Cosmology **NAME:**

Homework 5: Advance Solutions of the Friedmann Equation

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005 qfull 00630 1 3 0 easy math: Einstein universe, einstein universe

1. The Einstein universe (proposed by Einstein in 1917) was the first cosmological model derived consistently from a physical theory (i.e., general relativity) and was the beginning of modern cosmology. Einstein assumed the cosmological principle (i.e., a homogeneous, isotropic universe) and represented the mass-energy by a pressureless perfect fluid where the density scaled as a^{-3}. In modern cosmology jargon, this kind of perfect fluid is called "matter" and approximates ordinary baryonic matter and dark matter For cosmological purposes, matter has approximately zero kinetic energy relative its local comoving frame.

   Einstein believing in 1917 that the universe was one of stars (which seemed on average at rest) and not galaxies wanted a static model, but found that impossible with his field equations as originally formulated (O'Raifeartaigh et al. 2017). So he added the cosmological constant term Λ to the field equations which was the simplest possible modification and had no significant effect on smaller-than-cosmological-scale phenomena. The Einstein universe he obtained is a finite, boundless, positively curved universe or hyperspherical universe. It is geometrically the 3-dimensional surface of the a 3-sphere (which is actually a 4-dimensional sphere in Euclidean or flat space). The distance to return to the same point along a geodesic is $2\pi a_0$, where a_0 is the Gaussian curvature radius a hyperspherical universe. (CL-11–12). For considering the Einstein universe, a_0 is not the conventional dimensionless quantity but a physical proper distance with units of length.

   Einstein in 1931 abandoned the Einstein universe since observations showed an expanding universe and because the Einstein universe had been shown to be unstable by Eddington in 1930 (O'Raifeartaigh et al. 2017 p. 36, 41).

   Note that Einstein did not have the Friedmann equation and acceleration equation when he derived the Einstein universe. He used a general relativity directly and followed a "rough and winding road" (O'Raifeartaigh et al. 2017, p. 18).

   In this problem, we investigate the Einstein universe. There are parts a,b,c,d,e,f,g,h. In exam environments, do **ONLY** parts a,b,c,d.

   NOTE: This question has **MULTIPLE PAGES** on an exam.

   a) The Friedmann equation and acceleration equation in forms appropriate for solving for the Einstein universe and investigating its stability are

   $$H^2 = \left(\frac{\dot{x}}{x}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a_0^2 x^2} + \frac{\Lambda}{3} = \frac{8\pi G\rho_0}{3}\left[\Omega_{\rm M}x^{-3} + \Omega_k x^{-2} + \Omega_\Lambda\right]$$

   and

   $$\frac{\ddot{x}}{x} = -\frac{4\pi G\rho}{3} + \frac{\Lambda}{3} = -\frac{4\pi G\rho_0}{3}\left[\Omega_{\rm M}x^{-3} - 2\Omega_\Lambda\right]$$

   (Li-55 *mutatis mutandis*), where $x = a/a_0$, a_0 is the Gaussian curvature radius of the Einstein universe (as aforesaid), ρ_0 is the density of Einstein universe, $k = 1$ for a positive curvature universe,

   $$\Omega_{\rm M} = 1\ , \qquad \Omega_k = -\frac{kc^2}{a_0^2(8\pi G\rho_0/3)}\ , \qquad \text{and} \qquad \Omega_\Lambda = \frac{\Lambda}{8\pi G\rho_0}\ .$$

   Note we cannot use the Hubble parameter H in defining the density parameter Ω_i quantities since $H = 0$ for the Einstein universe.

   The Einstein universe has $x = 1$, $\dot{x} = 0$, $\ddot{x} = 0$ and $\rho = \rho_0$. Given the Einstein-universe values, determine formula for Λ from the first form of the acceleration equation and the numerical value of Ω_Λ from the second form.

   b) Given the Einstein-universe values, determine the formula for a_0 as function of ρ_0 and then the formula for a_0 as a function of Λ. **Hint:** Start from the second form of the Friedmann equation and recall the given formula for Ω_k.

   c) Given $G = 6.67430(15) \times 10^{-11}$ MKS, vacuum light speed $c = 2.99792458 \times 10^8$ m/s, $\rho_0 = 0.85 \times 10^{-26}$ kg/m^3 (which is suggest value of the critical density circa 2021), and 1 Gpc = $(3.085677581\ldots) \times 10^{25}$ m, calculate the Gaussian curvature radius a_0 in units of gigaparsecs (Gpc). You can use your phone for the calculations—but only for those.

d) Now write the Friedmann equation in the dimensionless form

$$\frac{dx}{d\tau} = \pm\sqrt{f(x)} \ ,$$

where the dimensionless time τ is given by

$$\tau = t\sqrt{\frac{8\pi G\rho_0}{3}} \ .$$

Sketch a plot the radicand $f(x)$ for $x \geq 0$ going left from $x = 1$ to $x = 0$ and right from $x = 1$ to $x = \infty$. Using the first two derivatives of $f(x)$ as a function of x (not τ) prove that the Einstein universe (i.e., the $x = 1$ case) is a unique static universe for $x \geq 0$.

e) For the initial condition x_1 greater/less than 1 at τ_1 and the positive/negative case for $x' = \pm\sqrt{f(x)}$, describe the evolution of x with τ increasing and in particular what happens if $x \to 0$. Explain the evolutions and describe the stability of the Einstein universe to perturbations in these cases. **Hint:** It might help to draw a figure of the evolutions.

f) For the initial condition x_1 greater/less than 1 at τ_1 and the negative/positive case for $x' = \mp\sqrt{f(x)}$, describe the probable evolution of x with $\tau \to \infty$. Prove these evolutions and describe the stability of the Einstein universe to perturbations in these cases. **Hint:** The proof requires that you show that all orders of derivative of x are zero when x is stationary. You will need to determine the x'', x''', and $x^{(4)}$, notice some things about these orders of derivative, and add some explanatory words. Also, it might help to draw a figure of the evolutions.

g) From parts (d) and (e), what is the stability of the Einstein universe to general perturbations of a? Note a solution is unstable to general perturbations if it is unstable to any kind of perturbations.

h) Given all the answers to the other parts, discuss how an Einstein universe filled with real gas (including dark matter gas) and/or stars might evolve.

SUGGESTED ANSWER:

a) From the first and second forms of the acceleration equation, respectively, by inspection,

$$\Lambda = 4\pi G\rho_0 \qquad \text{and} \qquad \Omega_\Lambda = \frac{1}{2}\Omega_{\rm M} = \frac{1}{2} \ .$$

b) From the second form of the Friedmann equation,

$$\begin{aligned} 0 &= \Omega_{\rm M} + \Omega_k + \Omega_\Lambda & \Omega_k &= -\frac{3}{2} \\ -a_0^2\left(\frac{8\pi G\rho_0}{3c^2}\right) &= -\frac{2}{3} & a_0^2\left(\frac{4\pi G\rho_0}{c^2}\right) &= 1 \\ a_0 &= \frac{c}{\sqrt{4\pi G\rho_0}} = \frac{c}{\sqrt{\Lambda}} \end{aligned}$$

(CL-28).

c) Behold:

$$\begin{aligned} a_0 &= \frac{c}{\sqrt{4\pi G\rho_0}} \\ &= [(3.6387557\ldots)\,{\rm Gpc}] \times \sqrt{\left(\frac{6.67430(15)\times 10^{-11}\,{\rm MKS}}{G}\right)\left(\frac{0.85\times 10^{-26}\,{\rm kg/s}}{\rho_0}\right)} \ . \end{aligned}$$

This Gaussian curvature radius is significantly smaller than the Λ-CDM model observable universe radius currently about $r = 14.25\,$Gpc.

d) Behold:

$$\begin{aligned} \dot{x} &= \pm x\sqrt{\frac{8\pi G\rho_0}{3}}\sqrt{\Omega x^{-3} + \Omega_k x^{-2} + \Omega_\Lambda} = \pm\sqrt{\frac{8\pi G\rho_0}{3}}\sqrt{\frac{1}{x} - \frac{3}{2} + \frac{x^2}{2}} \\ \frac{dx}{d\tau} &= \pm\sqrt{f(x)} \ , \end{aligned}$$

where the radicand and its first two derivatives are

$$f(x) = \frac{1}{x} - \frac{3}{2} + \frac{x^2}{2} \qquad \frac{df}{dx} = -\frac{1}{x^2} + x \qquad \frac{d^2 f}{dx^2} = \frac{2}{x^3} + 1 \ .$$

You will have to imagine the plot. At $x = 1$, f(x) has a zero: i.e., $f(x = 1) = 0$. To the left of $x = 1$, $f(x)$ rises to infinity asymptotically as $1/x$. To the right of $x = 1$, $f(x)$ rises to infinity asymptotically as x^2.

As aforesaid, $f(x)$ has a zero at $x = 1$. From df/dx, we find

$$x_{\text{stationary}} = e^{i(2\pi n/3)}$$

where unique values only occur for $n = 0$, $n = 1$, and $n = 2$. The only real stationary point is for $n = 1$ which gives

$$x_{\text{stationary}} = 1 \ .$$

which is the location of the known zero. Since $d^2 f/dx^2 > 0$ at $x = 1$, we know $f(x)$ is a minimum there and, of course, it is the only real minimum $f(x)$ has.

Now $x = 1$ is the Einstein universe itself. There can be no other zeros for for $x \geq 0$ since $x = 1$ is the unique minimum of $f(x)$. Since there are no other zeros for $x \geq 0$, we see that there are no other static universes, just the Einstein universe.

Note for $x \leq 0$, there are no stationary points and $f(x)$ going leftward rises asymptotically as $1/x$ from negative infinity at $x = 0$ to 0 at $x = -2$ (which cannot be a physical universe), and then rises to positive infinity asymptotically as x^2.

e) For x_1 greater/less than 1, the positive/negative x slope $\pm\sqrt{f(x)}$ causes the x function to increase/decrease to infinity/zero as τ increases. In the case where x decreases, the slope when $x \to 0$ is infinite: this is a Big Crunch in cosmology jargon. Because of the divergence of the solutions from $x = 1$ (which is the Einstein universe itself), the Einstein universe is unstable for perburbations to x_1 greater/less than 1 and positive/negative slope $\pm\sqrt{f(x)}$.

f) For x_1 greater/less than 1, the negative/positive x slope $\mp\sqrt{f(x)}$ probably causes the x function to decrease/increase to 1 at $\tau \to \infty$: i.e., in this case x converges asymptotically to the Einstein universe. To prove this evolution, consider

$$\begin{aligned}
x' &= \pm\sqrt{f(x)} \\
x'' &= \pm\left(\frac{1}{2}\right)\frac{(-1/x^2 + x)x'}{\sqrt{f(x)}} = \left(\frac{1}{2}\right)\left(-\frac{1}{x^2} + x\right) \\
x''' &= \left(\frac{1}{2}\right)\left(\frac{2}{x^3} + 1\right)x' \\
x^{(4)} &= -\left(\frac{3}{x^4}\right)(x')^2 + \left(\frac{1}{2}\right)\left(\frac{2}{x^3} + 1\right)x'' = -\left(\frac{3}{x^4}\right)(x')^2 + \left(\frac{1}{2}\right)\left(\frac{2}{x^3} + 1\right)\left(\frac{1}{2}\right)\left(-\frac{1}{x^2} + x\right) \ .
\end{aligned}$$

Note that the shown orders of derivative are all zero for $x = 1$. Note also that whenever a factor of x'' turns up in a higher order of derivative, it can replaced by $(1/2)(-1/x^2 + x)$ which is zero for $x = 1$. Always making this replacement, all terms in higher orders of derivative will equal expressions with at least one factor of x' or one factor of $(1/2)(-1/x^2 + x)$, will have no higher orders of derivative of x, and will have no factor that is infinite for $x = 1$. Thus, all orders of derivative higher than 4 are also zero for $x = 1$. There seems no elegant way to show this, but it's clear enough.

Now if the x solutions reach $x = 1$ at say finite τ_2, there would have to be some curvature at τ_2 (i.e., at least one non-zero order of derivative at τ_2) since there are no discontinuities in the x solutions (except for $x = 0$). We conclude that x solutions for specified cases only reach $x = 1$ asymptotically as $\tau \to \infty$. Thus our original description of the evolutions is correct. Because of the asymptotic convergence of the solutions to $x = 1$ as $\tau \to \infty$, the Einstein universe is stable for perburbations to x_1 greater/less than 1 and negative/positive slope $\mp\sqrt{f(x)}$.

g) Since the Einstein universe is unstable to perturbations of a of the cases of part (d), it is unstable to general perburations of a even though it is stable to the special case of perturbations of a of part (e).

h) A realistic Einstein universe consisting of a real gas and/or stars will probably be unstable to local perturbations: e.g., density, velocity, or curvature perturbations. The initial conditions of the realistic Einstein universe can be left unexplained if you like since there are always unexplained initial conditions in cosmology no matter what you do. Yours truly guesses that some regions of the Einstein universe will start expanding and others contracting. The contracting regions will probably become gravitationally bound at some point and be supported by rotational kinetic energy. They would become something like galaxies or dark matter halos. However, the cosmological constant term will drive an overall expansion between the bound systems.

Superficially at least, the post-Einstein universe (as it can be called) would resemble the Big Bang universe, but without a Big Bang. It could have a very long phase of slow evolution before the nonlinear growth of density perturbations lead to galaxies or dark matter halos. This slow evolution would have been an attractive feature before the 1950s. In those days, the Hubble constant was measured to be of order 500 km/s/Mpc (e.g., Tammann 2005, arXiv:astro-ph/0512584, p. 6) implying a Hubble time of order 2 Gyr which was smaller than the age of the Earth estimate of order 3 Gyr made in the 1930s (see Wikipedia: Age of the Earth: Radiometric dating). In a Big Bang universe with negative acceleration (which is what you get without a positive cosmological constant or constant dark energy), the age of the universe is less than the Hubble time. So back before the 1950s, there was a tension between age of the Earth Hubble constant in a Big Bang universe which could be avoided with the post-Einstein universe. However, the Hubble constant was revised down to less than 200 km/s/Mpc in the 1950s (e.g., Tammann 2005, arXiv:astro-ph/0512584, p. 6) and this eliminates the attractive early slow evolution of the post-Einstein universe. Of course, the Big Bang universe provides a natural explanation for the cosmic microwave background and the primordial cosmic abundances of H, D, He, and Li. The post-Einstein universe if it started dense and hot enough could explain the cosmic microwave background, but not the primordial cosmic abundances. Both universes have unexplained initial conditions without another theory. The leading other theory since circa 1980 is inflation which works for the Big Bang theory, but probably not the post-Einstein universe. But to follow up the point from above about initial conditions, what set the initial conditions for the pre-inflaction universe?

In fact, the post-Einstein universe under the name of the Lemaître-Eddington universe was proposed in 1925 by Lemaître and supported by Eddington (e.g., Bondi 1960, p. 84–85, 117–121, 159, 175, 180). The Lemaître-Eddington universe itself fell out of favor circa 1935 for whatever reason (Bondi 1960, p. 119). However, the Lemaître universe (1931; see Bondi 1960, p. 84–85, 120–122, 165ff, 176, 180) which proposed a Big Bang phase evolving into an Einstein phase and then evolving into accelerating phase had a vogue from the 1931 to the 1950s. In fact, the Lemaître universe is a lot like the Λ-CDM model except with an Einstein phase and positive curvature. Maybe the Lemaître universe will make a comeback and relieve the Hubble tension of circa 2017–2030. This seems an unlikely hypothesis.

Fortran-95 Code

```
      print*
      print*,'CPB.'
!      pi=acos(-1.0_np)
      pi=3.14159265358979323846264338327950288419716939937510_np
!
!!23456789a123456789b123456789c123456789d123456789e123456789f123456789g12
!        ! https://en.wikipedia.org/wiki/Pi#Approximate_value_and_digits 51
digits
      grav=6.67430e-11_np
! http://en.wikipedia.org/wiki/Gravitational_constant MKS error (15):
!                    ! so 4 digit accurate, but there is controversy
      clight=2.99792458e8_np    ! light speed in m/s
!                               ! https://en.wikipedia.org/wiki/Speed_of_light
!      pc_m=(1.49597870700e11_np/(pi/(180.0_np*3600.0_np)))
      pc_m=9.6939420213600000e+16_np/pi
!      !
https://en.wikipedia.org/wiki/Parsec#Calculating_the_value_of_a_parsec
!      ! http://en.wikipedia.org/wiki/Astronomical_unit
```

```
        xgpc_m=pc_m*1.e+9_np  ! Also the conversion Mpc to m
        rho_c=0.85e-26_np     !
    ! %
https://en.wikipedia.org/wiki/Observable_universe#Estimates_based_on_critical_density
        a_0=clight/sqrt(4.0_np*pi*grav*rho_c)/xgpc_m
        print*,'a_0'
        print*,a_0
    ! 3.6387557188704396337   Seems to be right.
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Redaction: Jeffery, 2018jan01

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005 qfull 00810 1 3 0 easy math: 1st order DE rule

2. First order (ordinary) differential equations that are autonomous (meaning they have no explicit dependence on the independent variable) can only have stationary points at infinity (i.e., plus or minus infinity) and each such stationary point corresponds to a static solution. Hereafter for brevity, we call such differential equations 1st order DEs and the rule they obey the 1st order DE rule. The form of these 1st order DEs is

$$x' = f(x) \ ,$$

where x is the dependent variable and t is the independent variable and we assume $f(x)$ is infinitely differentiable and contains no fractional roots. There are exceptions to the 1st order DE rule. The ones known to yours truly are of the form

$$x' = \pm[g(x)]^P \ ,$$

where $P = (1 - 1/n)$ with $n \in [2, \infty)$ and we assume $g(x)$ is infinitely differentiable with respect to x. Note $g(x)$ may go negative as a function of x, but we assume it does not negative as function of t at stationary points. The most obvious and most important exception is for $n = 2$ (i.e., $P = 1/2$) which gives

$$x' = \pm[g(x)]^{1/2} \ ,$$

which is exampled by the Friedmann equation. In fact for $n \geq 3$, yours truly know of no interesting cases at all. There may other exceptions to the 1st order DE rule yours truly knows not of. In this problem, we only treat the cases that obey the 1st order DE rule.

NOTE: There are parts a,b,c,d.

a) Given x_i (or in the time variable t_i) is a stationary point of $x' = f(x)$ (i.e., $x'(x_i) = f(x_i) = f[x(t_i)] = 0$), prove without words that $x''(x_i) = 0$.

b) The part (a) answer gives the base case (or 1st step) for a proof by induction that all orders of derivative of x with respect to t at x_i (or in the time variable t_i) are zero. The proof follows by inspection if your math intuition is good enough. However, do a formal proof by induction. **Hint:** For the proof, you do **NOT**, in fact, need the full general Leibniz rule for the derivative of a product (Ar-558)

$$\frac{d^m(fg)}{dx^m} = \sum_{k=0}^{m} \binom{m}{k} \frac{d^k f}{dx^k} \frac{d^{m-k} g}{dx^{m-k}} \ .$$

Using it actually makes the proof a bit more tricky to follow. But you do need to know that the nth order derivative of x (i.e., $x^{(n)}$) is obtained by applying the general Leibniz rule for $m = n - 2$ to the result of the part (a) answer and that highest derivative of x on the right-hand side of that application is $x^{(n-1)}$. Note that $f(x)$ is general to the degree specified in the preamble, and so the proof is unchanged if any order of derivative $f(x)$ with respect to x is zero at x_i.

c) Given the part (b) result, give an argument for why the stationary point t_i must be all points (i.e., is actually a static solution) or at time equals infinity.

d) A 1st order DE system given a small perturbation from a static solution either asymtotically goes back to it (i.e., is asymptotic to it at positive infinity, and so is called stable) or grows away from it (i.e., is asymptotic to it at negative infinity, and so is called unstable). Assuming the df/dx is nonzero at x_i, prove without words that a 1st order DE system given a small perturbation (i.e., a perturbation Δx_0 which requires only 1st order expansion of $f(x)$ in small $\Delta x = x - x_i$) varies exponentially and determine the condition for stability.

SUGGESTED ANSWER:

a) Behold:

$$1)\qquad x' = f(x) \qquad\qquad 2)\qquad x'' = \frac{df}{dx}x' \qquad\qquad 3)\qquad x''(x_i) = \frac{df}{dx}x'(x_i) = 0\ .$$

b) Part (a) gave the first step of the proof by induction: i.e., that $x''(x_i) = 0$. The second step is assuming $x^{(j)}(x_i) = 0$ for all $j \in [1, n-1]$ and then for the third step expanding

$$x^{(n)} = \frac{d^{n-2}\left[(df/dx)x'\right]}{dt^{n-2}} = \sum_{k=0}^{n-2}\binom{n-2}{k}\frac{d^k(df/dx)}{dt^k}\frac{d^{n-2-k}x'}{dt^{n-2-k}}$$
$$= \sum_{k=0}^{n-2}\binom{n-2}{k}\frac{d^{n-2-k}(df/dx)}{dt^{n-2-k}}\frac{d^k x'}{dt^k}$$
$$= \frac{d^{n-2}(df/dx)}{dt^{n-2}}x' + \ldots + \frac{df}{dx}(x')^{(n-2)} = \frac{d^{n-1}f)}{dx^{n-1}}(x')^{n-1} + \ldots + \frac{df}{dx}(x')^{(n-2)}$$
$$= \text{terms all with factors of } (x')^{(j)} \text{ with } j \in [1, n-2]$$
$$= \text{terms all with factors } x^{(j)} \text{ with } j \in [1, n-1]$$
$$\text{which are all zero for } x = x_i \text{ by assumption}$$
$$x^{(n)}(x_i) = 0 \qquad \text{QED.}$$

Since the result is for general n, we have $x^{(n)}(x_i) = 0$ for all $n \geq 1$.

c) If all orders of derivative are zero at t_i, the solution of x must be constant to $\pm\infty$ with value x_i (i.e., must be a static solution x_i) or it is asymptotically constant at one of $\pm\infty$ where it is asymptotic to asymptote $x = x_i$.

d) Behold:

$$x' = f(x) = f(x_i) + \Delta x \left.\frac{df}{dx}\right|_{x_i} + \ldots = 0 + \Delta x \left.\frac{df}{dx}\right|_{x_i} + \ldots = \Delta x \left.\frac{df}{dx}\right|_{x_i} + \ldots\ ,$$

where $\Delta x = x - x_i$ and hereafter we set $R = df/dx|_{x_i}$ for niceness. For perturbation Δx_0 sufficiently small, we have the approximate 1st order DE and solution

$$1)\quad \frac{d\Delta x}{dt} = \Delta x R \qquad\qquad 2)\quad \frac{d\Delta x}{\Delta x} = R\,dt \qquad\qquad 3)\quad \frac{d(\pm\Delta x)}{(\pm\Delta x)} = \frac{d(|\Delta x|)}{|\Delta x|} = R\,dt$$
$$4)\quad \ln\left(\left|\frac{\Delta x}{\Delta x_0}\right|\right) = Rt \qquad\qquad 5)\quad |\Delta x| = |\Delta x_0|e^{Rt} \qquad\qquad 6)\quad \Delta x = \Delta x_0 e^{Rt}$$

where the upper case is for $\Delta x_0 > 0$ and the lower case is for $\Delta x_0 < 0$. Note we did not need the upper/lower case stuff if we just knew that the antiderivative of $1/y$ is always $\ln(|y|)$. From expression (5), we see that the exponential variation is away from the static solution for $R > 0$ and toward the static solution if $R < 0$. Thus, the condition for stability is $R < 0$ and the condition for instability is $R > 0$. If $R = 0$, then one must check what happens for the first higher order expansion term n of $f(x)$ where the nth order derivative coefficient $(d^n f/dx^n)|_{x_i} \neq 0$.

Redaction: Jeffery, 2018jan01

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005 qfull 00812 1 3 0 easy math: perturbation solutions for 1st order DEs

3. Consider the 1st order (ordinary, autonomous) differential equation

$$x' = f(x)\ ,$$

where x is the dependent variable and t is the independent variable and we assume $f(x)$ is infinitely differentiable and contains no fractional roots. The 1st order DE rule (as yours truly calls it) applies to this DE. We have $f(x_i) = 0$ and therefore x_i yields a constant solution and a stationary point at either of $\pm\infty$.

NOTE: There are parts a,b.

a) Assuming $(df/dx)(x_i) \neq 0$, solve without words for the 1st order perturbation solution in small $\Delta x = x - x_i$. Let Δx_0 be the initial perturbation, time zero is 0, and $R_1 = (df/dx)(x_i)$ for compactness. What is the condition for convergence/divergence in the future to the constant solution? What is the condition for convergence/divergence in the past to the constant solution? **Hint:** Recall the antiderivative of $1/y$ is always $\ln(|y|)$.

b) Now assume the lowest order nonzero coefficient in the expansion of $f(x)$ in small δx is $(d^k f/dx^k)(x_i)$ where $k \geq 2$. The write the solution only in terms of $|\Delta x|$ and $|\Delta x_0|$ since that seems most clear and start from the differential form

$$\frac{d|\Delta x|}{|\Delta x|^k} = hR_k\, dt \ ,$$

where for k even $h = \pm 1$ with upper case for $\Delta x > 0$ and lower case for $\Delta x < 0$ and for k odd $h = 1$, and $R_k = (d^k f/dx^k)(x_i)$ for compactness. Show why this differential form is correct before you use it.

c) What happens as $hR_k t$ **INCREASES/DECREASES** from 0? At what time t is there an infinity?

SUGGESTED ANSWER:

a) Behold:

$$1)\ \ \frac{d\Delta x}{dt} = \Delta x R_1 \qquad 2)\ \ \frac{d\Delta x}{\Delta x} = R_1\, dt \qquad 3)\ \ \ln\left(\left|\frac{\Delta x}{\Delta x_0}\right|\right) = R_1 t$$
$$4)\ \ |\Delta x| = |\Delta x_0| \exp(R_1 t) \qquad 5)\ \ \Delta x = \Delta x_0 \exp(R_1 t) \ .$$

As expression (5) shows convergence (divergence) in the future is given for $R_1 < 0$ ($R > 0$). As expression (5) shows convergence (divergence) in the past is given for $R_1 > 0$ ($R < 0$).

b) Behold:

$$1)\ \ \frac{d\Delta x}{dt} = \Delta x^k R_k \qquad 2)\ \ \frac{d\Delta x}{\Delta x^k} = R_k\, dt \qquad 3)\ \ \frac{d(\pm\Delta x)}{(\pm\Delta x)^k} = hR_k\, dt$$
$$4)\ \ \frac{d|\Delta x|}{|\Delta x|^k} = hR_k\, dt \qquad 5)\ \ \left.\frac{|\Delta x|^{-k+1}}{-k+1}\right|_{\Delta x_0}^{\Delta x} = hR_k t$$
$$6)\ \ |\Delta x|^{-k+1} = |\Delta x_0|^{-k+1} - (k-1)hR_k t$$
$$7)\ \ |\Delta x| = \left[\frac{1}{1/|\Delta x_0|^{k-1} - (k-1)hR_k t}\right]^{1/(k-1)} \ .$$

Note that if k is even, then $(\pm\Delta x)^k = \Delta x^k$ and in order to turn the differential $d\Delta x$ into $(\pm\Delta x)$ we need to multiply the other side of the equation by $h = \pm 1$. If k is odd, then $(\pm\Delta x)^k = \pm\Delta x^k$ and in order to turn the differential $d\Delta x$ into $(\pm\Delta x)$ we just need to multiply top and bottom of $d\Delta x/\Delta x^k$ by ± 1 and in this case $h = 1$.

b) As $hR_k t$ increases/decreases from 0, Δx diverges/converges relative to the constant solution. In fact, the diverging solution goes to $+\infty$ at

$$t = \frac{1}{(k-1)hR_k|\Delta x_0|^{k-1}} \ .$$

Redaction: Jeffery, 2018jan01

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005 qfull 00820 1 3 0 easy math: main exception to the 1st order DE rule

4. First order (ordinary) differential equations that are autonomous (meaning they have no explicit dependence on the independent variable) can only have stationary points at infinity (i.e., plus or minus infinity) and each such stationary point corresponds to a static solution. Hereafter for brevity, we call such differential equations 1st order DEs and the rule they obey the 1st order DE rule. The form of these 1st order DEs is

$$x' = f(x) \ ,$$

where x is the dependent variable and t is the independent variable and we assume $f(x)$ is infinitely differentiable. There are exceptions to the 1st order DE rule. The ones known to yours truly are of the form

$$x' = \pm [g(x)]^P \ ,$$

where $P = (1 - 1/n)$ with $n \in [2, \infty)$ and we assume $g(x)$ is infinitely differentiable with respect to x. Note $g(x)$ may go negative as a function of x, but we assume it does not negative as function of t at stationary points. The most obvious and most important exception is for $n = 2$ (i.e., $P = 1/2$) which gives

$$x' = \pm\sqrt{g(x)} \ ,$$

which is exampled by the Friedmann equation. In fact for $n \geq 3$, yours truly know of no interesting cases at all. There may other exceptions to the 1st order DE rule yours truly knows not of. In this problem, we only treat the cases that obey the 1st order DE rule.

NOTE: There are parts a,b,c,d,e.

a) Given x_i (or in the time variable t_i) is a stationary point of $x' = \pm\sqrt{g(x)}$ (i.e., $x'(x_i) = \pm\sqrt{g(x_i)} = \pm\sqrt{g[x(t_i)]} = 0$), prove without words that $x''(x_i) \neq 0$ for $g(x_i) \neq 0$.

b) What does the part (a) answer imply about x_i? What does the part (a) answer imply about x_i given the sign of $dg/dx(x_i)$?

c) Given $(dg/dx)(x_i) = 0$, prove by induction that for general $n \in [1\infty]$ that $x^{(n)}(x_i) = 0$. **Hint:** Consider $x^{(4)}(x_i) = 0$ as step 1 (i.e., the base case) of the proof. Note that the right-hand side of the expressions in the proof will always have a derivative of x two orders lower than the left-hand side.

d) Given $(dg/dx)(x_i) = 0$, what does the part (c) answer imply about x_i?

e) Given $(dg/dx)(x_i) = 0$, and therefore there is a static solution $x = x_i$ for all time t, we can consider what the lowest order solution is for a small perturbation from the static solution. The expansion of the differential equation in small $\Delta x = x - x_i$ is

$$\frac{d\Delta x}{dt} = \pm\sqrt{\sum_{k=\ell}^{\infty} \Delta x^k \left[\frac{d^k g}{dx^k}(x_i)\right]} \ ,$$

where ℓ is the lowest power for which there is a nonzero coefficient $(d^\ell g/dx^\ell)(x_i)$. What possible signs can Δx when ℓ is even and $(d^\ell g/dx^\ell)(x_i) > 0$? What possible signs can Δx when ℓ is even and $(d^\ell g/dx^\ell)(x_i) < 0$? What possible signs can Δx when ℓ is odd?

SUGGESTED ANSWER:

a) Behold:

$$1)\quad x' = \pm\sqrt{g} \qquad 2)\quad x'' = \frac{1}{2}\frac{1}{(\pm\sqrt{g})}\frac{dg}{dx}x' \qquad 3)\quad x'' = \frac{1}{2}\frac{1}{(\pm\sqrt{g})}\frac{dg}{dx}(\pm\sqrt{g})$$

$$4)\quad x'' = \frac{1}{2}\frac{dg}{dx} \qquad 5)\quad x''(x_i) = \frac{1}{2}\frac{dg}{dx}(x_i) \neq 0 \ ,$$

given that $g(x_i) \neq 0$.

b) The point x_i (or t_i in the time variable) is a stationary point of $x(t)$. If $dg/dx(x_i)$ is positive/negative, the stationary point is a minimum/maximum.

c) From part (a), we obtain

$$1)\quad x^{(3)} = \frac{1}{2}\frac{d^2 g}{dx^2}x' \qquad 2)\quad x^{(4)} = \frac{1}{2}\left[\frac{d^3 g}{dx^3}(x')^2 + \frac{d^2 g}{dx^2}x''\right] \ ,$$

where expressions (1) and (2) are zero for $x = x_i$ since $x'(x_i) = 0$ by hypothesis and $x''(x_i) = 0$ by part (a) plus the hypothesis that $(dg/dx)(x_i) = 0$. Expression (1) is actually the first step of the proof since it implies every higher derivative $x^{(n)}$ can be obtained if you know all the derivatives between $x^{(1)}$ and $x^{(n-2)}$. In any case, we explicitly differentiate expression (1) $(n-3)$ times to obtain

$$x^{(n)} = \frac{1}{2}\left[\frac{d^{n-1}g}{dx^{n-1}}(x')^{n-2} + \ldots + \frac{d^2g}{dx^2}(x')^{(n-2)}\right] \ .$$

All the terms on the right-hand side have factors of $(x')^j$ with $j \in [1, n-2]$. As the second step for the proof, we assume all $(x')^j(x_i) = 0$ for $j \in [1, n-2]$. The third step for the proof is by noting that given the first two steps the last expression gives $x^{(n)}(x_i) = 0$ for $n \in [1, \infty]$.

d) Since $x^{(n)}(x_i) = 0$ for $n \in [1, \infty]$, $x(t)$ must be constant to $\pm\infty$ with value x_i (i.e., must be a static solution x_i) or it is asymptotically constant at one of $\pm\infty$ where it is asymptotic to asymptote $x = x_i$.

e) When ℓ is even and $(d^\ell g/dx^\ell)(x_i) > 0$, Δx can be either positive or negative. This is actually the case for small perturbations from the Einstein universe and the radiation-positive curvature-Λ universe (which is the radiation analogue to the Einstein universe which is the matterpositive curvature-Λ universe).

When ℓ is even and $(d^\ell g/dx^\ell)(x_i) < 0$, there are no possible perturbation solutions for real numbers. There is just the static solution itself isolated in solution land. An example of this case is when $g(x) = -\Delta x^2$ which implies $\ell = 2$

When ℓ is odd and $(d^\ell g/dx^\ell)(x_i) > 0$, we can only have $\Delta x > 0$. An example of this case is when $g(x) = \Delta x^3$ which implies $\ell = 3$.

When ℓ is odd and $(d^\ell g/dx^\ell)(x_i) < 0$, we can only have $\Delta x < 0$. An example of this case is when $g(x) = -\Delta x^3$ which implies $\ell = 3$.

Redaction: Jeffery, 2018jan01

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005 qfull 00910 1 3 0 easy math: logistic function

5. The logistic function (called that for a darn good reason) turns up in many contexts looking like:

$$f(x) = \begin{cases} \dfrac{f_{\rm M}}{1+e^{-r(x-x_0)}} = \dfrac{f_{\rm M}}{1+(f_{\rm M}/f_0-1)e^{-rx}} & \text{in general form;} \\ \dfrac{1}{1+e^{-x}} = \dfrac{e^x}{e^x+1} = \dfrac{1}{2}\left[\tanh(x/2)+1\right] & \text{in natural or reduced form.} \end{cases}$$

In this question, we only use the natural form for simplicity and elegance.

There are parts a,b,c,d. **NOTE:** This question has **MULTIPLE PAGES** on an exam.

a) Determine f' (which is, in fact, called the logistic distribution), f'' (also write it as an explicitly even function which it is), the antiderivative of f (easy if you write f in terms of e^x), and the integral of f' from $-x$ to x. Use the natural form of the function.

b) Determine stationary points of f and f' and the values of f and f' at those points. Use the natural form of the function.

c) The logistic function can be used as a smooth replacement for the Heaviside step function:

$$H(x) = \begin{cases} 0 & x < 0; \\ 1/2 & x = 0; \\ 1 & x > 0. \end{cases}$$

Show that logistic function becomes the that Heaviside step function with the appropriate limiting procedure. **Hint:** This is really easy.

d) The logistic function is actually the solution of a 1st order nonlinear differential equation. This equation shows up, for example, in population dynamics. Say you have population N that grows at rate (per population) r with unlimited resources. However, the rate with resources limited by carry capacity (or maximum population) K is modeled as $r(1-N/K)$ which is zero when $N \to K$. The growth differential equation for N, sometimes called the Verhulst-Pearl equation, is

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N \ ,$$

Reduce this equation to natural form and find the solution. Then write the solution out in population-dynamics form for general initial population N_0 at $t = 0$ and show the small N/K and $t \to \infty$ asymptotic limiting cases explicitly. **Hint:** You'll need a table integral.

SUGGESTED ANSWER:

a) Behold:

$$f(x) = \frac{1}{1+e^{-x}}$$

$$f'(x) = \frac{e^{-x}}{(1+e^{-x})^2} = \frac{1}{(e^{x/2}+e^{-x/2})^2} = \left(\frac{1}{4}\right)\left[\frac{1}{\cosh^2(x/2)}\right] \geq 0 \qquad \text{which is the logistic distribution;}$$

$$f''(x) = \frac{2e^{-2x}}{(1+e^{-x})^3} - \frac{e^{-x}}{(1+e^{-x})^2} = \frac{e^{-x}(e^{-x}-1)}{(1+e^{-x})^3} \leq 0$$

$$\int f(x)\,dx = \int \frac{e^x}{1+e^x}\,dx = \ln(1+e^x)$$

$$\int_{-x}^{x} f'(x)\,dx = \frac{1}{1+e^{-x}} - \frac{1}{1+e^{x}} = \frac{e^{x/2}}{e^{x/2}+e^{-x/2}} - \frac{e^{-x/2}}{e^{x/2}+e^{-x/2}} = \tanh(x/2) = \begin{cases} 1 & \text{for } x = \infty; \\ 0 & \text{for } x = 0. \end{cases}$$

b) Behold:

$$f(x) = \begin{cases} \dfrac{1}{1+e^{-x}} & \text{in general;} \\ 0 & \text{for } f \text{ minimum at } x = -\infty; \\ 1 & \text{for } f \text{ maximum at } x = \infty; \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{e^{-x}}{(1+e^{-x})^2} = \dfrac{1}{(e^{x/2}+e^{-x/2})^2} \geq 0 & \text{in general;} \\ 0 & \text{for } f \text{ stationary points at } x = \pm\infty; \\ 0 & \text{for } f' \text{ minima at } x = \pm\infty; \\ \dfrac{1}{4} & \text{for } f' \text{ maxima at } x = 0; \end{cases}$$

$$f''(x) = \begin{cases} \dfrac{e^{-x}(e^{-x}-1)}{(1+e^{-x})^3} \leq 0 & \text{in general;} \\ 0 & \text{for stationary points at } x = 0 \text{ and } x = \pm\infty; \end{cases}$$

c) Behold:

$$\lim_{r\to\infty} f(x) = \lim_{r\to\infty} \frac{1}{1+e^{-rx}} = \left\{ \begin{array}{ll} 0 & x < 0; \\ 1/2 & x = 0; \\ 1 & x > 0 \end{array} \right\} = H(x) \ .$$

d) Let $x = N/K$ and $\tau = rt$. The Verhulst-Pearl equation now reduced form and solution follow:

$$\frac{dx}{d\tau} = x(1-x) \qquad \frac{dx}{x(1-x)} = d\tau$$

$$\ln\left(\frac{x}{1-x}\right) = \tau - C \qquad \frac{x}{1-x} = Ce^{\tau} \qquad x(1+Ce^{\tau}) = Ce^{\tau} \qquad x = \frac{1}{1+Ce^{-\tau}}$$

$$N = \begin{cases} \dfrac{K}{1+(K/N_0-1)e^{-rt}} & \text{in general;} \\ N_0 e^{rt} & \text{for } N_0/K < N/K << 1 \text{ which is exponential growth;} \\ K[1-(K/N_0-1)e^{-rt}] & \text{asymptotically as } t \to \infty. \end{cases}$$

Redaction: Jeffery, 2018jan01

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005 qfull 00950 1 3 0 easy math: The matter-positive-curvature universe

Extra keywords: Need to rewrite in scaled form throughout, but no time 2023nov26.

6. The Friedmann equation is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$$

(Li-55). Let's consider the matter-positive-curvature universe (i.e., a universe with $\rho \propto 1/a^3$, $k > 0$, $\Lambda = 0$). The geometry of this universe is the surface of hypersphere (specifically a 3-sphere) which is finite, but unbounded. Here, however, we are only interested in the solution for cosmic scale factor a, not in the geometry.

There are parts a,b,c,d,e.

a) Rewrite the Friedmann the form $\dot{a} = f(a)$ with $\Lambda = 0$, $\rho = \rho_{\rm M}(a_{\rm M}/a)^3$. We define $a_{\rm M}$ to be the a value for the minimum density $\rho_{\rm M}$ that is allowed by the differential equation. Determine the value for k in terms of the minimum density $\rho_{\rm M}$. What is $a_{\rm M}$ in the solution $a(t)$?

b) Given that the Friedmann equation is of the form $f' = \pm\sqrt{g(f)}$ and that for small a we must have the Einstein-de-Sitter universe behavior ($a \propto t^{2/3}$ assuming $a(t = 0) = 0$), describe what the solution must look like qualitatively.

c) Rewrite the Friedmann equation in natural units: $\sqrt{k}\,t \to t$ and $a/a_{\rm M} \to a$.

d) An approximate simple analytic solution for the Friedmann equation (in natural units) suggested by part (b) is

$$a = \sin^{2/3}\left(\frac{\pi}{2}\frac{t}{t_{\rm M}}\right) ,$$

where $t_{\rm M}$ is the location of the maximum. This approximate solution is an interpolation formula since it gives the right behavior at the endpoints and the maximum. But $t_{\rm M}$ has to be determined. What are natural guesses for $t_{\rm M}$? Now use a 1-step Euler method to obtain a reasonable estimate of a good value for the approximate solution.

e) Actually, an exact analytic solution can be obtained to the differential equation in terms of a new independent variable η. One needs a trick:

$$\dot{a} = \frac{da}{d\eta}\dot{\eta} = \frac{da}{d\eta}\frac{1}{a} \qquad \text{with requirement} \qquad \dot{\eta} = \frac{1}{a} .$$

The trick gets rid of an a in a denominator, but in the way that clairvoyance says is the Tao. Using the trick solve for $a(\eta)$ using a table integral and with the constant of integration chosen so that $a(\eta = 0) = 0$. Then find $t(\eta)$. What the limits of η? Why can we write an analytic formula for $a(t)$? but it has no analytic form

SUGGESTED ANSWER:

a) Behold:

$$\dot{a} = \pm\sqrt{\frac{8\pi G}{3}\rho_{\rm M} a_{\rm M}^2\left(\frac{a_{\rm M}}{a}\right) - k}$$
$$k = \frac{8\pi G}{3}\rho_{\rm M} a_{\rm M}^2 ,$$

which has the right dimensions the adopted k: i.e., the inverse square of time like $G\rho$. Note the minimum density occurs, of course, when the radicant equals zero. The $a_{\rm M}$ quantity is the maximum of $a(t)$. Any bigger value makes the radicand negative and gives a complex differential equation.

b) Given the form of the differential equation, the solution must be symmetric about the single stationary point (assuming the solution is not a periodic function) which is where the solution changes branches from the $f' = \sqrt{g(f)}$ case to the $f = -\sqrt{g(f)}$. The stationary point is, in fact, where the radicand is zero: i.e., where $a = a_{\rm M}$. We will label this point $t_{\rm M}$. Since a can go to zero (where it will have an infinite slope, but that's OK), but not to infinity for a real solution, the stationary point must be maximum which we already inferred in part (a). Given the Einstein-de-Sitter universe behavior, we must have $a \propto t^{2/3}$ for $t \to 0$ and by symmetry $a \propto (2t_{\rm M} - t)^{2/3}$ for $t \to 2t_{\rm M}$. There is no real solution for a beyond the endpoints $t = 0$

and $t = 2t_{\rm M}$. Since the differential equation has a simple structure, we can expect a simple structure for the soltuion $a(t)$ with no funny wiggles, etc. Give the solution near the endpoints bulges upward relative to a straight line and no funny wiggles, etc., the solution is probably a simple convex-up symmetric curve.

c) Behold:

$$\dot a = \pm\sqrt{\frac{1}{a} - 1}\ .$$

d) Well since the solution exists only over a limited range and we are using natural units, $t_{\rm M} = 1$ is one natural guess. Another natural guess given the form of the approximate solution is $t_{\rm M} = \pi/2$.

Applying the 1-step Euler method to the differential equation equation gives

$$\frac{1}{t_{\rm M}} = \sqrt{\frac{1}{a_{\rm E}} - 1}\ ,$$

where the left-hand side is the slope of the line replacement for the solution between $t = 0$ and $t_{\rm M}$ and $a_{\rm E}$ has to be estimated to give a good result. The ideal choice of $a_{\rm E}$ would be the one that makes the slope given by the right-hand side exactly equal to the slope of the correct line replacement for the solution. But don't know what that ideal choice is. The natural choice if we knew nothing of the solution is $a_{\rm E} = 1/2$. But since the solution is convex-up, maybe $a_{\rm E} = 2/3$ or $a_{\rm E} = 3/4$ could be better. Let's try them all the suggested possibilities:

$$t_{\rm M} = \begin{cases} \dfrac{1}{\sqrt{(1/a_{\rm E}) - 1}} & \text{in general;} \\ 1 & \text{for } a_{\rm E} = 1/2; \\ \sqrt{2} = 1.414\ldots < \pi/2 & a_{\rm E} = 2/3; \\ \sqrt{3} = 1.732\ldots > \pi/2 & a_{\rm E} = 3/4. \end{cases}$$

Since the two possibly better 1-step Euler method values give $t_{\rm M}$ values close to $\pi/2$ (a natural guess for $t_{\rm M}$) and their average approximately equals it $((\sqrt{2} + \sqrt{3})/2 = 1.5731\ldots \approx \pi/2 = 1.5707963\ldots)$, we'll adopt $t_{\rm M} = \pi/2$. In fact, this is fortuitiously exactly correct.

e) Behold:

$$\dot a = \frac{da}{d\eta}\dot\eta = \frac{da}{d\eta}\frac{1}{a} = \sqrt{\frac{1}{a} - 1}$$

$$\frac{da}{d\eta} = \sqrt{a - a^2} \qquad \frac{da}{\sqrt{a - a^2}} = d\eta \qquad -\cos^{-1}(2a - 1) = \eta + C$$

$$a = \frac{1}{2}[1 + \cos(\eta + C)] \qquad a = \frac{1}{2}[1 + \cos(\eta + \pi)] \qquad a = \frac{1}{2}[1 - \cos(\eta)]\ .$$

Now

$$\dot a = \frac{da}{dt} = \frac{da}{d\eta}\frac{1}{a} \qquad dt = a\,d\eta \qquad t = \int_0^\eta \frac{1}{2}[1 - \cos(\eta')]\,d\eta' = \frac{1}{2}[\eta - \sin(\eta)]\ .$$

So finally we obtain

$$a = \frac{1}{2}[1 - \cos(\eta)] \qquad t = \frac{1}{2}[\eta - \sin(\eta)]$$

with limits $\eta \in [0, 2\pi]$.

The function $t(\eta)$ cannot be analytically inverted. So an analytic $a(t)$ is not possible. However, an analytic $t(a)$ exists:

$$a = \frac{1}{2}[1 - \cos(\eta)] \qquad \eta = \cos^{-1}(1 - 2a) \qquad t = \frac{1}{2}\left\{\cos^{-1}(1 - 2a) - \sin\left[\cos^{-1}(1 - 2a)\right]\right\}\ .$$

Actually, what is analytic or not depends on definition. If we define $\eta(t)$ to be analytic, then an $a(t)$ analytic exists by definition. But by convention, only common transcental functions are defined to be analytic: so we don't define $\eta(t)$ to be analytic.

Fortran-95 Code

```
print*
piL2=0.5_np*acos(-1.0_np)
a=0.5_np*(sqrt(2.0_np)+sqrt(3.0_np))
print*,'piL2,a'
print*,piL2,a
! 1.57079632679489661926          1.57313218497098617117
```

Redaction: Jeffery, 2018jan01