Cosmology & Galaxies **NAME:**

Homework 24: Cosmic Present Star-Forming Galaxies (SFGs)

---

024 qmult 00180 1 1 3 easy memory: Milk Way stellar mass and virial mass

1. In solar mass units, the Milky Way stellar mass (i.e., the mass in stars M_*) is $\sim$ ____________ $M_\odot$ and its virial mass ($M_{\rm vir}$: i.e., fiducial total mass which is mostly dark matter) is $\sim$ ____________ $M_\odot$. At least these values were standard circa 2023. However, a downward revision may have become accepted just about that year.

   a) 10^{12}; 10^{10} b) 5×10^{10}; 5×10^{10} c) 5×10^{10}; 10^{12} d) 10^9; 5×10^{10}
   e) 10^9; 5×10^8

   SUGGESTED ANSWER: (c) See Ci-55. See Ou et al. (2023, arXiv:2303.12838) for a downward revision of the virial mass to $\sim 2\times10^{11}\,M_\odot$

   Wrong answers:
   a) As Lurch would say AAAarrgh.

   Redaction: Jeffery, 2008jan01

---

024 qmult 00230 1 4 2 easy deducto-memory: exponential profile for face-on spiral galaxies

2. "Let's play *Jeopardy*! For \$100, the answer is:

$$I_\lambda = I_{\lambda,0} e^{-(R/R_{\rm d})} ,$$

   where I_λ is the surface brightness, $I_{\lambda,0}$ is the central surface brightness, R is the radius coordinate, and $R_{\rm d}$ disk scale length (and not the effective or half-light radius)."

   What is the standard ____________ surface brightness profile, Alex?

   a) edge-on spiral disc b) face-on spiral disc c) elliptical d) dwarf irregular
   e) general Sérsic

   SUGGESTED ANSWER: (b)

   Wrong answers:
   a) As Lurch would say AAAARGH.

   Redaction: Jeffery, 2008jan01

---

024 qmult 00600 1 1 4 easy memory: two main classes of galaxy bulges

3. There are two main classes of galaxy bulges:

   a) classical bulges and non-classical bulges b) big bulges and disc-like bulges
   c) little bulges and disc-like bulges d) classical bulges and disc-like bulges
   e) little bulges and big bulges

   SUGGESTED ANSWER: (d)

   Wrong answers:
   a) Seems reasonable.

   Redaction: Jeffery, 2008jan01

---

024 qmult 00620 1 4 2 easy deducto-memory: Schmidt-Kennicutt law

4. "Let's play *Jeopardy*! For \$100, the answer is:

$$\Sigma_{\rm SFR} = B\left(\frac{\Sigma_{\rm gas}}{1\,M_\odot/{\rm pc}^2}\right)^\alpha M_\odot/{\rm yr}/{\rm kpc}^2 ,$$

   where SFR means star formation rate, $\Sigma_{\rm SFR}$ is surface star formation rate in units of $M_\odot/{\rm yr}/{\rm kpc}^2$, $\Sigma_{\rm gas}$ is gas surface density in units $M_\odot/{\rm pc}^2$ (the denominator below $\Sigma_{\rm gas}$ makes the overall factor dimensionless), $B\approx10^{-4}$ is an empirical constant, and $\alpha = 1.40(15)$ is another empirical constant with some theoretical understanding.

What is the ____________, Alex?

a) Press-Kennicutt law b) Schmidt-Kennicutt law c) Press-Schechter law
d) Martin-Schmidt law e) Martin-Schmidt-Kennicutt law

SUGGESTED ANSWER: (b) See Ci-83.

Wrong answers:
c) As Lurch would say AAAARGH.

Redaction: Jeffery, 2008jan01

---

024 qfull 00100 1 3 0 easy math: inclined circle analyzed: On exams, do all parts with minimal words.

5. An inclined circle (ideal disc galaxy is) is seen in projection as an ellipse.

NOTE: There are parts a,b. On exams, do all parts with minimal words.

a) The equation for a circle is written elaborately is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1 \ ,$$

where a is the radius. Find the explicit formula for y. The circle is rotated on its x-axis to inclination angle i where inclination angle is measured from the direction to the observer to a normal to the circle. What is the projected height of every y point (i.e., what is the inclined y_i)? Prove the inclined circle (i.e., projected circle) is an ellipse and find its semi-minor axis b.

b) The area of an ellipse is $A = \pi ab$ and the circularized radius an ellipse created by inclination is defined by

$$R_i = \sqrt{ab} = a\sqrt{\cos(i)} \ .$$

Prove that the differential area of an inclined circle is

$$dA = 2\pi R_i \, dR_i \ .$$

SUGGESTED ANSWER:

a) Behold:

$$y = \pm a\sqrt{1 - \left(\frac{x}{a}\right)^2}$$

By inspection of a diagram you will have to imagine, every inclined y_i is given by $y_i = y\cos(i)$. If you multiple the circle formula for y by $\cos(i)$, you get

$$y_i = \pm a\cos(i)\sqrt{1 - \left(\frac{x}{a}\right)^2} = \pm b\sqrt{1 - \left(\frac{x}{a}\right)^2} \ ,$$

where we have defined $b = a_i \cos(i)$. The new formula identified by inspection as that of an ellipse where y_i is the y coordinate and b is the semi-minor axis. Thus, the inclined circle is an ellipse with semi-minor axis b

b) Behold:

$$dA = \pi \, d(ab) = d\left[a^2\cos(i)\right] = 2\pi a \, da \cos(i) = 2\pi a\sqrt{\cos(i)} \, da\sqrt{\cos(i)} = 2\pi R_i \, dR_i \qquad \text{QED.}$$

Redaction: Jeffery, 2018jan01

---

028 qfull 00350 1 3 0 easy math: free-fall time and collapse to star time: On exams, only do parts a,b,c.

6. The free-fall time for a straight line fall of a particle of mass m starting from rest to a point source or spherically symmetric source of mass M (always interior to the infalling particle) is

$$t_{\rm ff} = \frac{t_{\rm orbit}}{2} = \frac{\pi}{\sqrt{G(M+m)}}\left(\frac{r}{2}\right)^{3/2} \ ,$$

where $t_{\rm orbit}$ is the orbital period predicted by the Newtonian physics version of Kepler's 3rd law and r is the initial distance from the particle to the source center and is twice the relative semi-major axis of an elliptical orbit of the particle to the source (Wikipedia: Free-fall time; Wikipedia: Kepler's laws of planetary motion Third law; Ci-246). The Kepler's 3rd law orbital period is independent of eccentricity $e \leq 1$, and so half of it is the free-fall time.

NOTE: There are parts a,b,c,d,e,f. On exams, only do parts a,b,c.

a) What is the free-fall time for test particle (i.e., one of negligible mass)?

b) What is the free-fall time as a function of r for a spherical mass distribution with initially constant density ρ and outer radius $r \leq R$. The matter is initiall all at rest and there is zero pressure at all times. Assume the (infinitely thin) shells of matter in the distribution at all the r values never cross during free fall which is true and plausible, but seems tricky to prove. Describe the order of arrival of the shells at the center?

HINT: Remember the shell theorem

$$\vec{g} = -\frac{GM(r)}{r^2}\hat{r}$$

where the mass distribution is sperically symmetric and $M(r)$ is the interior mass to radius r. Note $M(r)$ must increase monotonically since there is no negative mass, but it can be zero out some radius r.

NOTE: For all subsequent parts, we assume a spherically symmetric mass distribution at all times with initial outer radius R and there is zero pressure at all times.

c) Say that the interior mass $M(r)$ to radius r obeys a power law $M(r) = M_0(r/r_0)^\alpha$ where $\alpha \leq 3$. When does the mass all collapse to the center assuming that it magically all stops there on arrival and the shells of matter at all the r values never cross during free fall which is true and plausible, but seems tricky to prove.

d) For star formation, we want to relate density ρ to the particle density n which can be measured more directly. The relating formula is

$$n = \rho\left(\sum_i \frac{X_i}{A_i m_{\rm p}}\right) ,$$

where X_i is the mass fraction of species i (which could be any atom or a molecule including those that are distinct due to their isotopic nature), A_i is the atomic mass number (which could be a molecular mass number), and $m_{\rm p} = 1.67262192369(51) \times 10^{-24}\,$g is the proton mass. Note this special case atomic mass number is in units of proton masses, not daltons (symbol u or Da and AKA atomic mass units). The fact is most of the universe is made of hydrogen (which made of protons) and not made of daltonium (which is made of daltons). Worrying about corrections due to electron masses, binding energies, and isotopes abundances (which aside from hydrogen and helium are rather uncertain) is below the level of accuracy of this problem. The mean atomic mass is defined by

$$\mu^{-1} = \sum_i \frac{X_i}{A_i}$$

which gives

$$n = \frac{\rho}{\mu m_{\rm p}} \qquad \text{or} \qquad \rho = n\mu m_{\rm p} \ .$$

Fiducial cosmic values for X_i are: $X = 0.73$ for H, $Y = 0.25$ for He, and $Z = 0.02$ for metals. Two fiducial mean atomic masses are given by

$$\mu_{\rm H_1,dominated} = \left(\frac{X}{1} + \frac{Y}{2} + \frac{Z}{30}\right)^{-1} \qquad \text{and} \qquad \mu_{\rm H_2,dominated} = \left(\frac{X}{2} + \frac{Y}{2} + \frac{Z}{30}\right)^{-1} ,$$

where the atomic mass for Z is a rough fiducial average based on the fiducial atomic masses of very abundant metals: i.e., $A_{\rm C,6} = 12$, $A_{\rm O,8} = 16$, $A_{\rm Si,14} = 28$, and $A_{\rm Fe,28} = 56$. Compute the $\mu_{\rm H_1,dominated}$ and $\mu_{\rm H_2,dominated}$ values to 3-digit precision which probably 1 more digit than is significant, but it is useful to know insignificant digits sometimes to check for consistency between different calculations.

HINT: Write a small computer program to do the calculation.

f) The part (b) answer gives a fiducial lower limit for the formation time for a star. It is just a fiducial lower limit since real initial clouds of molecular gas do not have uniform density, are not spherically symmetric, and do not have zero pressure and zero initial kinetic energy. It is just a lower limit since the pressure force and kinetic energy in the molecular cloud resist collapse during the collapse process and delay collapse to a star sized object. However, it is useful to rewrite the part (b) answer in terms fiducial values: particle density $10^3\,\mathrm{cm}^{-3}$, $\mu_{\mathrm{H_2,dominated}}$ from part (e), and Julian years (i.e., 365.25 days). Do the rewrite.

HINT: Write a small computer program to do the calculation.

SUGGESTED ANSWER:

a) Behold:

$$t_{\mathrm{ff}} = \frac{\pi}{\sqrt{GM}}\left(\frac{r}{2}\right)^{3/2} .$$

b) Note a shell always has the same mass interior to it during free fall and that mass always acts as a point mass at the center by the shell theorem. The exterior shells have no affect by the shell theorem and since they never cross first mentioned shell. Thus, the free-fall time for all shells is

$$t_{\mathrm{ff}}(r) = \frac{\pi}{\sqrt{G(4\pi/3)\rho r^3}}\left(\frac{r}{2}\right)^{3/2} = \sqrt{\frac{3\pi}{32G\rho}} .$$

Since t_{ff} is, in fact, independent of r, the shells all arrive simultaneously at the center.

c) In this case,

$$t_{\mathrm{ff}} = \frac{\pi}{\sqrt{GM(r)}}\left(\frac{r}{2}\right)^{3/2} = \frac{\pi}{\sqrt{GM(r)}}\left(\frac{r_0}{2}\right)^{3/2}\left(\frac{r}{r_0}\right)^{3/2} = \frac{\pi}{\sqrt{GM_0}}\left(\frac{r_0}{2}\right)^{3/2}\left(\frac{r}{r_0}\right)^{(3-\alpha)/2} ,$$

where $(3-\alpha) \geq 0$. Assuming the shells never cross, the time when all test particles are at the center is where $r = R$, its maximum value. Thus,

$$t_{\mathrm{ff}} = \frac{\pi}{\sqrt{GM_0}}\left(\frac{r_0}{2}\right)^{3/2}\left(\frac{R}{r_0}\right)^{(3-\alpha)/2} .$$

d) The two fiducial mean atomic masses are

$$\mu_{\mathrm{H_1,dominated}} = \left(\frac{X}{1}+\frac{Y}{2}+\frac{Z}{30}\right)^{-1} = 1.26 \quad \text{and} \quad \mu_{\mathrm{H_2,dominated}} = \left(\frac{X}{2}+\frac{Y}{2}+\frac{Z}{30}\right)^{-1} = 2.34 .$$

f) Behold:

$$t_{\mathrm{ff}}(r) = \sqrt{\frac{3\pi}{32G\rho}} = (1.07\times10^6\,\mathrm{Jyr})\left(\frac{\mu_{\mathrm{H_2,dominated}}}{2.34}\right)^{-1/2}\left(\frac{n_{H_2}}{10^3\,\mathrm{cm}^{-3}}\right)^{-1/2} .$$

Fortran-95 Code

```
print*
print*,'Fiducial mean atomic masses and fiducial free-fall time'
x=0.73_np
xm=1.0_np
xm2=2.0_np
y=0.25_np
ym=4.0_np
z=0.02_np
zm=30.0_np
```

```
         xmu=1.0_np/(x/xm+y/ym+z/zm)
         xmu2=1.0_np/(x/xm2+y/ym+z/zm)
         print*,'xmu,xmu2'
         print*,xmu,xmu2
   !     1.2607690691321706241        2.3355391202802646944
         pi=acos(-1.0_np)
         pi=3.14159265358979323846264338327950288419716939937510_np
   !
!!23456789a123456789b123456789c123456789d123456789e123456789f123456789g12
   !           ! https://en.wikipedia.org/wiki/Pi#Approximate_value_and_digits 51
digits
         gcon=6.67430e-11_np  ! (15)
https://physics.nist.gov/cuu/Constants/Table/allascii.txt mks
         xmp=1.67262192369e-27_np  ! (51)
https://physics.nist.gov/cuu/Constants/Table/allascii.txt mks
         xn=1.e+3_np*(1.e+6_np)   ! from cm**(-3) to m**(-3)
         den=xmu2*xmp*xn
         xjy=365.25_np
         daysec=86400.0_np
         con=1.0_np/(xjy*daysec)
         tfid=sqrt(3.0_np*pi/(32.0_np*gcon*den))*con
         print*,'tfid'
         print*,tfid  ! 1065028.6238804413767
```

Redaction: Jeffery, 2018jan01

---

028 qfull 00360 1 3 0 easy math: Free-fall time and shells crossing: On exams, do all parts.

7. Consider free-falling spherical shells of matter that only interact gravitationally.

   NOTE: There are parts a,b,c. On exams, do all parts, but answer with minimal words.

   a) First we consider a single infinitely thin spherical shell of radius $r_{\rm s}$ and mass m. What is the gravitational field $\vec{g}$ at $r < r_{\rm s}$? What is the gravitational field $\vec{g}$ at $r > r_{\rm s}$? Justify your answers.

   b) What of the gravitational field $\vec{g}$ at $r_{\rm s}$? In one sense, the field is indeterminate since there is a discontinuity in the field r and which value you get depends on the direction you take the limit in. However, a limiting value often depends on the limiting process and some limiting processes are physically realistic and others are not. A physically realistic limit gravitational field at r does exist. The trick is consider tiny cylinder Gaussian surface (see Wikipedia: Gaussian surface) placed on the shell of radius r that extends inward and outward from r and whose top and bottom are parallel to the shell surface. In the small limit, the cylindar straddles an infinite infinitely thin plane of surface mass density $\sigma = m/(4\pi r_{\rm s}^2)$. Determine the gravitational field on the top and bottom of the cylinder just due to the enclosed mass. Then find the gravitational field due the rest of the shell on enclosed mass in the cylinder for all r including $r = r_{\rm s}$. That gravitational field is the gravitational field that can accelerate the enclosed mass treating it as test particle.

   c) Do infalling spherical shells ever cross for any possible mass distribution? Prove your answer. **HINT:** Recall, the free-fall time for a straight line fall of a particle of mass m starting from rest to a point source or spherically symmetric source of mass M (always interior to the infalling particle) is

$$t_{\rm ff} = \frac{t_{\rm orbit}}{2} = \frac{\pi}{\sqrt{G(M+m)}}\left(\frac{r}{2}\right)^{3/2} ,$$

   where $t_{\rm orbit}$ is the orbital period predicted by the Newtonian physics version of Kepler's 3rd law and r is the initial distance from the particle to the source center and is twice the relative semi-major axis of an elliptical orbit of the particle to the source (Wikipedia: Free-fall time; Wikipedia: Kepler's laws of planetary motion Third law; Ci-246). The Kepler's 3rd law orbital period is independent of eccentricity $e \leq 1$, and so half of it is the free-fall time.

SUGGESTED ANSWER:

a) By the shell theorem,

$$\vec{g} = \begin{cases} 0 & \text{for } r < r_{\rm s}; \\ -\dfrac{Gm}{r^2}\hat{r} & \text{for } r > r_{\rm s}. \end{cases}$$

b) If the cylinder is sufficiently small, it encloses a bit of an infinitely thin plane of mass. Let the enclosed mass be $\Delta m = \sigma A$, where A is the area of the top and bottom of the cylinder Symmetry dictates that the field is parallel to the sides of the cylinder, and so the sides contribute nothing to the Gauss' law integral over the cylinder. Initially, we assume that the top and bottom are at equal distance from the plane, so symmetry guarantees the magnitude of $\vec{g}$ is the same on top and bottom, but this assumption turns out to be unnecessary. Symmetry dictates that the differential area vectors $d\vec{A}$ of the top and bottom point opposite to the gravitational field on both sides of the plane. We now find

$$1)\quad \oint \vec{g}\cdot d\vec{A} = -4\pi G\Delta m \qquad 2)\quad -g\Delta A - g\Delta A = -4\pi G\sigma\Delta A \qquad 3)\quad 2g = 4\pi G\sigma$$

$$4)\quad 2g = 4\pi\frac{Gm}{4\pi r_{\rm s}^2} \qquad 5)\quad 2g = \frac{Gm}{r_{\rm s}^2} \qquad 6)\quad \vec{g}_{\rm cyl} = \frac{1}{2}\frac{Gm}{r_{\rm s}^2}(\pm\hat{r}) ,$$

where the upper case is for bottom and the lower case for the top and since the result does not depend distance from the plane the top and bottom could be at any distance from the plane as long the approximating the enclosed bit of shell as part of infinite infinitely thin plane is valid. The total field is given by part (a) for $r \neq r_{\rm s}$. Thus, the gravitational field due to the rest of the shell (the shell not counting except the enclosed mass) is

$$\vec{g}_{\text{shell rest}} = \vec{g} - \vec{g}_{\rm cyl} = \begin{cases} -\left(\dfrac{1}{2}\right)\dfrac{Gm}{r^2}\hat{r} & \text{for small } \delta r = r - r_{\rm s} < 0; \\ -\left(\dfrac{1}{2}\right)\dfrac{Gm}{r^2}\hat{r} & \text{for small } \delta r = r - r_{\rm s} > 0; \\ -\left(\dfrac{1}{2}\right)\dfrac{Gm}{r^2}\hat{r} & \text{for } r = r_{\rm s} \\ & \text{which is the physically realistic limit.} \end{cases}$$

Remarkably, the effective gravitational field on any differential bit of an infinitely thin shell is the average of the gravitational field of the whole shell for $< r_{\rm s}$ and $r > r_{\rm s}$.

Let us go beyond the required answer. The above result is convincing, yours truly believes, for isolated infinitely thin shells. But one can ask what of the limit of infinitely thin shells that make up a continuum: i.e., finite thickness shells stuck together. Say a finite thickness shell has mean radius $r_{\rm s}$, thickness Δr, and mass m. As boundaries between finite thickness shells are made closer together, the physically realistic limit is that shells have constant density. So to 1st order in small Δr, one has

$$m = 4\pi\rho r_{\rm s}^2\Delta r \qquad \text{and constant density} \qquad \rho = \frac{m}{4\pi r_{\rm s}^2\Delta r} .$$

The inward force on a differential solid angle $d\Omega$ of the finite thickness shell due to itself to 1st order in Δr is

$$dF = \int_{r-\Delta r/2}^{r+\Delta r/2} \frac{G(4\pi/3)[\rho r^3 - (r_{\rm s}-\Delta r/2)^3]}{r^2}\rho r^2\,dr\,d\Omega = \int_0^{\Delta r}\frac{G(4\pi\rho r_{\rm s}^2 x)}{r_{\rm s}^2}\rho r_{\rm s}^2\,dx\,d\Omega$$
$$= G(4\pi)\rho^2 r_{\rm s}^2\frac{\Delta r^2}{2}\,d\Omega = \frac{1}{2}\frac{Gm^2}{r_{\rm s}^2}\frac{d\Omega}{4\pi} ,$$

where we have used the shell theorem. Thus the 1st order force on the whole finite thickness shell is

$$F = \frac{1}{2}\frac{Gm^2}{r_{\rm s}^2} .$$

In the last result holds in the limit that $\Delta r \to 0$, and thus the infinitely thin shell in this derivation does self-gravitate in same physically realistic limit way that we would have expected from the first result above.

c) The answer is yes. Consider the two free-fall times for two infinitely thin spherical shells 1 and 2 where shell 1 is smaller than shell 2:

$$t_1 = \frac{\pi}{\sqrt{G(m_1/2)}}\left(\frac{r_1}{2}\right)^{3/2} \qquad \text{and} \qquad t_2 = \frac{\pi}{\sqrt{G[m_1+(m_2/2)]}}\left(\frac{r_2}{2}\right)^{3/2} ,$$

where the 1/2 factors are the self-gravitations corrections for shells which we can take as being made of test particles and which we found in part (b). If $t_1 > t_2$, the shells must cross. Now $t_1 > t_2$ implies

$$\sqrt{\frac{[m_1+(m_2/2)]}{m_1/2}}\left(\frac{r_1}{r_2}\right)^{3/2} > 1 .$$

Clearly, the inequality holds if m_2 is made large enough. Thus, there are mass distributions where infalling shells must cross. But what happens then they cross requires further analysis.

Redaction: Jeffery, 2018jan01

---

024 qfull 01030 1 3 0 easy math: galaxy potential energy and escape velocity: On exams, only do parts a,b,c,d.

8. In this question, we consider escape velocities from galaxies. The path is long if one does not gloss over tricky points like Ci-86–87.

NOTE: There are parts a,b,c,d,e,f. On exams, only do parts a,b,c,d and answer with minimal words.

a) From introductory physics, the change mechanical energy of particle is

$$\Delta E = \Delta KE + \Delta PE = W_{\rm noncon} ,$$

where KE is kinetic energy, PE is potential energy, and $W_{\rm noncon}$ work done by nonconservative forces. If there are no nonconservative forces, mechanical energy is conserved and

$$1)\quad \Delta E = 0 \qquad 2)\quad \Delta KE = -\Delta PE \qquad 3)\quad E = KE + PE \quad \text{is constant.}$$

The escape velocity from some point (with no nonconservative forces) can be found from some point noting that $KE = 0$ at infinity where the gravitational potential Φ (which is potential energy PE per unit mass) is defined to be zero. Find the general formula for escape velocity $v_{\rm esc}$ given that kinetic energy is initially KE and gravitational potential is initially Φ.

b) Assume a spherically symmetric mass distribution for a galaxy which seems to be often approximately true since dark matter halos are often quite spherically symmetric it seems though not always. Let the density profile be a power law

$$\rho = \rho_{\rm s}\left(\frac{r}{r_{\rm s}}\right)^{-\alpha} = \rho_{\rm s} x^{-\alpha} ,$$

where $\rho_{\rm s}$ is a scale density, $r_{\rm s}$ is a scale radius, $x = r/r_{\rm s}$ is a dimensionless radius, and α is the power. Determine the formula for interior mass $M(r)$ (i.e., mass interior to radius r) in terms of a scale $M_{\rm s}$ and x assuming $\alpha < 3$.

c) Why can't a galaxy have pure power law density profile from $r = 0$ to $r = \infty$, in fact? **HINT:** Consider the divergence behavior of the interior mass formula.

d) There is a tricky point in considering potential change. When integrating up the potential energy of a gravitating sphere, we use

$$PE(r) = \int_0^r \left[\frac{-GM(r)}{r}\right] 4\pi r^2 \rho\, dr ,$$

where $M(r)$ is the interior mass and $\Phi = -GM(r)/r$ is the gravitational potential r. This is the right thing to do, but $-GM(r)/r$ is not the potential at r in the fully assembled gravitating sphere.

Why not? Show what the potential at r is (relative to infinity which is zero) for a gravitating sphere of total radius R. **HINT:** Getting the signs right for potential is tricky. You have to do the sign on every step right—or chance of being right is only 50 %.

e) Making use of the part (b) and the part (d) answers find the potential from $x \leq X$ for $\alpha < 3$. Show explicitly the cases for 1) $\alpha \neq 2$, 2) $\alpha \in (2,3)$ and $x << X$, 3) $\alpha < 2$ and nd $x << X$, and 4) $\alpha = 2$.

f) From the part (e) answer from the escape velocity formula for the case of $\alpha \in (2,3)$ and $x << X$ in terms of the circular velocity for scaled radius $x = 1$. What is the escape velocity if circular velocity is 200 km/s and $\alpha = 9/8$? Why are galactic outflows hard to understand if α gets very close to 2? Having α close to 2 is what is implied by the flat velocity curve ranges of observed disc galaxies.

SUGGESTED ANSWER:

a) Given $\Delta E = 0$, $\Delta KE = -KE$, and $\Delta\Phi = -\Phi$, we find

$$1)\quad 0 = \Delta KE + m\Delta\Phi = -KE - m\Phi \qquad 2)\quad KE = \frac{1}{2}mv_{\rm esc}^2 = -m\Phi \qquad 3)\quad v_{\rm esc} = \sqrt{2(-\Phi)} = \sqrt{2|\Phi|}\ .$$

b) Behold:

$$M(r) = \int_0^r 4\pi r'^2 \rho\, dr' = 4\pi r_{\rm s}^3 \rho_s \int_0^x x'^{2-\alpha}\, dx' = 4\pi r_{\rm s}^3 \rho_s \left(\frac{x^{3-\alpha}}{3-\alpha}\right) = M_{\rm s} x^{3-\alpha}$$

where we have assumed that $\alpha < 3$.

c) If $\alpha < 3$, the integral diverges for $x \to \infty$. If $\alpha = 3$, the integral is a logarithm and diverges for both $x \to 0$ and $x \to \infty$. If $\alpha > 3$, the integral diverges for $x \to 0$. The upshot is that no real galaxy can have a pure power law density profile.

d) The integral is a process of adding mass to the growing sphere of radius r and mass $M(r)$ with nothing above r. The gravitational potential $\Phi = -GM(r)/r$ is exactly right for the surface of the growing sphere relative to infinity. You bring differential mass $4\pi r^2 \rho\, dr$ from infinity and add it the growing sphere with the correct contribution to potential energy. But after you have added a finite amount of mass above r to radius R, the potential at r is given by

$$\begin{aligned}
\Phi &= -\int_\infty^r \vec{F}\cdot d\vec{r}' = \int_r^\infty \vec{F}\cdot d\vec{r}' = -\int_r^\infty \frac{GM(r')}{r'^2}\, dr' \\
&= -\int_r^R \frac{GM(r')}{r'^2}\, dr' - \int_R^\infty \frac{GM(R)}{r'^2}\, dr' = -\int_r^R \frac{GM(r')}{r'^2}\, dr' - GM(R)\left(-\frac{1}{r}\right)\Bigg|_R^\infty \\
&= -\int_r^R \frac{GM(r')}{r'^2}\, dr' - \frac{GM(R)}{R}\ .
\end{aligned}$$

Note that signs are infernally tricky with gravitational potential. You have do every step exactly right or you will make random sign errors and only be right 50 % of the time.

e) Recalling that we are requiring $\alpha < 3$, we find

$$\Phi = -\int_r^R \frac{GM(r')}{r'^2}\, dr' - \frac{GM(R)}{R}$$

$$\Phi = -\frac{GM_{\rm s}}{r_{\rm s}} \int_x^X x'^{1-\alpha}\, dx' - \frac{GM_{\rm s}}{R} X^{3-\alpha}$$

$$\Phi = \begin{cases}
-\dfrac{GM_{\rm s}}{r_{\rm s}}\left(\dfrac{X^{2-\alpha} - x^{2-\alpha}}{2-\alpha}\right) - \dfrac{GM_{\rm s}}{R} X^{3-\alpha} & \text{for } \alpha \neq 2; \\[2ex]
-\dfrac{GM_{\rm s}}{r_{\rm s}}\left[-\left(\dfrac{x^{2-\alpha}}{2-\alpha}\right)\right] = -\dfrac{GM_{\rm s}}{r_{\rm s}}\left(\dfrac{x^{2-\alpha}}{\alpha-2}\right) & \text{for } \alpha \in (2,3) \text{ and } x << X; \\[2ex]
-\dfrac{GM_{\rm s}}{r_{\rm s}}\left(\dfrac{X^{2-\alpha}}{2-\alpha}\right) - \dfrac{GM_{\rm s}}{R} X^{3-\alpha} & \text{for } \alpha < 2 \text{ and } x << X; \\[2ex]
-\dfrac{GM_{\rm s}}{r_{\rm s}} \ln\left(\dfrac{X}{x}\right) - \dfrac{GM_{\rm s}}{R} X^{3-\alpha} & \text{for } \alpha = 2
\end{cases}$$

We see that for $\alpha \in (2,3)$ and $x << X$, the result is independent of X. This is an easy result to guess, but takes some care to prove.

f) For $\alpha \in (2,3)$ and $x << X$, we find the escape velocity formula to be

$$v_{\rm esc} = \begin{cases} \sqrt{\dfrac{2GM_{\rm s}}{r_{\rm s}}\left(\dfrac{x^{2-\alpha}}{\alpha-2}\right)} = v_{\rm cir}\sqrt{\dfrac{2x^{2-\alpha}}{\alpha-2}} & \text{in general;} \\ (200\,{\rm km/s})\times\sqrt{\dfrac{2}{\alpha-2}} & \text{for } x=1 \text{ and } v_{\rm cir}=200\,{\rm km/s;} \\ 800\,{\rm km/s} & \text{for } x=1,\ v_{\rm cir}=200\,{\rm km/s, and } \alpha=9/8. \end{cases}$$

Since disc galaxies often have large density profile ranges where $\alpha \approx 2$, it is clear the escape velocities can be very high. For example the escape speed from the center of the Milky Way is $\sim 800\,{\rm km/s}$ (Ci-87). Such high escape velocities makes understanding galactic outflows challenging since stellar winds and supernovae do not typically reach such velocities for the bulk of their material.

Redaction: Jeffery, 2018jan01

---

024 qfull 01050 1 3 0 easy math: metallicity saturation in galaxies: On exams, do only parts a,b,d,e.

9. The metallicity of galaxies does not generally increase with cosmic time, but reaches an (approximate) plateau due to gas inflow from the intergalactic/circumgalactic medium (which if intergalactic is of nearly primordial gas: primordial cosmic gas fiducial mass fractions $X = 0.75$ H, $Y = 0.25$ He, $Z = 0.001$ metallicity which is overwhelmingly deuterium counted as a metal: Wikipedia: Big Bang: Abundance of primordial elements) and the outflow of metal enriched gas from stellar evolution (i.e., stellar winds and supernovae) back to the intergalactic/circumgalactic medium or into compact astro-bodies (compact remnants, long-lived small mass stars, brown dwarts, planets, and smaller astro-bodies). The plateau phase will probably not last forever since cosmological constant acceleration isolates all bound systems not participating in the mean expansion of the universe from fresh primordial gas. So a slow metallicity increase should occur despite gas inflow/outflows as the overall isolated bound system gas gradually enriches. However, this enrichment seems very slow since cosmic time $\sim 5\,{\rm Gyr}$ after the Big Bang (Weinberg 2016, arXiv:1604.07434) and will gradually turn off with the end of the stelliferous era (theoretically cosmic time ~ 0.15–10^5 Gyr: Wikipedia: Graphical timeline of the Stelliferous Era; Wikipedia: Future of an expanding universe: The Stelliferous Era). In this question, will do a simple modeling of the plateauing of galaxy metallicity.

NOTE: There are parts a,b,c,d,e,f. On exams, do only parts a,b,d,e and answer using minimal words.

a) Write a (1st order ordinary autonomous) differential equation for galaxy gas density ρ (assumed to be uniform) in terms of a constant inflow rate of gas $F = (d\rho/dt)_{\rm inflow}$ (not necessarily primordial gas) and an outflow rate $-\kappa\rho = -\rho/\tau$, where κ is the rate constant and $\tau = 1/\kappa$ is the time constant. The outflow rate includes both outflow of gas back to the intergalactic/circumgalactic medium and into compact objects.

b) Using an integrating factor solve the differential equation of part (a) with ρ_0 as the initial density at time zero (i.e., $t = 0$). Give the 1st-order-in-small-t solution and the asymptotic solution as $t \to \infty$ (which is also the constant solution of the differential equation). What name can be given to the time constant τ?

c) Why do we get an asymptotic solution in part (b)?

d) Write a (1st order ordinary autonomous) differential equation for galaxy gas metal density $Z\rho$ (assumed to be uniform) in terms of a constant inflow rate of metal-only gas $Z_{\rm in}F = Z_{\rm in}(d\rho/dt)_{\rm inflow}$, where $Z_{\rm in} \in [0,1]$. Let the outflow rate be the same as in part (b): i.e., $-\kappa\rho = -\rho/\tau$, where κ is the rate constant and $\tau = 1/\kappa$ is the time constant. There is also a rate constant γ for the creation metal-only gas in the galaxy from zero-metallicity gas with density $(1-Z)\rho$.

e) The differential equation in part (c) can be solved for Z for general time t using the solution of part (b), but it seems a bit tedious to get this solution. However, finding the asymptotic solution $Z_{\rm asy}$ as $t \to \infty$ is easy. Find it. Check that $Z_{\rm asy}$ is dimensionally correct and show that it satisfies $Z_{\rm asy} \in [0,1]$.

f) We can make a crude estimate of current cosmic $Z_{\rm asy}$. First, let

$$\kappa = \frac{(d\rho/dt)_{\rm outflow}}{\rho} = \frac{3\,{\rm M}_\odot/{\rm yr}}{\rho} ,$$

where $3\,{\rm M}_\odot/{\rm yr}$ is roughly the rate of star formation for a galaxy like the Milky Way (Ci-383) and we assume this is of order the overall gas loss rate due gas outflow back to the intergalactic/circumgalactic medium and locking up of gas in compact astro-bodies. Second, let

$$\gamma = \frac{[d(Z\rho)/dt]_{\rm metal\,creation}}{\rho} = \frac{[5\,{\rm SNe}/(100\,{\rm yr})]\times(1\,{\rm M}_\odot\ {\rm metals/per\,SNe})}{\rho} ,$$

where $5\,{\rm SNe}/(100\,{\rm yr})$ is roughly the rate of supernovae for a galaxy like the Milky Way (Wikipedia: Supernova: Milky Way candidates) and we assume that this is of order the metal creation given that each supernovae yields of order $1\,{\rm M}_\odot$ of metals. Let $Z_{\rm in} = 0.001$ the fiducial primordial cosmic metallicity. Calculate $Z_{\rm asy}$ with these values and discuss whether the result is reasonable or not.

SUGGESTED ANSWER:

a) Behold:

$$\frac{d\rho}{dt} = F - \kappa\rho .$$

b) Behold:

$$1)\quad \frac{d\rho}{dt} = F - \kappa\rho \qquad 2)\quad \frac{d\rho}{dt} + \kappa\rho = F \qquad 3)\quad e^{\kappa t}\frac{d\rho}{dt} + e^{\kappa t}\kappa\rho = Fe^{\kappa t}$$
$$4)\quad e^{\kappa t}\rho|_0^t = F\tau(e^{\kappa t}-1) \qquad 5)\quad \rho = F\tau(1-e^{-\kappa t}) + \rho_0 e^{-\kappa t} .$$

Alternatively since F does not depend on ρ, we could solve as follows:

$$1)\quad \frac{d\rho}{dt} = F - \kappa\rho \qquad 2)\quad \frac{d\rho}{\rho - F\tau} = -\frac{dt}{\tau} \qquad 3)\quad \ln|\rho' - F\tau|\Big|_{\rho_0}^{\rho} = -\frac{t}{\tau}$$
$$4)\quad \ln\left|\frac{\rho - F\tau}{\rho_0 - F\tau}\right| = -\frac{t}{\tau} \qquad 5)\quad \rho - F\tau = (\rho_0 - F\tau)e^{-t/\tau}$$
$$6)\quad \rho = F\tau(1-e^{-t/\tau}) + \rho_0 e^{-t/\tau} .$$

Thus, we have

$$\rho = \begin{cases} F\tau(1-e^{-\kappa t}) + \rho_0 e^{-\kappa t} & \\ \qquad = F\tau(1-e^{-t/\tau}) + \rho_0 e^{-t/\tau} & \text{in general;} \\ Ft + \rho_0(1-\kappa t) = Ft + \rho_0\left(1 - \frac{t}{\tau}\right) & \text{for small } t; \\ F\tau & \text{for } t\to\infty\text{:} \\ & \text{this is the asymptotic solution.} \end{cases}$$

The time constant τ can be called the e-folding time.

c) So why do we get an asymptotic solution in part (b)? There is more than one answer. The most immediately cogent answer is that that is what the mathematical solution gives us. However, a more general answer is that the differential equation gives a singular constant solution for $d\rho/dt = 0$: i.e., constant solution $\rho = F/\kappa = F\tau$. Now a constant solution has all orders of derivative equal to zero. Therefore, a non-constant solution that is infinitely differentiable (which is all we can get from our differential equation) can never be equal to the constant solution, except asymptotically at $t = \pm\infty$. Now the differential equation derivative $d\rho/dt$ is positive/negative for ρ less/greater than the constant solution $F\tau$, and therefore ρ must increase/decrease forever as time advances and since there is only one possible constant asymptotic solution for it to approach, it must approach that one asymptotically: i.e., the constant asymptotic solution $F\tau$.

A perspective on the asymptotic solution (which is not part of the answer) is as follows. The initial density just gives a transient solution (which unless it is large relative to the asymptotic solution) will be relatively small in a few e-folding times. The ratio R of the driven solution outflow to the inflow rate is

$$R = \frac{\kappa F\tau(1-e^{-t/\tau})}{F} = \begin{cases} 1-e^{-t/\tau} & \text{in general;} \\ \dfrac{t}{\tau} & \text{for } t/\tau << 1; \\ 1 & \text{for } t \to \infty. \end{cases}$$

When the driven inflow and outflow rates are equal (which is only asymptotically at $t \to \infty$), the density must be unchanging: i.e., must be a constant asymptotic solution.

d) Behold:

$$\frac{d(Z\rho)}{dt} = Z_{\text{in}}F - \kappa Z\rho + \gamma(1-Z)\rho = Z_{\text{in}}F - (\kappa+\gamma)Z\rho + \gamma\rho \ .$$

e) The asymptotic solution is found by setting $d(Z\rho)/dt = 0$. We get

$$Z_{\text{asy}} = \frac{Z_{\text{in}}F + \gamma\rho_{\text{asy}}}{(\kappa+\gamma)\rho_{\text{asy}}} = \frac{Z_{\text{in}}F + \gamma F\tau}{(\kappa+\gamma)F\tau} = \frac{Z_{\text{in}} + \gamma\tau}{(\kappa+\gamma)\tau} = \frac{Z_{\text{in}} + \gamma\tau}{1+\gamma\tau} \ ,$$

where we have used the fact that $\kappa\tau = 1$. Note the formula is dimensionally correct since the terms in the numerator and denominator are all dimensionless and yield a dimensionless Z_{asy} as they should. Note a quantity being dimensionless does not mean not having a physical nature. It just means the quantity is written in terms of natural units. Since $Z_{\text{in}} \in [0,1]$, we have $Z_{\text{asy}} \in [0,1]$ by inspection.

Though not required by the question, we note

$$Z_{\text{asy}} = \begin{cases} \dfrac{Z_{\text{in}}+\gamma\tau}{1+\gamma\tau} & \text{in general;} \\ Z_{\text{in}} & \text{for } \gamma\tau = 0; \\ 1 & \text{for } \gamma\tau \to \infty. \\ \dfrac{\gamma\tau}{1+\gamma\tau} & \text{for } Z_{\text{in}} = 0; \\ 1 & \text{for } Z_{\text{in}} = 1; \end{cases}$$

Note if $\tau = 0$, the gas inflowed is instantly outflowed and there is no way its metallicity can be increased above Z_{in}. On the other hand, if $\tau = \infty$, the density goes to infinity asymptotically and so does the metallicity.

e) First, with the given values

$$\gamma\tau = \frac{\gamma}{\kappa} = \frac{[5\,\text{SNe}/(100\,\text{yr})] \times (1\,\text{M}_\odot\,\text{metals/per SNe})}{3\,\text{M}_\odot/\text{yr}} \approx 0.017 \ .$$

Now

$$Z_{\text{asy}} = \frac{Z_{\text{in}}+\gamma\tau}{1+\gamma\tau} \approx \frac{0.001+0.017}{1+0.017} \approx 0.017 \ .$$

The solar (surface) metallicity is fiducially 0.02 though one precise determination puts it at 0.0134 (Wikipedia: Metallicity: Mass fraction). The solar value is thought to be of order of the cosmic present metallicity, and so our calculated value is of order correct. The fact that is so close to 0.02 must be considered an accident since our input values have an estimated error of a factor of 2 at least.

Redaction: Jeffery, 2018jan01