Cosmology **Name:**

Homework 10: The Cosmic Microwave Background

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002 qfull 00500 1 3 0 easy math: nu,lambda,hybrid represention

1. Specific intensity and related quantities (e.g., energy density per unit wavelength) are conventionally given in three representations: photon energy representation I_E, frequency represenation I_ν, and wavelength represenation I_λ. These represenations are related by differential expression

$$I_E\,dE = I_\nu\,d\nu = I_\lambda\,(-d\lambda)\ ,$$

where the minus sign is occasionally omitted if one knows what one means—which is that a differential increase in photon energy/frequency corresponds to a differential decrease in wavelength.

a) As well as the three conventional representations, there is a hybrid representation

$$EI_E = \nu I_\nu = \lambda I_\lambda$$

which has the same value whichever of E, ν, or λ is used as the independent variable. Prove the hybrid representation equality. **Hint:** You will have use differentials of the logarithm of the independent variables (e.g., $d[\ln(E)]$) and make use of the de Broglie relations $E = h\nu = hc/\lambda$.

b) Suggest two or three reasons why people might want to use the hybrid represenation for graphing.

c) Planck's law (AKA the blackbody specific intensity spectrum) in the frequency represenation is

$$B_\nu = \frac{2hv^3}{c^2}\frac{1}{e^x - 1}\ , \qquad \text{where} \qquad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda}\ .$$

Derive the energy representation B_E, wavelength represenation B_λ, and the hybrid represenation $EB_E = \nu B_\nu = \lambda B_\lambda$ in E, ν and λ forms.

d) Derive the Rayleigh-Jeans law (small x, small E, small ν, large λ approximation) and the Wien approximation (large x, large E, large ν, small λ approximation) for B_E, B_ν, and B_λ **Hint:** This pretty easy albeit tedious.

SUGGESTED ANSWER:

a) First,

$$I_E\,dE = I_\nu\,d\nu = I_\lambda\,(-d\lambda)$$
$$EI_E\,d[\ln(E)] = \nu I_\nu\,d[\ln(\nu)] = \lambda I_\lambda\,\{-d[\ln(\lambda)]\}\ .$$

Second,

$$E = h\nu = hc/\lambda$$
$$\ln(E) = \ln(h\nu) = \ln(hc/\lambda)$$
$$d[\ln(E)] = d[\ln(\nu)] = -d[\ln(\lambda)]\ .$$

Dividing the first result by the second gives the required result:

$$EI_E = \nu I_\nu = \lambda I_\lambda \qquad \text{QED.}$$

b) First, since $EI_E = \nu I_\nu = \lambda I_\lambda$, there is no wondering about how the values would differ if you graphed the one instead of the other since they are all the same. They hybrid represenation is neutral. Second, if you use a logarithmic horizontal axis (which is often convenient for large energy/frequency/wavelength bands), you can integrate up energy by eye which is useful for quick estimates. Third, for the energy and frequency representations, there is often an exponential decline as you go beyond the peak. Among other things, this is due to the inverse exponential behavior of the Planck spectrum beyond the peak: so thermal or semi-thermal will exhibit a rapid decline beyond the peak. If there is a rapid decline beyond the peak, using the hybrid representation can flatten the spectrum and save you from needing an ugly large vertical range to see the whole spectrum.

c) Behold:

$$B_E = B_\nu \frac{d\nu}{dE} = \frac{2E^3}{h^3c^2}\frac{1}{e^x - 1} \qquad \text{and} \qquad B_\lambda = -B_\nu \frac{d\nu}{d\lambda} = \frac{2hc^2}{\lambda^5}\frac{1}{e^x - 1} \ ,$$

and so

$$EB_E = \frac{2E^4}{h^3c^2}\frac{1}{e^x - 1} = \nu B_\nu = \frac{2hv^4}{c^2}\frac{1}{e^x - 1} = \lambda B_\lambda = \frac{2hc^2}{\lambda^4}\frac{1}{e^x - 1} \ .$$

d) Behold:

$$B_E = \begin{cases} \frac{2E^3}{h^3c^2}\frac{1}{e^x - 1} & \text{in general;} \\ \frac{2E^3}{h^3c^2 x} = \frac{2E^2}{h^3c^2}kT & \text{for } x << 1\text{: Rayleigh-Jeans law;} \\ \frac{2E^3}{h^3c^2}e^{-x} & \text{for } x >> 1\text{: Wien approximation;} \end{cases}$$

$$B_\nu = \begin{cases} \frac{2hv^3}{c^2}\frac{1}{e^x - 1} & \text{in general;} \\ \frac{2hv^3}{c^2 x} = \frac{2v^2}{c^2}kT & \text{for } x << 1\text{: Rayleigh-Jeans law;} \\ \frac{2hv^3}{c^2}e^{-x} & \text{for } x >> 1\text{: Wien approximation;} \end{cases}$$

$$B_\lambda = \begin{cases} \frac{2hc^2}{\lambda^5}\frac{1}{e^x - 1} & \text{in general;} \\ \frac{2hc^2}{\lambda^5 x} = \frac{2c}{\lambda^4}kT & \text{for } x << 1\text{; Rayleigh-Jeans law;} \\ \frac{2hc^2}{\lambda^5}e^{-x} & \text{for } x >> 1\text{: Wien approximation;} \end{cases}$$

Redaction: Jeffery, 2018jan01

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002 qfull 00510 1 3 0 easy math: Debye function and blackbody radiation results

2. The total Debye function (i.e., the sum of the first and second Debye functions) is

$$D_z = \int_0^\infty \frac{x^z}{e^x - 1}\,dx = z!\zeta(z+1) \ ,$$

(e.g., Wolfram Mathworld: Debye functions; Wikipedia: Debye function) where the factorial function

$$z! = \begin{cases} \int_0^\infty x^z e^{-x}\,dx = z(z-1)! & \text{for } z \text{ not a negative integer and also not 0 for the second form;} \\ n! & \text{for integer } n \geq 0; \\ \sqrt{\pi} & \text{for } z = -1/2; \\ \frac{(2z)!!}{2^{(z+1/2)}}\sqrt{\pi} & \text{for half-integer } z \geq 1/2; \end{cases}$$

and Riemann zeta function (without anayltic continuation considered)

$$\zeta(s) = \begin{cases} \displaystyle\sum_{\ell=1}^{\infty} \frac{1}{\ell^s} & \text{in general;} \\ \zeta(1) = \displaystyle\sum_{\ell=1}^{\infty} \frac{1}{\ell} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots & \text{the divergent harmonic series (Ar-279);} \\ \zeta(2) = \dfrac{\pi^2}{6} = \dfrac{\pi^2}{2\cdot 3} = 1.6449340668482264364724151666460\ldots & \\ \zeta(3) = 1.2020569031595942853997381615114\ldots & \\ \zeta(4) = \dfrac{\pi^4}{90} = \dfrac{\pi^4}{2\cdot 3^2\cdot 5} = 1.0823232337111381915160036965411\ldots & \\ \zeta(5) = 1.0369277551433699263313654864570\ldots & \\ \zeta(6) = \dfrac{\pi^6}{945} = \dfrac{\pi^6}{3^3\cdot 5\cdot 7} = 1.0173430619844491397145179297909\ldots & \\ \zeta(7) = 1.0083492773819228268397975498497\ldots & \\ \zeta(8) = \dfrac{\pi^8}{9450} = \dfrac{\pi^8}{2\cdot 3^3\cdot 5^2\cdot 7} = 1.0040773561979443393786852385087\ldots & \\ \zeta(9) = 1.0020083928260822144178527692324\ldots & \\ \approx 1 + \displaystyle\int_{3/2}^{\infty} x^{-s}\,dx = 1 + \frac{(2/3)^{s-1}}{s-1} & \text{integral approximation;} \\ 1 + \dfrac{1}{2^s} & \text{asymptotic form as } s \to \infty. \end{cases}$$

(e.g., Wikipedia: Riemann zeta function; OEIS: Riemann zeta function).

a) Prove $D_z = z!\zeta(z+1)$.

b) Determine the general moment formula M_n (where n is the moment power) for the distribution $f(x) = Ax^z/(e^x - 1)$, where A is the normalization constant which you must determine too. Specialize for $n = 0$ (the normalization), $n = 1$ (the mean), and $n = 2$. Determine the general formula for the variance σ^2

c) From the Planck's law specific intensity,

$$B_\nu = \frac{2hv^3}{c^2}\frac{1}{e^x - 1}\ , \qquad \text{where} \qquad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda}\ ,$$

show the total energy density of a blackbody radiation field is

$$\epsilon = aT^4\ ,$$

where

$$a = \frac{8\pi^5 k^4}{15h^3c^3} = (7.5657332500339\ldots) \times 10^{-16}\,\mathrm{J/m^3/K^4} = 1\,\mathrm{J/m^3} \times \left(\frac{1}{6029.6164961230K}\right)^4$$

is the radiation density constant. Remember to change an isotropic specific intensity into a density you must multiply by $4\pi/c$.

d) Show that the mean photon energy of blackbody radiation field is

$$E = \frac{\zeta(4)}{\zeta(3)}(3kT) = (2.70117803291906\ldots) \times kT$$
$$= 2.32769513 \times 10^{-4}\,\mathrm{eV} \times T = 1\,\mathrm{eV} \times \left(\frac{T}{4296.09525\ldots\,\mathrm{K}}\right)\ ,$$

where $k = (0.8617333262\ldots) \times 10^4\,\mathrm{eV/K}$.

e) It is quite possible to have a radiation field with a Planck law spectrum, but not a blackbody radiation field energy density. Recall, an isotropic blackbody radiation field has energy density

$$\epsilon_\nu = \frac{4\pi}{c} B_\nu \ .$$

Now say for example, say you have blackbody radiator sphere of radius R and you are a distance $r \geq R$ from the center. The energy density at r is $W = \Omega/(4\pi)$ times that of isotropic blackbody radiation field where Ω is the solid angle subtended by the sphere at r. The effect is called geometrical dilution and, of course, is approximately true of stars. Show that the geometrical dilution factor

$$W = \frac{\Omega}{4\pi} = \frac{1}{2}\left[1 - \sqrt{1 - \left(\frac{R}{r}\right)^2}\,\right]$$

(Mi-120). **Hint:** Drawing a diagram may help.

SUGGESTED ANSWER:

a) Behold:

$$\begin{aligned} D_z &= \int_0^\infty \frac{x^z}{e^x - 1}\,dx = \int_0^\infty x^z e^{-x}\left(\sum_{\ell=0}^\infty e^{-\ell x}\right)dx = \sum_{\ell=0}^\infty \int_0^\infty x^z e^{-(\ell+1)x}\,dx \\ &= \sum_{\ell=0}^\infty \frac{1}{(\ell+1)^{z+1}}\int_0^\infty t^z e^{-t}\,dt = z!\sum_{\ell=1}^\infty \frac{1}{\ell^{z+1}} = z!\zeta(z+1) \qquad \text{QED.} \end{aligned}$$

b) Behold:

$$M_n = A\int_0^\infty \frac{x^{z+n}}{e^x - 1}\,dx = \begin{cases} \dfrac{(z+n)!\zeta(z+n+1)}{z!\zeta(z+1)} & \text{in general;} \\ 1 & \text{for normalization } n = 0; \\ (z+1)\dfrac{\zeta(z+2)}{\zeta(z+1)} & \text{for the mean } n = 1; \\ (z+2)(z+1)\dfrac{\zeta(z+3)}{\zeta(z+1)} & \text{for } n = 2. \end{cases}$$

For the variance,

$$\sigma^2 = M_2 - M_1^2 = \left[\frac{(z+1)}{\zeta(z+1)}\right]^2\left[\left(\frac{z+2}{z+1}\right)\zeta(z+3)\zeta(z+1) - \zeta(z+2)^2\right] \ .$$

c) Behold:

$$\begin{aligned} \epsilon &= \frac{4\pi}{c}\int_0^\infty B_\nu\,d\nu = \frac{4\pi}{c}\int_0^\infty \frac{2h\nu^3}{c^2}\frac{1}{e^x - 1}\,d\nu \\ &= \frac{4\pi}{c}\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \frac{4\pi}{c}\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4[3!\zeta(4)] = \frac{4\pi}{c}\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4\frac{\pi^4}{15} \\ &= \frac{8\pi^5 k^4}{15h^3c^3} \qquad \text{QED.} \end{aligned}$$

d) Behold:

$$\begin{aligned} E &= \frac{\int_0^\infty B_\nu\,d\nu}{\int_0^\infty B_\nu/(h\nu)\,d\nu} = kT\frac{3!\zeta(4)}{2!\zeta(3)} = \frac{\zeta(4)}{\zeta(3)}(3kT) = (2.70117803291906\ldots) \times kT \\ &= (2.32769513\ldots) \times 10^{-4}\,\mathrm{eV} \times T = 1\,\mathrm{eV} \times \left(\frac{T}{4296.09525\ldots\,\mathrm{K}}\right) \qquad \text{QED.} \end{aligned}$$

e) Let $\mu = \cos\theta$, where θ is the angle measured from the radial direction from the sphere center. Note that

$$d\Omega = \sin\theta\, d\theta\, d\phi = -d\mu\, d\phi$$

and the cosine of the angle from the radial direction to the limb of the sphere is

$$\mu = \cos\theta = \frac{\sqrt{r^2 - R^2}}{r} = \sqrt{1 - \left(\frac{R}{r}\right)^2} \; .$$

Behold:

$$W = \frac{\Omega}{4\pi} = \frac{1}{4\pi}\int_\mu^1 \int_0^{2\pi} d\mu\, d\phi = \frac{1}{2}(1-\mu) = \frac{1}{2}\left[1 - \sqrt{1 - \left(\frac{R}{r}\right)^2}\right] \qquad \text{QED.}$$

Fortran-95 Code

```
          print*
          sigma=5.670374419e-8_np    ! exact MKS
https://physics.nist.gov/cuu/Constants/Table/allascii.txt
          clight=2.99792458e8_np     ! exact
https://physics.nist.gov/cuu/Constants/Table/allascii.txt
          boltev=0.8617333262e-4_np  ! exact but irr
https://physics.nist.gov/cuu/Constants/Table/allascii.txt
    !            1 23456789a1
          radcon=4.0_np*sigma/clight
          tem_radcon=radcon**(-0.25_np)
          print*,'radcon,tem_radcon'
          print*,radcon,tem_radcon
    ! 7.56573325003392847185E-0016   6029.6164961230119483
    ! 1 23456789a123456              1234 56789a123456
          zeta3=1.2020569031595942853997381615114_np
          zeta4=1.0823232337111381915160036965411_np
          coef=3.0_np*zeta4/zeta3
          coefev=coef*boltev
          tem_fid=1/coefev
          print*,'coef,coefev,tem_fid'
          print*,coef,coefev,tem_fid
    ! 2.7011780329190638961          2.32769513096571802645E-0004
4296.0952518945998619
    ! 1 23456789a123456              1 23456789a1                   1234 56789a1
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Redaction: Jeffery, 2018jan01

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010 qfull 00110 1 3 0 easy math: The cosmic evolution of CMB/CRF

3. The cosmic background radiation (CBR)(which in the modern observable universe is mostly the cosmic microwave background (CMB)) conserves photon number to good approximation. This means photon number density n varies with time.

   a) Prove that the energy density of the CBR obeys

   $$\epsilon = \epsilon_0 \left(\frac{a_0}{a}\right)^4 \; ,$$

   where 0 refers to the modern observable universe or any other reference cosmic time and a is the cosmic scale factor.

   b) Assume that the CBR can be parameterized by

   $$\epsilon = a_{\rm R} T^4 \; ,$$

where $a_{\rm R}$ is the radiation density constant (usually symbolized by a) and T is a parameter that would be temperature if the CBR had a Planck-law (i.e., blackbody) spectrum. Show that

$$T = T_0 \left(\frac{a_0}{a}\right) \ .$$

c) Planck's law (AKA the blackbody specific intensity spectrum) in the frequency represenation is

$$B_\nu = \frac{2hv^3}{c^2}\frac{1}{e^x - 1} \ , \qquad \text{where} \qquad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda} \ .$$

Show that the CBR obeys this law as the observable universe evolves provided it obeys it at the fiducial time and we define temperature evolution to obey the rule found in part (b). **Hint:** The photons in a frequency bin stay in that frequency bin as the universe evolves, and so obey the same energy scaling as the overall CBR. Thus at a general time, we have

$$I_\nu \, d\nu = \left(\frac{a_0}{a}\right)^4 B_{\nu_0} \, d\nu_0 \ ,$$

where we have indeed assumed the fiducial time has a Planck-law spectrum. The proof requires showing that $I_\nu \, d\nu = B_\nu \, d\nu$ with the temperature evolution obeying the rule found in part (b).

SUGGESTED ANSWER:

a) Since photon number is conserved, photon density n goes as $1/a^3$. Now since photon energy E goes as $1/a$, we must have

$$\epsilon \propto nE \propto \frac{1}{a^3}\frac{1}{a} = \frac{1}{a^4} \ , \qquad \text{and so} \qquad \epsilon = \epsilon_0 \left(\frac{a_0}{a}\right)^4 \qquad \text{QED.}$$

b) We have,

$$\epsilon = a_{\rm R}T^4 \qquad a_{\rm R}T^4 = \epsilon_0 \left(\frac{a_0}{a}\right)^4 \qquad T \propto \frac{1}{a} \qquad T = T_0 \left(\frac{a_0}{a}\right) \qquad \text{QED.}$$

c) Behold:

$$\begin{aligned} I_\nu \, d\nu &= \left(\frac{a_0}{a}\right)^4 B_{\nu_0} \, d\nu_0 = \left(\frac{a_0}{a}\right)^4 \frac{2h\nu_0^3}{c^2}\frac{1}{e^{x_0} - 1} \, d\nu_0 \\ &= \frac{2h\nu^3}{c^2}\frac{1}{e^{x_0} - 1} \, d\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^x - 1} \, d\nu \\ &= B_\nu \, d\nu \ , \end{aligned}$$

provided we define T by

$$x_0 = \frac{h\nu_0}{kT_0} = \frac{h\nu(a/a_0)}{kT_0} = \frac{h\nu}{kT_0(a_0/a)} = \frac{h\nu}{kT} \qquad \text{or} \qquad T = T_0 \left(\frac{a_0}{a}\right)$$

which is just the rule we found in part (b): QED.

Redaction: Jeffery, 2018jan01

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010 qfull 00310 1 3 0 easy math: recombination studied

4. Let's consider the recombination of the cosmic radiation field: i.e., recombination.

a) Consider the differential equation

$$\frac{dN_e}{dt} = -CN_e^2 + CN_{\rm I}(N_{\rm H} - N_e) \ .$$

This is very simplified equation for recombination assuming a pure hydrogen gas with number density $N_{\rm H}$ and ionizing photon density $N_{\rm I}$: both we assume to be constant over the short time scales. The N_e is the electron density which is also the hydrogen ion density by charge conservation. The two C's are rate coefficients which are equal by a detailed balancing argument that yours truly is none too certain of. The products of the densities arise since the reactions are fluxes of one kind of particle on density of another. Find the steady-state solution in terms of $X = N_e/N_{\rm H}$ and $R = N_{\rm I}/N_{\rm H}$ and argue why it must be asymptotically approached as time goes to infinity.

Actually, the idea is that the steady-state solution is really a quasistatic process: "a thermodynamic process that happens slowly enough for the system to remain in internal equilibrium." We are crudely/vaguely attempting to understand recombination in this question. But we don't get too far.

b) Find the limiting forms of solution X for $R \to 0$ (to 1st order in small R), $R = 1$, and $R \to \infty$ to first order in small $1/R$). What is special about $X(R = 1)$ from a number point of view?

c) For the nonce, let's define the recombination temperature of the cosmic radiation field by $R(T) = 1$. Let N be the photon density, we have

$$1 = R = \frac{N_{\rm I}}{N_{\rm H}} = \frac{N_{\rm I}/N}{N_{\rm H}/N} = \frac{1}{\eta} f_{\rm I} = \frac{1}{\eta} \frac{D_2^{(2)}(x)}{D_n} \approx \frac{1}{\eta} \frac{e^{-x} x^2}{2\zeta(3)} ,$$

where we have approximated the second Debye function by leading term which is valid for $x >> 1$ and where $x = E_{\rm R}/(kT)$ where $E = 13.605693009(84)\,{\rm eV}$ is the Rydberg energy (i.e., the ionization energy of hydrogen) and T is the recombination temperature that we are solving for. The baryon-to-photon ratio $\eta = 6 \times 10^{-10}$ for a fiducial value, $\zeta(3) = 1.202056903159594285399738161511 4\ldots$, and $k = 0.86173303 \times 10^{-4}\,{\rm eV}$.

Solve for x by iteration and then determine T. Remember a iteration formula tends to converge/diverge when its slope is low/high relative to 1. You could write a small computer program to do the solution. **Hint:** In a test *mise en scène*, just do the zeroth order solution: i.e., no iteration.

SUGGESTED ANSWER:

a) Behold:

$$0 = -CN_e^2 + CN_{\rm I}(N_{\rm H} - N_e) \qquad 0 = X^2 + R(1 - X) \qquad X = \frac{R \pm \sqrt{R^2 + 4R}}{-2} \qquad X = \frac{\sqrt{R^2 + 4R} - R}{2} ,$$

where only the positive solution is physically relevant.

First, since the differential equation is a 1st order one with no special features that would give a stationary point at a finite time, it can only have stationary points at $t = \pm\infty$: i.e., asymptotic solutions. Second, we note that if right-hand side is positive/negative, then N_e will increase/decrease with time but then the right-hand side will go to zero and the N_e will have a stationary point. But by the first point, this can only be at $t = \infty$. So our solution is the asymptotic solution: i.e., the steady-state solution.

b) Behold:

$$X = \begin{cases} \dfrac{\sqrt{R^2 + 4R} - R}{2} & \text{in general;} \\[2ex] \dfrac{R}{2} & \text{for } R << 1\text{, and note the ionized fraction is not } R \text{ in this limit;} \\[2ex] \dfrac{\sqrt{5} - 1}{2} = 0.618033988749894848204586834365638117720\ldots & \text{which is the golden ratio minus 1;} \\[2ex] 1 - \dfrac{1}{R} & \text{for } R \to \infty. \end{cases}$$

c) A good guess at convergent iteration formula is

$$x = -\ln[2\zeta(3)\eta] + 2\ln(x) \qquad \text{since} \qquad \frac{dx}{dx} = \frac{2}{x} << 1$$

for x rather large which it probably is. The computer program is displayed below. The values obtained are:

$$x = 26.9444659685049923755 \qquad \text{and} \qquad T = 5859.74\,\text{K} .$$

The recombination temperature (as we've defined it for the nonce) is not decoupling temperature, but should be of the same order of magnitude. The fiducial decoupling temperature is 3000 K. So our result is, indeed, not so far off.

Fortran-95 Code

```
          print*
          zeta3=1.2020569031595942853997381615114_np !
http://oeis.org/wiki/Riemann_zeta_function
          eryd=13.605 693 009_np  !  (84)
https://physics.nist.gov/cuu/Constants/Table/allascii.txt
          boltev=0.86173303e-4_np  ! (50)
https://physics.nist.gov/cuu/Constants/Table/allascii.txt
          eta=6e-10_np  ! fiducial baryon-to-photon ratio
                        !
https://en.wikipedia.org/wiki/Big_Bang_nucleosynthesis#Characteristics
          x0=-log(2.0_np*zeta3*eta)
          x=x0
          i=0
          do
           i=i+1
           xold=x
           x=x0+2.0_np*log(x)
           if(abs(xold-x)/x .le. 1.e-12_np) exit
           print*,i,xold,x
          end do
          trec=eryd/(boltev*x)
          print*,'x0,x,trec'
          print*,x0,x,trec
    ! 20.3569101047609651092         26.9444659685049923755
5859.74001252925990268
```

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