Cosmology **NAME:**

Homework 5: Advanced Solutions of the Friedmann Equation

1. The Friedmann equation for the radiation-matter universe (which applies to the observable universe up to of order 10 Gyr) in general scaled form is

$$\left(\frac{\dot{x}}{x}\right)^2 = \Omega_{4,0}x^{-4} + \Omega_{3,0}x^{-3}$$

where x is the cosmic scale factor with $x_0 = 1$ for cosmic present, $\tau = H_0 t$ is the scaled cosmic time with t being cosmic time in time units and H_0 being the Hubble constant, $\Omega_{4,0}$ is the radiation density parameter for cosmic present, and $\Omega_{3,0}$ is the matter density parameter for cosmic present.

NOTE: There are parts a,b,c,d,e,f,g. On exams, do **ONLY** parts a,b,c. The parts a,b,c can be done independently, and so don't stop if you can't do one.

a) Determine the radiation-matter equality scale factor $x_{\rm eq}$: i.e., the x value that makes the radiation and matter mass-energy equal.

b) Defining $y = x/x_{\rm eq}$, rewrite the Friedmann equation into a nice integrable form $dw = f(y)\,dy$ (i.e., a special case scaled form), where $w = \tau/\tau_{\rm sc}$ is rescaled time and the form has no constants. What is $\tau_{\rm sc}$ in terms of the density parameters?

c) Solve the Friedmann equation form found in part (b) for $w(y)$ with $w(y = 0) = 0$. You will need the table integral

$$\int \frac{y\,dy}{\sqrt{1+y}} = \frac{2}{3}(y-2)\sqrt{1+y}\ .$$

d) For $w(y)$, write out the special cases $w(y = 0)$ $w(y)$ to 2nd order in small y, $w(y = 1)$ (at the radiation-matter equality) $w(y = 2)$ (at 2 times the radition-matter equality) $w(y = 3)$ (at 3 times the radition-matter equality which is where the exact y(w) formula changes form), and $w(y >> 1)$ (the large y asymptotic limit).

d) Solve for the asymptotic limiting small w and large w forms of $y(w)$.

f) Transform the limiting forms found in part (d) into the general scaled forms: i.e., into $x(\tau)$ forms.

g) This a challenging part if you have some time. Yours truly has probably spent more time than it is worth trying to find good analytic approximate for solutions $x(\tau)$ for cases where no exact solution exists or the exact solution exists, but is too complex for easy understanding. In fact, the V model solutions (Jeffery 2025) provide understandable exact solutions which are analogues to the standard traditional, but non-exact, solutions for the Friedmann equation found by Alexander Alexandrovich Friedmann (1888–1925), Georges Lemaitre (1894–1966), Willem de Sitter (1872–1934), and others long ago. There may be no better way in general to understand those standard traditional, but non-exact, solutions than using those V model solution analogues. However, in special cases, there may be. One special case, is the radiaton-matter universe. In fact, an exact solution for $y(w)$ exists with two mathematically equivalent formulae that look rather different (Jeffery 2026). But both formulae are too complex for easy understanding. However, a fairly accurate, easy-to-understand interpolation formula does exist that agrees asymptotically with the symptotic limiting small w and large w forms of $y(w)$ found in part (d). See if you can find it. **HINT:** The formula uses $\arctan[y_{\rm small}(w)]$ and it takes some playing around to find it.