

Cosmology

Name:

Homework 10: The Cosmic Microwave Background

1. Specific intensity and related quantities (e.g., energy density per unit wavelength) are conventionally given in three representations: photon energy representation I_E , frequency representation I_ν , and wavelength representation I_λ . These representations are related by differential expression

$$I_E dE = I_\nu d\nu = I_\lambda (-d\lambda) ,$$

where the minus sign is occasionally omitted if one knows what one means—which is that a differential increase in photon energy/frequency corresponds to a differential decrease in wavelength.

- a) As well as the three conventional representations, there is a hybrid representation

$$EI_E = \nu I_\nu = \lambda I_\lambda$$

which has the same value whichever of E , ν , or λ is used as the independent variable. Prove the hybrid representation equality. **Hint:** You will have use differentials of the logarithm of the independent variables (e.g., $d[\ln(E)]$) and make use of the de Broglie relations $E = h\nu = hc/\lambda$.

- b) Suggest two or three reasons why people might want to use the hybrid representation for graphing.
c) Planck's law (AKA the blackbody specific intensity spectrum) in the frequency representation is

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^x - 1} , \quad \text{where} \quad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda} .$$

Derive the energy representation B_E , wavelength representation B_λ , and the hybrid representation $EB_E = \nu B_\nu = \lambda B_\lambda$ in E , ν and λ forms.

- d) Derive the Rayleigh-Jeans law (small x , small E , small ν , large λ approximation) and the Wien approximation (large x , large E , large ν , small λ approximation) for B_E , B_ν , and B_λ **Hint:** This pretty easy albeit tedious.
2. The total Debye function (i.e., the sum of the first and second Debye functions) is

$$D_z = \int_0^\infty \frac{x^z}{e^x - 1} dx = z! \zeta(z+1) ,$$

(e.g., Wolfram Mathworld: Debye functions; Wikipedia: Debye function) where the factorial function

$$z! = \begin{cases} \int_0^\infty x^z e^{-x} dx = z(z-1)! & \text{for } z \text{ not a negative integer and also not } 0 \text{ for the second form;} \\ n! & \text{for integer } n \geq 0; \\ \sqrt{\pi} & \text{for } z = -1/2; \\ \frac{(2z)!!}{2^{(z+1/2)}} \sqrt{\pi} & \text{for half-integer } z \geq 1/2; \end{cases}$$

and Riemann zeta function (without analytic continuation considered)

$$\zeta(s) = \left\{ \begin{array}{l} \sum_{\ell=1}^{\infty} \frac{1}{\ell^s} \\ \zeta(1) = \sum_{\ell=1}^{\infty} \frac{1}{\ell} = 1 + \frac{1}{2} + \frac{1}{3} + \dots \\ \zeta(2) = \frac{\pi^2}{6} = \frac{\pi^2}{2 \cdot 3} = 1.644934066848226436472415166646\dots \\ \zeta(3) = 1.2020569031595942853997381615114\dots \\ \zeta(4) = \frac{\pi^4}{90} = \frac{\pi^4}{2 \cdot 3^2 \cdot 5} = 1.082323233711138191516003696541\dots \\ \zeta(5) = 1.036927755143369926331365486457\dots \\ \zeta(6) = \frac{\pi^6}{945} = \frac{\pi^6}{3^3 \cdot 5 \cdot 7} = 1.0173430619844491397145179297909\dots \\ \zeta(7) = 1.008349277381922826839797549849\dots \\ \zeta(8) = \frac{\pi^8}{9450} = \frac{\pi^8}{2 \cdot 3^3 \cdot 5^2 \cdot 7} = 1.004077356197944339378685238508\dots \\ \zeta(9) = 1.002008392826082214417852769232\dots \\ \approx 1 + \int_{3/2}^{\infty} x^{-s} dx = 1 + \frac{(2/3)^{s-1}}{s-1} \\ 1 + \frac{1}{2^s} \end{array} \right. \begin{array}{l} \text{in general;} \\ \text{the divergent} \\ \text{harmonic series} \\ \text{(Ar-279);} \\ \\ \\ \\ \\ \\ \\ \\ \text{integral} \\ \text{approximation;} \\ \text{asymptotic form as} \\ s \rightarrow \infty. \end{array}$$

(e.g., Wikipedia: Riemann zeta function; OEIS: Riemann zeta function).

- Prove $D_z = z!\zeta(z+1)$.
- Determine the general moment formula M_n (where n is the moment power) for the distribution $f(x) = Ax^z/(e^x - 1)$, where A is the normalization constant which you must determine too. Specialize for $n = 0$ (the normalization), $n = 1$ (the mean), and $n = 2$. Determine the general formula for the variance σ^2 .
- From the Planck's law specific intensity,

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^x - 1}, \quad \text{where} \quad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda},$$

show the total energy density of a blackbody radiation field is

$$\epsilon = aT^4,$$

where

$$a = \frac{8\pi^5 k^4}{15h^3 c^3} = (7.5657332500339\dots) \times 10^{-16} \text{ J/m}^3/\text{K}^4 = 1 \text{ J/m}^3 \times \left(\frac{1}{6029.6164961230\text{K}} \right)^4$$

is the radiation density constant. Remember to change an isotropic specific intensity into a density you must multiply by $4\pi/c$.

- Show that the mean photon energy of blackbody radiation field is

$$\begin{aligned} E &= \frac{\zeta(4)}{\zeta(3)}(3kT) = (2.70117803291906\dots) \times kT \\ &= 2.32769513 \times 10^{-4} \text{ eV} \times T = 1 \text{ eV} \times \left(\frac{T}{4296.09525\dots \text{K}} \right), \end{aligned}$$

where $k = (0.8617333262\dots) \times 10^4$ eV/K.

- e) It is quite possible to have a radiation field with a Planck law spectrum, but not a blackbody radiation field energy density. Recall, an isotropic blackbody radiation field has energy density

$$\epsilon_\nu = \frac{4\pi}{c} B_\nu .$$

Now say for example, say you have blackbody radiator sphere of radius R and you are a distance $r \geq R$ from the center. The energy density at r is $W = \Omega/(4\pi)$ times that of isotropic blackbody radiation field where Ω is the solid angle subtended by the sphere at r . The effect is called geometrical dilution and, of course, is approximately true of stars. Show that the geometrical dilution factor

$$W = \frac{\Omega}{4\pi} = \frac{1}{2} \left[1 - \sqrt{1 - \left(\frac{R}{r}\right)^2} \right]$$

(Mi-120). **Hint:** Drawing a diagram may help.

3. The cosmic background radiation (CBR)(which in the modern observable universe is mostly the cosmic microwave background (CMB)) conserves photon number to good approximation. This means photon number density n varies with time.

- a) Prove that the energy density of the CBR obeys

$$\epsilon = \epsilon_0 \left(\frac{a_0}{a}\right)^4 ,$$

where 0 refers to the modern observable universe or any other reference cosmic time and a is the cosmic scale factor.

- b) Assume that the CBR can be parameterized by

$$\epsilon = a_R T^4 ,$$

where a_R is the radiation density constant (usually symbolized by a) and T is a parameter that would be temperature if the CBR had a Planck-law (i.e., blackbody) spectrum. Show that

$$T = T_0 \left(\frac{a_0}{a}\right) .$$

- c) Planck's law (AKA the blackbody specific intensity spectrum) in the frequency representation is

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^x - 1} , \quad \text{where} \quad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda} .$$

Show that the CBR obeys this law as the observable universe evolves provided it obeys it at the fiducial time and we define temperature evolution to obey the rule found in part (b). **Hint:** The photons in a frequency bin stay in that frequency bin as the universe evolves, and so obey the same energy scaling as the overall CBR. Thus at a general time, we have

$$I_\nu d\nu = \left(\frac{a_0}{a}\right)^4 B_{\nu_0} d\nu_0 ,$$

where we have indeed assumed the fiducial time has a Planck-law spectrum. The proof requires showing that $I_\nu d\nu = B_\nu d\nu$ with the temperature evolution obeying the rule found in part (b).

4. Let's consider the recombination of the cosmic radiation field: i.e., recombination.

- a) Consider the differential equation

$$\frac{dN_e}{dt} = -CN_e^2 + CN_I(N_H - N_e) .$$

This is very simplified equation for recombination assuming a pure hydrogen gas with number density N_H and ionizing photon density N_I : both we assume to be constant over the short time

scales. The N_e is the electron density which is also the hydrogen ion density by charge conservation. The two C 's are rate coefficients which are equal by a detailed balancing argument that yours truly is none too certain of. The products of the densities arise since the reactions are fluxes of one kind of particle on density of another. Find the steady-state solution in terms of $X = N_e/N_H$ and $R = N_I/N_H$ and argue why it must be asymptotically approached as time goes to infinity.

Actually, the idea is that the steady-state solution is really a quasistatic process: “a thermodynamic process that happens slowly enough for the system to remain in internal equilibrium.” We are crudely/vaguely attempting to understand recombination in this question. But we don't get too far.

- b) Find the limiting forms of solution X for $R \rightarrow 0$ (to 1st order in small R), $R = 1$, and $R \rightarrow \infty$ to first order in small $1/R$. What is special about $X(R = 1)$ from a number point of view?
- c) For the nonce, let's define the recombination temperature of the cosmic radiation field by $R(T) = 1$. Let N be the photon density, we have

$$1 = R = \frac{N_I}{N_H} = \frac{N_I/N}{N_H/N} = \frac{1}{\eta} f_I = \frac{1}{\eta} \frac{D_2^{(2)}(x)}{D_n} \approx \frac{1}{\eta} \frac{e^{-x} x^2}{2\zeta(3)},$$

where we have approximated the second Debye function by leading term which is valid for $x \gg 1$ and where $x = E_R/(kT)$ where $E = 13.605693009(84)$ eV is the Rydberg energy (i.e., the ionization energy of hydrogen) and T is the recombination temperature that we are solving for. The baryon-to-photon ratio $\eta = 6 \times 10^{-10}$ for a fiducial value, $\zeta(3) = 1.2020569031595942853997381615114\dots$, and $k = 0.86173303 \times 10^{-4}$ eV.

Solve for x by iteration and then determine T . Remember a iteration formula tends to converge/diverge when its slope is low/high relative to 1. You could write a small computer program to do the solution. **Hint:** In a test *mise en scène*, just do the zeroth order solution: i.e., no iteration.