to the north-west. In this problem, treat the distances and directions as exact: e.g., 40 km is the exact distance from Gris-Nez to Folkestone north-west is exactly 45° west of north, and south-west is exactly 135° west of north. Also treat the Earth as flat. **HINT:** Smear your body all over with fats to prevent hypothermia.

    a) If you swim at 4.0 km/h, how long will it take you?

    b) As it turns out you land at New Romney about 20 km south-west of Folkestone. What is the average channel current velocity during your swim?

    c) What was your average velocity (note velocity, not just speed) during the swim relative to the fixed Earth? As noted in the preamble, assume that Folkestone is exactly north-west of Gris-Nez and New Romney is exactly south-west of Folkestone. Give the velocity in magnitude-direction format. **HINT:** Velocity requires a direction specification too.

33. Deep in the Amazonian jungle you wish to cross a river livid with piranha and crocodiles. Let x be the coordinate **ALONG** the river and y the coordinate **PERPENDICULAR** to the river. The river width is $y_{\text{max}} = 20\,\text{m}$. You are going to paddle across (on an unstable rotting log—using your bare hands) and your paddling speed is $v_{\text{paddle}} = 0.50\,\text{m/s}$.

    a) How long does it take you to cross assuming that you (wisely) aim straight in the y-direction?

    b) Assuming the river flows at a steady velocity $v_x = 0.20\,\text{m/s}$ how far downstream (i.e., in the x direction) have you gone while crossing?

    c) What is your velocity relative to the shore during the crossing? Give this velocity in magnitude-direction format. **HINT:** Remember to specify direction.

    d) Now let's make the not-too-likely assumption that the river velocity in the x-direction is a linear function of y, the distance in the y-direction. Let

    $$v_x(y) = v_{x\,\text{max}} \frac{y}{y_{\text{max}}} \ ,$$

where $y = 0$ at the starting shore and $v_{x\,\mathrm{max}} = 0.8\,\mathrm{m/s}$. Given that everything, except the river velocity, is the same as in the earlier parts of the question, how far downstream (i.e., in the x direction) do you move?

34. Let's do a general treatment—like it or not—of circular motion in polar coordinates.

a) The radial and angular unit vectors of polar coordinates are, respectively,

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \qquad \text{and} \qquad \hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y} \ ,$$

where the Cartesian unit vectors $\hat{x}$ and $\hat{y}$ are constants and θ is angular position. Note that $\hat{r}$ is the unit vector that points in the θ direction. Both polar coordinate unit vectors are position dependent, unlike the Cartesian unit vectors $\hat{x}$ and $\hat{y}$. Show $\hat{\theta}$ points $\pi/2$ counterclockwise from $\hat{r}$. **HINT:** Evaluate $\hat{r}$ for $\theta + \pi/2$ (i.e., evaluate the unit vector that points in the $\theta + \pi/2$ direction) and make use of standard trig identities

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \qquad \text{and} \qquad \sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \ .$$

b) Differentiate $\hat{r}$ and $\hat{\theta}$ with respect to time and write the results in terms of $\hat{r}$ and $\hat{\theta}$ and the angular velocity $\omega = d\theta/dt$. **HINT:** Use the chain rule.

c) The position vector for circular motion in polar coordinates is

$$\vec{r} = r\hat{r} \ ,$$

where r is a constant (since the motion is cirular). Derive the velocity and acceleration formulae making use of the results of the part (b) answer. Note that angular acceleration is given the symbol α: thus,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \ .$$

d) Write down the formulae for the tangential and radial components of velocity and acceleration. The radial component of the acceleration is called the centripetal acceleration which means center seeking acceleration since for circular motion it always points toward the center (i.e., is always negative). Write the centripetal acceleration in terms of ω and tangential velocity component.

e) The position vector for in polar coordinates in general (i.e., for varying direction and magnitude) is

$$\vec{r} = r\hat{r} \ ,$$

Derive the general velocity and acceleration formulae making use of the results of the part (b) answer and using the symbols for angular velocity (ω) and angular acceleration (α) for simplification.

35. Consider a TGV (Train à Grande Vitesse).

a) Say it has a speed of $216\,\mathrm{km/h}$ and is moving in a circle. What is the minimum radius of the circle if the centripetal acceleration is to be kept less than $0.05g$? Explain why it must be the minimum radius. **HINT:** Schematically plot centripetal acceleration versus r.

b) If the radius is $1.00\,\mathrm{km}$, what is the maximum speed (in m/s and km/hr) before the acceleration exceeds $0.05g$? Explain why it is the maximum speed. **HINT:** Schematically plot centripetal acceleration versus v.

NOTE: Here we use $g = 9.8\,\mathrm{m/s^2}$ as a unit of acceleration. An accelerated frame is exactly like being gravitational field so far a mechanical experiments internal to the frame are concerned. This is actually a profound observation that started Einstein on his way to his general theory of relativity. The pseudo-gravitational force points opposite to the acceleration and has magnitude ma. This pseudo or inertial force is sometimes called the g-force and is specified as a force per unit mass in units of g. Thus circling with acceleration $0.05g$ is just like being in gravitational field of $0.05g$ where the gravitational force points radially outward.

Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

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1 Constants

$$c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 2.998 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \qquad \text{exact by definition}$$

$$e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$G = 6.67428(67) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(2006, CODATA)}$$

$$g = 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 8.987551787\ldots \times 10^9 \approx 8.99 \times 10^9 \approx 10^{10}\,\mathrm{N\,m^2/C^2}\text{exact by definition}$$

$$k_{\mathrm{Boltzmann}} = 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} = 0.8617343(15) \times 10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,\mathrm{eV/K}$$

$$m_e = 9.10938215(45) \times 10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV}$$

$$m_p = 1.672621637(83) \times 10^{-27}\,\mathrm{kg} = 938.272013(23), \mathrm{MeV}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = 8.8541878176\ldots \times 10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11} \qquad \text{vacuum permittivity (exact by definition)}$$

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad \text{exact by definition}$$

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2 Geometrical Formulae

$$C_{\mathrm{cir}} = 2\pi r \qquad A_{\mathrm{cir}} = \pi r^2 \qquad A_{\mathrm{sph}} = 4\pi r^2 \qquad V_{\mathrm{sph}} = \frac{4}{3}\pi r^3$$

$$\Omega_{\mathrm{sphere}} = 4\pi \qquad d\Omega = \sin\theta\,d\theta\,d\phi$$

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3 Trigonometry Formulae

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}$$

$$f(\theta) = f(\theta + 360^\circ)$$

$$\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin(\theta+90^\circ)=\cos(\theta) \qquad \cos(\theta+90^\circ)=-\sin(\theta) \qquad \tan(\theta+90^\circ)=-\tan(\theta)$$

$$\sin(180^\circ-\theta)=\sin(\theta) \qquad \cos(180^\circ-\theta)=-\cos(\theta) \qquad \tan(180^\circ-\theta)=-\tan(\theta)$$

$$\sin(90^\circ-\theta)=\cos(\theta) \qquad \cos(90^\circ-\theta)=\sin(\theta) \qquad \tan(90^\circ-\theta)=\frac{1}{\tan(\theta)}=\cot(\theta)$$

$$\sin(a+b)=\sin(a)\cos(b)+\cos(a)\sin(b) \qquad \cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b)$$

$$\sin(2a)=2\sin(a)\cos(a) \qquad \cos(2a)=\cos^2(a)-\sin^2(a)$$

$$\sin(a)\sin(b)=\frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right] \qquad \cos(a)\cos(b)=\frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right]$$

$$\sin(a)\cos(b)=\frac{1}{2}\left[\sin(a-b)+\sin(a+b)\right]$$

$$\sin^2\theta=\frac{1}{2}[1-\cos(2\theta)] \qquad \cos^2\theta=\frac{1}{2}[1+\cos(2\theta)] \qquad \sin(a)\cos(a)=\frac{1}{2}\sin(2a)$$

$$\cos(x)-\cos(y)=-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos(x)+\cos(y)=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x)+\sin(y)=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

4 Approximation Formulae

$$\frac{\Delta f}{\Delta x}\approx\frac{df}{dx} \qquad \frac{1}{1-x}\approx 1+x\ :\ (x<<1)$$

$$\sin\theta\approx\theta \qquad \tan\theta\approx\theta \qquad \cos\theta\approx 1-\frac{1}{2}\theta^2 \qquad \text{all for } \theta<<1$$

5 Quadratic Formula

$$\text{If} \qquad 0=ax^2+bx+c\ , \qquad \text{then} \qquad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=-\frac{b}{2a}\pm\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}}$$

6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

Taylor's series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) \\ &= f(x_0) + (x - x_0) f^{(1)}(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x - x_0)^3}{3!} f^{(3)}(x_0) + \ldots \end{aligned}$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

8 One-Dimensional Kinematics

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2 x}{dt^2}$$

$$v = at + v_0 \qquad x = \frac{1}{2}at^2 + v_0 t + x_0 \qquad v^2 = v_0^2 + 2a(x - x_0)$$

$$x = \frac{1}{2}(v_0 + v)t + x_0 \qquad x = -\frac{1}{2}at^2 + vt + x_0 \qquad g = 9.8\,\mathrm{m/s^2}$$

$$x_{\rm rel} = x_2 - x_1 \qquad v_{\rm rel} = v_2 - v_1 \qquad a_{\rm rel} = a_2 - a_1$$

$$x' = x - v_{\rm frame}t \qquad v' = v - v_{\rm frame} \qquad a' = a$$

9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\rm avg} = \frac{\Delta\vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\rm avg} = \frac{\Delta\vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

10 Projectile Motion

$$x = v_{x,0}t \qquad y = -\frac{1}{2}gt^2 + v_{y,0}t + y_0 \qquad v_{x,0} = v_0\cos\theta \qquad v_{y,0} = v_0\sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0\cos\theta} \qquad y = y_0 + x\tan\theta - \frac{x^2 g}{2v_0^2\cos^2\theta}$$

$$x_{\text{for } y \text{ max}} = \frac{v_0^2\sin\theta\cos\theta}{g} \qquad y_{\rm max} = y_0 + \frac{v_0^2\sin^2\theta}{2g}$$

$$x(y=y_0) = \frac{2v_0^2\sin\theta\cos\theta}{g} = \frac{v_0^2\sin(2\theta)}{g} \qquad \theta_{\text{for max}} = \frac{\pi}{4} \qquad x_{\rm max}(y=y_0) = \frac{v_0^2}{g}$$

$$x(\theta=0) = \pm v_0\sqrt{\frac{2(y_0-y)}{g}} \qquad t(\theta=0) = \sqrt{\frac{2(y_0-y)}{g}}$$

11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad \vec{a}_{\rm centripetal} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\rm centripetal} = \frac{v^2}{r} = r\omega^2 \qquad a_{\rm tan} = r\alpha$$