Cosmology & Galaxies NAME:

Homework 23: Cosmic Present Galaxies as a Benchmark for Evolutionary Studies

---

023 qmult 00110 1 4 5 easy deducto-memory: specific intensity and surface brightness

1. "Let's play *Jeopardy*! For $100, the answer is: It and surface brightness are the same physical quantities though in some conventions surface brightness has an extra factor of 4π. The name used just depends on context."

   What is _____, Alex?

   a) radiant flux b) absolute magnitude c) apparent magnitude d) mean intensity
   e) specific intensity

   SUGGESTED ANSWER: (e) That 4π factor in the view of a radiative transfer specialist is a useless complication.

   Wrong answers:
   b) As Lurch would say AAAARGH.
   d) Mean intensity is the angle-averaged specific intensity.

   Redaction: Jeffery, 2008jan01

---

006 qmult 00110 1 1 3 easy memory: hybrid representation of specific intensity

2. The hybrid representation (AKA logarithmic representation of specific intensity satisfies equation:

   a) $I_E = I_\nu = I_\lambda$.
   b) $I_E/E = I_\nu/\nu = I_\lambda/\lambda$.
   c) $EI_E = \nu I_\nu = \lambda I_\lambda$.
   d) $I_E = I_\nu = 1/I_\lambda$.
   e) $I_E = 1/I_\nu = 1/I_\lambda$.

   SUGGESTED ANSWER: (c)

   Wrong answers:
   a) Exactly wrong.
   b) Exactly wrong too.

   Redaction: Jeffery, 2008jan01

---

023 qmult 00420 1 4 2 easy deducto-memory: the Sérsic profile specified

3. "Let's play *Jeopardy*! For $100, the answer is:

$$I_\lambda = I_{\lambda,0} \exp\left(bx^{1/n}\right) = I_{\lambda,\mathrm{e}} \exp\left[b(x^{1/n} - 1)\right] \ ,$$

   where I_λ is the surface brightess as a function x, $x = R/R_\mathrm{e}$ is the radius (elliptical or circularized radius) in units of the effective radius R_e, $I_{\lambda,0} = I_\lambda(x = R/R_\mathrm{e} = 0)$, $I_{\lambda,\mathrm{e}} = I_\lambda(x = R/R_\mathrm{e} = 1)$, n is an index parameter typically in the range $[1, 2.5]$ for star forming galaxies (SFGs) and in the range $[2.5, 10]$ for early type galaxies (ETGs), and b is a function of n (i.e., $b = b(n)$).

   What is the _____, Alex?

   a) de Vaucouleurs profile b) Sérsic profile c) Navarro-Frenk-White profile (NFW profile)
   d) Burkert profile e) Brownstein profile

   SUGGESTED ANSWER: (b) The Sérsic profile and the Sérsic index n are discussed by Ci-31.

   Wrong answers:
   a) As Lurch would say AAAARGH.

   Redaction: Jeffery, 2008jan01

---

006 qfull 00110 1 3 0 easy math: nu,lambda,hybrid represention: On exams omit part d. Use minimal words.

4. Specific intensity and related quantities (e.g., energy density per unit wavelength) are conventionally given in three representations: photon energy representation I_E, frequency representation I_ν, and wavelength representation I_λ. These representations are related by differential expression

$$I_E\, dE = I_\nu\, d\nu = I_\lambda\, (-d\lambda) \ ,$$

where the minus sign is occasionally omitted if one knows what one means—which is that a differential increase in photon energy/frequency corresponds to a differential decrease in wavelength.

There are parts a,b,c,d. On exams, omit part d.

a) As well as the three conventional representations, there is a hybrid representation (AKA logarithmic representation)

$$EI_E = \nu I_\nu = \lambda I_\lambda$$

which has the same value whichever of E, ν, or λ is used as the independent variable. Prove the hybrid representation equality. **Hint:** You will have to use differentials of the logarithm of the independent variables (e.g., $d[\ln(E)]$) and make use of the de Broglie relations $E = h\nu = hc/\lambda$.

b) Suggest two or three reasons why people might want to use the hybrid representation for graphing.

c) Planck's law (AKA the blackbody specific intensity spectrum) in the frequency representation is

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^x - 1} \ , \qquad \text{where} \qquad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda} \ .$$

Derive the explicit energy representation B_E, wavelength representation B_λ, and hybrid representation $EB_E = \nu B_\nu = \lambda B_\lambda$ in all three of the E, ν and λ forms.

d) Derive the Rayleigh-Jeans law (small x, small E, small ν, large λ approximation) and the Wien approximation (large x, large E, large ν, small λ approximation) for B_E, B_ν, and B_λ **Hint:** This pretty easy albeit tedious.

SUGGESTED ANSWER:

a) First,

$$\begin{aligned} I_E\, dE = I_\nu\, d\nu &= I_\lambda\, (-d\lambda) \\ EI_E\, d[\ln(E)] = \nu I_\nu\, d[\ln(\nu)] &= \lambda I_\lambda\, \{-d[\ln(\lambda)]\} \ . \end{aligned}$$

Second,

$$\begin{aligned} E = h\nu &= hc/\lambda \\ \ln(E) = \ln(h\nu) &= \ln(hc/\lambda) \\ d[\ln(E)] = d[\ln(\nu)] &= -d[\ln(\lambda)] \ . \end{aligned}$$

Dividing the first result by the second gives the required result:

$$EI_E = \nu I_\nu = \lambda I_\lambda \qquad \text{QED.}$$

b) First, since $EI_E = \nu I_\nu = \lambda I_\lambda$, there is no wondering about how the values on graphs of them would differ no matter which independent variable is used to evaluate them. The hybrid represention is neutral. Of course, a graph using wavelength would have an inversion relative to those using energy and frequency. Second, if you use a logarithmic horizontal axis (which is often convenient for large energy/frequency/wavelength bands), you can integrate up energy by eye which is useful for quick estimates. Third, for the energy and frequency representations, there is often an exponential decline as you go beyond the peak. Among other things, this is due to the inverse exponential behavior of the Planck spectrum beyond the peak: so thermal or semi-thermal specific intensity can exhibit a rapid decline beyond the peak. If there is a rapid decline beyond the peak, using the hybrid representation may flatten the spectrum and save you from needing an ugly large vertical range in order to see the whole spectrum.

c) Behold:

$$B_E = B_\nu \frac{d\nu}{dE} = \frac{2hv^3}{c^2}\frac{1}{e^x-1}\left(\frac{1}{h}\right) = \frac{2v^3}{c^2}\frac{1}{e^x-1} = \frac{2E^3}{h^3c^2}\frac{1}{e^x-1}$$

and

$$B_\lambda = -B_\nu\frac{d\nu}{d\lambda} = -\frac{2hv^3}{c^2}\frac{1}{e^x-1}\left(-\frac{c}{\lambda^2}\right) = -\frac{2hc^3}{c^2\lambda^3}\frac{1}{e^x-1}\left(-\frac{c}{\lambda^2}\right) = \frac{2hc^2}{\lambda^5}\frac{1}{e^x-1}\ ,$$

and so

$$EB_E = \frac{2E^4}{h^3c^2}\frac{1}{e^x-1} = \nu B_\nu = \frac{2hv^4}{c^2}\frac{1}{e^x-1} = \lambda B_\lambda = \frac{2hc^2}{\lambda^4}\frac{1}{e^x-1}\ .$$

d) Behold:

$$B_E = \begin{cases} \dfrac{2E^3}{h^3c^2}\dfrac{1}{e^x-1} & \text{in general;} \\ \dfrac{2E^3}{h^3c^2x} = \dfrac{2E^2}{h^3c^2}kT & \text{for } x << 1\text{: Rayleigh-Jeans law;} \\ \dfrac{2E^3}{h^3c^2}e^{-x} & \text{for } x >> 1\text{: Wien approximation;} \end{cases}$$

$$B_\nu = \begin{cases} \dfrac{2hv^3}{c^2}\dfrac{1}{e^x-1} & \text{in general;} \\ \dfrac{2hv^3}{c^2x} = \dfrac{2v^2}{c^2}kT & \text{for } x << 1\text{: Rayleigh-Jeans law;} \\ \dfrac{2hv^3}{c^2}e^{-x} & \text{for } x >> 1\text{: Wien approximation;} \end{cases}$$

$$B_\lambda = \begin{cases} \dfrac{2hc^2}{\lambda^5}\dfrac{1}{e^x-1} & \text{in general;} \\ \dfrac{2hc^2}{\lambda^5x} = \dfrac{2c}{\lambda^4}kT & \text{for } x << 1\text{; Rayleigh-Jeans law;} \\ \dfrac{2hc^2}{\lambda^5}e^{-x} & \text{for } x >> 1\text{: Wien approximation;} \end{cases}$$

Redaction: Jeffery, 2018jan01

---

002 qfull 00850 1 3 0 easy math: iteration equation solution convergence: On exams, only do parts f,g,h.

5. Say you need to find a root to equation

$$g(y) = 0$$

and no analytic solution is available. The equation my be transcendental: i.e., no finite number operations results in a solution. There are many sophisticated of doing this (e.g., Pr-340ff), but a simple one is by an iteration function suitable if you can constrain the root you are looking for to some interval $y \in [a, b]$. First reaarange the equation as iteration equation

$$y = f(y)$$

and then iterate by feeding the output of function $f(y)$ back into function $f(y)$ as an argument or input. The iteration starts with an initial estimate solution y_0 and proceeds via iterates y_1, y_2, ..., y_{i-1}, y_i, etc. using equation

$$y_i = f(y_{i-1})\ .$$

But how do you know you will get convergence and not divergence or just wandering. We will investigate convergence in this question.

Note the iteration equation approach (assuming it converges) may be very slow both in computer time and iterations especially if you are trying to converge to high machine precision and, of course, for transcendental equations you will never find exact numerical solution. Faster methods are available (e.g., the Brent method (Pr-352) and Newton-Raphson method (Pr-355)), but if you are just solving a

simple one-off problem, the iteration equation method may be fine. In vast multiple variable problems like astrophysical atmosphere problems, a multivariable iteration "equation" may be all you have.

HINT: Drawing diagrams as needed helps.

NOTE: There are parts a,b,c,d,e,f,g,h,i. On exams, only do parts f,g,h.

a) First, without loss of generality adjust the variables such that root is zero. Of course, course you cannot do this in an actual problem unless you already know the answer, but for the proof you can assume you do know the answer. Define two functions

$$y = \pm x \qquad \text{and} \qquad y = f(x) \ .$$

The first function divides the Cartesian plane into 4 quadrants. Show that if $f(x)$ is confined to the side quadrants and never tounches the lines defined by $y = \pm x$ in interval $[-a, a]$ (except at the origin itself which is in the interval $[-a, a]$) that convergence is guaranteed for zeroth iterate $y_0 \in [-a, a]$.

b) In a real problem the interval surrounding the root may not be symmetric about the root. This can lead to divergence with some easily imagined bad behavior in the side-quadrant-confined iteration function $f(x)$. How is divergence easily prevented?

c) In terms of sufficient and necessary conditions for convergence how would you describe the side-quadrant-confined iteration function $f(x)$ condition?

d) What is a simple sufficient, but not necessary, condition side-quadrant-confined iteration function $f(x)$ to give convergence?

e) How would iterates behave if side-quadrant-confined iteration function $f(x)$ were monotonically increasing/decreasing?

f) What makes an iteration function to solve for a root (AKA a zero) better thinking in the simplest sense? Think of the ideal limit.

g) Consider the transcendental equation

$$\frac{1}{2} = (x + 1)e^{-x} \ .$$

Find an iteration function to solve for a that is probably divergent at a first guess. Note this is a real problem, and so the solution is not the origin.

h) For the transcendental equation of part (g)

$$\frac{1}{2} = (x + 1)e^{-x}$$

find an iteration function guaranteed to converge for some interval about the solution. Find the interval of convergence and prove convergence in the interval.

i) Try to solve your convergent iteration equation from part (h) by series expansion in small x. You may have to consult Wikipedia (Wikipedia: Natural logarithm) to see where the series expansion is covergent and where a truncated version is a valid approximation. Then just use the Wikipedia plot to estimate the solution: i.e., the point where $y = x$ and the y value from the iteration function.

j) If you know how to code, iterate to function you found in part (h) to convergence to within machine precision and give the number iteration needed and the result.

SUGGESTED ANSWER:

a) Behold:

$$|y_i| = |f(x = y_{i-1})| \leq |y_{i-1}| \leq |y_0| < a$$

with the equality holding only at the root $y = 0$ itself. The iteration must converge.

b) Just use the safe iteration function in iteration steps: i.e.,

$$y_i = \min[\max(a, f(x = y_{i-1}), b]$$

which guarantees that the iterates never leave the interval of convergence.

c) It is sufficient for convergence and necessary strictly decreasing discrepancy from the solution root. There are probably special cases with safe iteration function where convergence can be guaranteed without side-quadrant-confined iteration function $f(x)$ condition. However, if the iteration function were only in the side quandrants at the solution itself, it would diverge always without using the safe iteration function and with it just oscillate.

d) If

$$\left|\frac{df}{dx}\right| < 1 \ ,$$

then $f(x)$ is necessarily a side-quadrant-confined iteration function $f(x)$.

e) The iterates would keep/switch their sign with every iteration step.

f) The flatter an iteration function is the better. The ideal iteration function is perfectly flat. Every iteration converges exactly in one iteration.

g) Well,

$$x = f(x) = \frac{e^x}{2} - 1$$

looks promising to diverge since the exponential function can easily iterate out of the side quadrants about the solution x. To be definite, note

$$\frac{df}{dx} = \frac{e^x}{2} > 1$$

for $x > \ln(2) = 0.6931\ldots$. So if the solution $x > \ln(2)$, the derivative df/dx has slope greater than 1 in an interval surrounding the solution x. Therefore $f(x)$ is not in the side quadrants in the interval surrounding. the solution x. Now note

$$f[x = \ln(2)] = 0 \ ,$$

and so the solution must actually be greater than the $\ln(2)$. The upshot is the iteration equation is divergent everywhere since it is divergent in an interval surrounding the solution x.

h) Well,

$$x = f(x) = \ln[2(x+1)]$$

will converge for all $x > 0$ since

$$\frac{df}{dx} = \frac{1}{x+1}$$

is always between 0 and 1 and it is only 1 for $x = 0$ which is obviously not the solution x. Just to be clear, the iteration function must lead to convergence for $x \in (0, \infty)$ since it is entirely confined in the two side quadrants since the absolute value of the slope is always less than 1 for $x > 0$.

i) From Wikipedia (Wikipedia: Natural logarithm: Series)

$$x = f(x) = \ln([2(x+1)] = \ln(2) + \ln(1+x) = \ln(2) + \sum_{k=1}^{\infty} \frac{x^k}{k} = \ln(2) + x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

which is convergent only for range $(-1, 1]$, but is likely to be factor of 2ish accurate for $x \lesssim 2$ if truncated to 2nd order. Truncating to 2nd order gives

$$1) \qquad x = \ln(2) + x - \frac{x^2}{2} \qquad\qquad 2) \qquad x = \sqrt{2\ln(2)} = 1.17741002251547469\ldots \ .$$

Since the result is greater than 1, it is not consistent with a convergent Taylor's series, but it is probably factor of 2ish accurate. Just eye-balling Wikipedia's plot, suggests $y = x$ and $y = \ln([2(x+1)] = \ln(2) + \ln(1+x)$ intersect at $y \approx 1.6$.

j) Using the a fortran-95 code, it took 43 iterations to reach the machine precision answer $x = 1.6783469900166606528$. Machine precision is 18: i.e., the relative error in a number is of order 10^{-18}. The answer is not far from what eye-balling Wikipedia's plot gives.

Fortran-95 Code

```
        print*
        print*,'Iteration function solution example:'
        xtrun=sqrt(2.0_np*log(2.0_np))
        print*,'The Taylor series approximate solution:'
        print*,xtrun  ! 1.1774100225154746910
        iprec=precision(1.0_np)
        prec=10.0_np**(-iprec)
        print*,'iprec,prec'
        print*,iprec,prec
        i=0
        con=log(2.0_np)
        x=con
        print*,'i,x0,x'
        do
         i=i+1
         x0=x
         x=con+log(1.0_np+x)
         print*,i,x0,x
         if(abs((x0-x)/x) .le. prec) exit
        end do
!
! Iteration function solution example:
!  The Taylor series approximate solution:
!   1.1774100225154746910
! iprec,prec
!          18  9.9999999999999999978E-0019
! 'i,x0,x'
!           1  0.69314718055994530943         1.2197362146989897914
!           2   1.2197362146989897914         1.4905355471765533580
!           3   1.4905355471765533580         1.6056449470977198813
!
!          42   1.6783469900166606494         1.6783469900166606520
!          43   1.6783469900166606520         1.6783469900166606528
```

Redaction: Jeffery, 2018jan01