Cosmology & Galaxies **NAME:**

Homework 24: Cosmic Present Star-Forming Galaxies (SFGs)

1. In solar mass units, the Milky Way stellar mass (i.e., the mass in stars M_*) is $\sim$ ____________ $M_\odot$ and its virial mass ($M_{\rm vir}$: i.e., fiducial total mass which is mostly dark matter) is $\sim$ ____________ $M_\odot$. At least these values were standard circa 2023. However, a downward revision may have become accepted just about that year.

   a) 10^{12}; 10^{10} b) 5×10^{10}; 5×10^{10} c) 5×10^{10}; 10^{12} d) 10^{9}; 5×10^{10}
   e) 10^{9}; 5×10^{8}

2. "Let's play *Jeopardy*! For \$100, the answer is:

$$I_\lambda = I_{\lambda,0} e^{-(R/R_{\rm d})} ,$$

   where I_λ is the surface brightness, $I_{\lambda,0}$ is the central surface brightness, R is the radius coordinate, and $R_{\rm d}$ disk scale length (and not the effective or half-light radius)."

   What is the standard ____________ surface brightness profile, Alex?

   a) edge-on spiral disc b) face-on spiral disc c) elliptical d) dwarf irregular
   e) general Sérsic

3. There are two main classes of galaxy bulges:

   a) classical bulges and non-classical bulges b) big bulges and disc-like bulges
   c) little bulges and disc-like bulges d) classical bulges and disc-like bulges
   e) little bulges and big bulges

4. "Let's play *Jeopardy*! For \$100, the answer is:

$$\Sigma_{\rm SFR} = B \left(\frac{\Sigma_{\rm gas}}{1\, M_\odot/{\rm pc}^2} \right)^{\alpha} M_\odot/{\rm yr}/{\rm kpc}^2 ,$$

   where SFR means star formation rate, $\Sigma_{\rm SFR}$ is surface star formation rate in units of $M_\odot/{\rm yr}/{\rm kpc}^2$, $\Sigma_{\rm gas}$ is gas surface density in units $M_\odot/{\rm pc}^2$ (the denominator below $\Sigma_{\rm gas}$ makes the overall factor dimensionless), $B \approx 10^{-4}$ is an empirical constant, and $\alpha = 1.40(15)$ is another empirical constant with some theoretical understanding.

   What is the ____________, Alex?

   a) Press-Kennicutt law b) Schmidt-Kennicutt law c) Press-Schechter law
   d) Martin-Schmidt law e) Martin-Schmidt-Kennicutt law

5. An inclined circle (ideal disc galaxy is) is seen in projection as an ellipse.
   NOTE: There are parts a,b. On exams, do all parts with minimal words.

   a) The equation for a circle is written elaborately is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1 ,$$

   where a is the radius. Find the explicit formula for y. The circle is rotated on its x-axis to inclination angle i where inclination angle is measured from the direction to the observer to a normal to the circle. What is the projected height of every y point (i.e., what is the inclined y_i)? Prove the inclined circle (i.e., projected circle) is an ellipse and find its semi-minor axis b.

   b) The area of an ellipse is $A = \pi ab$ and the circularized radius an ellipse created by inclination is defined by

$$R_i = \sqrt{ab} = a\sqrt{\cos(i)} .$$

   Prove that the differential area of an inclined circle is

$$dA = 2\pi R_i \, dR_i .$$

6. The free-fall time for a straight line fall of a particle of mass m starting from rest to a point source or spherically symmetric source of mass M (always interior to the infalling particle) is

$$t_{\rm ff} = \frac{t_{\rm orbit}}{2} = \frac{\pi}{\sqrt{G(M+m)}}\left(\frac{r}{2}\right)^{3/2} ,$$

where $t_{\rm orbit}$ is the orbital period predicted by the Newtonian physics version of Kepler's 3rd law and r is the initial distance from the particle to the source center and is twice the relative semi-major axis of an elliptical orbit of the particle to the source (Wikipedia: Free-fall time; Wikipedia: Kepler's laws of planetary motion Third law; Ci-246). The Kepler's 3rd law orbital period is independent of eccentricity $e \leq 1$, and so half of it is the free-fall time.

NOTE: There are parts a,b,c,d,e,f. On exams, only do parts a,b,c.

a) What is the free-fall time for test particle (i.e., one of negligible mass)?

b) What is the free-fall time as a function of r for a spherical mass distribution with initially constant density ρ and outer radius $r \leq R$. The matter is initiall all at rest and there is zero pressure at all times. Assume the (infinitely thin) shells of matter in the distribution at all the r values never cross during free fall which is true and plausible, but seems tricky to prove. Describe the order of arrival of the shells at the center?

HINT: Remember the shell theorem

$$\vec{g} = -\frac{GM(r)}{r^2}\hat{r}$$

where the mass distribution is sperically symmetric and $M(r)$ is the interior mass to radius r. Note $M(r)$ must increase monotonically since there is no negative mass, but it can be zero out some radius r.

NOTE: For all subsequent parts, we assume a spherically symmetric mass distribution at all times with initial outer radius R and there is zero pressure at all times.

c) Say that the interior mass $M(r)$ to radius r obeys a power law $M(r) = M_0(r/r_0)^\alpha$ where $\alpha \leq 3$. When does the mass all collapse to the center assuming that it magically all stops there on arrival and the shells of matter at all the r values never cross during free fall which is true and plausible, but seems tricky to prove.

d) For star formation, we want to relate density ρ to the particle density n which can be measured more directly. The relating formula is

$$n = \rho\left(\sum_i \frac{X_i}{A_i m_{\rm p}}\right) ,$$

where X_i is the mass fraction of species i (which could be any atom or a molecule including those that are distinct due to their isotopic nature), A_i is the atomic mass number (which could be a molecular mass number), and $m_{\rm p} = 1.67262192369(51) \times 10^{-24}\,$g is the proton mass. Note this special case atomic mass number is in units of proton masses, not daltons (symbol u or Da and AKA atomic mass units). The fact is most of the universe is made of hydrogen (which made of protons) and not made of daltonium (which is made of daltons). Worrying about corrections due to electron masses, binding energies, and isotopes abundances (which aside from hydrogen and helium are rather uncertain) is below the level of accuracy of this problem. The mean atomic mass is defined by

$$\mu^{-1} = \sum_i \frac{X_i}{A_i}$$

which gives

$$n = \frac{\rho}{\mu m_{\rm p}} \qquad \text{or} \qquad \rho = n\mu m_{\rm p} .$$

Fiducial cosmic values for X_i are: $X = 0.73$ for H, $Y = 0.25$ for He, and $Z = 0.02$ for metals. Two fiducial mean atomic masses are given by

$$\mu_{\rm H_1,dominated} = \left(\frac{X}{1}+\frac{Y}{2}+\frac{Z}{30}\right)^{-1} \qquad \text{and} \qquad \mu_{\rm H_2,dominated} = \left(\frac{X}{2}+\frac{Y}{2}+\frac{Z}{30}\right)^{-1} ,$$

where the atomic mass for Z is a rough fiducial average based on the fiducial atomic masses of very abundant metals: i.e., $A_{\rm C,6} = 12$, $A_{\rm O,8} = 16$, $A_{\rm Si,14} = 28$, and $A_{\rm Fe,28} = 56$. Compute the $\mu_{\rm H_1,dominated}$ and $\mu_{\rm H_2,dominated}$ values to 3-digit precision which probably 1 more digit than is significant, but it is useful to know insignificant digits sometimes to check for consistency between different calculations.

HINT: Write a small computer program to do the calculation.

f) The part (b) answer gives a fiducial lower limit for the formation time for a star. It is just a fiducial lower limit since real initial clouds of molecular gas do not have uniform density, are not spherically symmetric, and do not have zero pressure and zero initial kinetic energy. It is just a lower limit since the pressure force and kinetic energy in the molecular cloud resist collapse during the collapse process and delay collapse to a star sized object. However, it is useful to rewrite the part (b) answer in terms fiducial values: particle density $10^3\,{\rm cm}^{-3}$, $\mu_{\rm H_2,dominated}$ from part (e), and Julian years (i.e., 365.25 days). Do the rewrite.

HINT: Write a small computer program to do the calculation.

7. Consider free-falling spherical shells of matter that only interact gravitationally.

NOTE: There are parts a,b,c. On exams, do all parts, but answer with minimal words.

a) First we consider a single infinitely thin spherical shell of radius $r_{\rm s}$ and mass m. What is the gravitational field $\vec{g}$ at $r < r_{\rm s}$? What is the gravitational field $\vec{g}$ at $r > r_{\rm s}$? Justify your answers.

b) What of the gravitational field $\vec{g}$ at $r_{\rm s}$? In one sense, the field is indeterminate since there is a discontinuity in the field r and which value you get depends on the direction you take the limit in. However, a limiting value often depends on the limiting process and some limiting processes are physically realistic and others are not. A physically realistic limit gravitational field at r does exist. The trick is consider tiny cylinder Gaussian surface (see Wikipedia: Gaussian surface) placed on the shell of radius r that extends inward and outward from r and whose top and bottom are parallel to the shell surface. In the small limit, the cylindar straddles an infinite infinitely thin plane of surface mass density $\sigma = m/(4\pi r_{\rm s}^2)$. Determine the gravitational field on the top and bottom of the cylinder just due to the enclosed mass. Then find the gravitational field due the rest of the shell on enclosed mass in the cylinder for all r including $r = r_{\rm s}$. That gravitational field is the gravitational field that can accelerate the enclosed mass treating it as test particle.

c) Do infalling spherical shells ever cross for any possible mass distribution? Prove your answer. **HINT:** Recall, the free-fall time for a straight line fall of a particle of mass m starting from rest to a point source or spherically symmetric source of mass M (always interior to the infalling particle) is

$$t_{\rm ff} = \frac{t_{\rm orbit}}{2} = \frac{\pi}{\sqrt{G(M+m)}}\left(\frac{r}{2}\right)^{3/2} ,$$

where $t_{\rm orbit}$ is the orbital period predicted by the Newtonian physics version of Kepler's 3rd law and r is the initial distance from the particle to the source center and is twice the relative semi-major axis of an elliptical orbit of the particle to the source (Wikipedia: Free-fall time; Wikipedia: Kepler's laws of planetary motion Third law; Ci-246). The Kepler's 3rd law orbital period is independent of eccentricity $e \leq 1$, and so half of it is the free-fall time.

8. In this question, we consider escape velocities from galaxies. The path is long if one does not gloss over tricky points like Ci-86–87.

NOTE: There are parts a,b,c,d,e,f. On exams, only do parts a,b,c,d and answer with minimal words.

a) From introductory physics, the change mechanical energy of particle is

$$\Delta E = \Delta KE + \Delta PE = W_{\rm noncon} ,$$

where KE is kinetic energy, PE is potential energy, and $W_{\rm noncon}$ work done by nonconservative forces. If there are no nonconservative forces, mechanical energy is conserved and

$$1)\quad \Delta E = 0 \qquad 2)\quad \Delta KE = -\Delta PE \qquad 3)\quad E = KE + PE \quad \text{is constant.}$$

The escape velocity from some point (with no nonconservative forces) can be found from some point noting that $KE = 0$ at infinity where the gravitational potential Φ (which is potential energy PE

per unit mass) is defined to be zero. Find the general formula for escape velocity $v_{\rm esc}$ given that kinetic energy is initially KE and gravitational potential is initially Φ.

b) Assume a spherically symmetric mass distribution for a galaxy which seems to be often approximately true since dark matter halos are often quite spherically symmetric it seems though not always. Let the density profile be a power law

$$\rho = \rho_{\rm s}\left(\frac{r}{r_{\rm s}}\right)^{-\alpha} = \rho_{\rm s}x^{-\alpha} \ ,$$

where $\rho_{\rm s}$ is a scale density, $r_{\rm s}$ is a scale radius, $x = r/r_{\rm s}$ is a dimensionless radius, and α is the power. Determine the formula for interior mass $M(r)$ (i.e., mass interior to radius r) in terms of a scale $M_{\rm s}$ and x assuming $\alpha < 3$.

c) Why can't a galaxy have pure power law density profile from $r = 0$ to $r = \infty$, in fact? **HINT:** Consider the divergence behavior of the interior mass formula.

d) There is a tricky point in considering potential change. When integrating up the potential energy of a gravitating sphere, we use

$$PE(r) = \int_0^r \left[\frac{-GM(r)}{r}\right] 4\pi r^2 \rho \, dr \ ,$$

where $M(r)$ is the interior mass and $\Phi = -GM(r)/r$ is the gravitational potential r. This is the right thing to do, but $-GM(r)/r$ is not the potential at r in the fully assembled gravitating sphere. Why not? Show what the potential at r is (relative to infinity which is zero) for a gravitating sphere of total radius R. **HINT:** Getting the signs right for potential is tricky. You have to do the sign on every step right—or chance of being right is only 50 %.

e) Making use of the part (b) and the part (d) answers find the potential from $x \leq X$ for $\alpha < 3$. Show explicitly the cases for 1) $\alpha \neq 2$, 2) $\alpha \in (2,3)$ and $x << X$, 3) $\alpha < 2$ and nd $x << X$, and 4) $\alpha = 2$.

f) From the part (e) answer from the escape velocity formula for the case of $\alpha \in (2,3)$ and $x << X$ in terms of the circular velocity for scaled radius $x = 1$. What is the escape velocity if circular velocity is 200 km/s and $\alpha = 9/8$? Why are galactic outflows hard to understand if α gets very close to 2? Having α close to 2 is what is implied by the flat velocity curve ranges of observed disc galaxies.

9. The metallicity of galaxies does not generally increase with cosmic time, but reaches an (approximate) plateau due to gas inflow from the intergalactic/circumgalactic medium (which if intergalactic is of nearly primordial gas: primordial cosmic gas fiducial mass fractions $X = 0.75$ H, $Y = 0.25$ He, $Z = 0.001$ metallicity which is overwhelmingly deuterium counted as a metal: Wikipedia: Big Bang: Abundance of primordial elements) and the outflow of metal enriched gas from stellar evolution (i.e., stellar winds and supernovae) back to the intergalactic/circumgalactic medium or into compact astro-bodies (compact remnants, long-lived small mass stars, brown dwarts, planets, and smaller astro-bodies). The plateau phase will probably not last forever since cosmological constant acceleration isolates all bound systems not participating in the mean expansion of the universe from fresh primordial gas. So a slow metallicity increase should occur despite gas inflow/outflows as the overall isolated bound system gas gradually enriches. However, this enrichment seems very slow since cosmic time ~ 5 Gyr after the Big Bang (Weinberg 2016, arXiv:1604.07434) and will gradually turn off with the end of the stelliferous era (theoretically cosmic time ~ 0.15–10^5 Gyr: Wikipedia: Graphical timeline of the Stelliferous Era; Wikipedia: Future of an expanding universe: The Stelliferous Era). In this question, will do a simple modeling of the plateauing of galaxy metallicity.

NOTE: There are parts a,b,c,d,e,f. On exams, do only parts a,b,d,e and answer using minimal words.

a) Write a (1st order ordinary autonomous) differential equation for galaxy gas density ρ (assumed to be uniform) in terms of a constant inflow rate of gas $F = (d\rho/dt)_{\rm inflow}$ (not necessarily primordial gas) and an outflow rate $-\kappa\rho = -\rho/\tau$, where κ is the rate constant and $\tau = 1/\kappa$ is the time constant. The outflow rate includes both outflow of gas back to the intergalactic/circumgalactic medium and into compact objects.

b) Using an integrating factor solve the differential equation of part (a) with ρ_0 as the initial density at time zero (i.e., $t = 0$). Give the 1st-order-in-small-t solution and the asymptotic solution as

$t \to \infty$ (which is also the constant solution of the differential equation). What name can be given to the time constant τ?

c) Why do we get an asymptotic solution in part (b)?

d) Write a (1st order ordinary autonomous) differential equation for galaxy gas metal density $Z\rho$ (assumed to be uniform) in terms of a constant inflow rate of metal-only gas $Z_{\rm in}F = Z_{\rm in}(d\rho/dt)_{\rm inflow}$, where $Z_{\rm in} \in [0,1]$. Let the outflow rate be the same as in part (b): i.e., $-\kappa\rho = -\rho/\tau$, where κ is the rate constant and $\tau = 1/\kappa$ is the time constant. There is also a rate constant γ for the creation metal-only gas in the galaxy from zero-metallicity gas with density $(1-Z)\rho$.

e) The differential equation in part (c) can be solved for Z for general time t using the solution of part (b), but it seems a bit tedious to get this solution. However, finding the asymptotic solution $Z_{\rm asy}$ as $t \to \infty$ is easy. Find it. Check that $Z_{\rm asy}$ is dimensionally correct and show that it satisfies $Z_{\rm asy} \in [0,1]$.

f) We can make a crude estimate of current cosmic $Z_{\rm asy}$. First, let

$$\kappa = \frac{(d\rho/dt)_{\rm outflow}}{\rho} = \frac{3\,{\rm M}_\odot/{\rm yr}}{\rho} ,$$

where $3\,{\rm M}_\odot/{\rm yr}$ is roughly the rate of star formation for a galaxy like the Milky Way (Ci-383) and we assume this is of order the overall gas loss rate due gas outflow back to the intergalactic/circumgalactic medium and locking up of gas in compact astro-bodies. Second, let

$$\gamma = \frac{[d(Z\rho)/dt]_{\rm metal\,creation}}{\rho} = \frac{[5\,{\rm SNe}/(100\,{\rm yr})] \times (1\,{\rm M}_\odot\ {\rm metals/per\ SNe})}{\rho} ,$$

where $5\,{\rm SNe}/(100\,{\rm yr})$ is roughly the rate of supernovae for a galaxy like the Milky Way (Wikipedia: Supernova: Milky Way candidates) and we assume that this is of order the metal creation given that each supernovae yields of order $1\,{\rm M}_\odot$ of metals. Let $Z_{\rm in} = 0.001$ the fiducial primordial cosmic metallicity. Calculate $Z_{\rm asy}$ with these values and discuss whether the result is reasonable or not.