**Cosmology** **NAME:**

**Homework 5: Advance Solutions of the Friedmann Equation**

1. The Einstein universe (proposed by Einstein in 1917) was the first cosmological model derived consistently from a physical theory (i.e., general relativity) and was the beginning of modern cosmology. Einstein assumed the cosmological principle (i.e., a homogeneous, isotropic universe) and represented the mass-energy by a pressureless perfect fluid where the density scaled as $a^{-3}$. In modern cosmology jargon, this kind of perfect fluid is called "matter" and approximates ordinary baryonic matter and dark matter For cosmological purposes, matter has approximately zero kinetic energy relative its local comoving frame.

   Einstein believing in 1917 that the universe was one of stars (which seemed on average at rest) and not galaxies wanted a static model, but found that impossible with his field equations as originally formulated (O'Raifeartaigh et al. 2017). So he added the cosmological constant term $\Lambda$ to the field equations which was the simplest possible modification and had no significant effect on smaller-than-cosmological-scale phenomena. The Einstein universe he obtained is a finite, boundless, positively curved universe or hyperspherical universe. It is geometrically the 3-dimensional surface of the a 3-sphere (which is actually a 4-dimensional sphere in Euclidean or flat space). The distance to return to the same point along a geodesic is $2\pi a_0$, where $a_0$ is the Gaussian curvature radius a hyperspherical universe. (CL-11–12). For considering the Einstein universe, $a_0$ is not the conventional dimensionless quantity but a physical proper distance with units of length.

   Einstein in 1931 abandoned the Einstein universe since observations showed an expanding universe and because the Einstein universe had been shown to be unstable by Eddington in 1930 (O'Raifeartaigh et al. 2017 p. 36, 41).

   Note that Einstein did not have the Friedmann equation and acceleration equation when he derived the Einstein universe. He used a general relativity directly and followed a "rough and winding road" (O'Raifeartaigh et al. 2017, p. 18).

   In this problem, we investigate the Einstein universe. There are parts a,b,c,d,e,f,g,h. In exam environments, do **ONLY** parts a,b,c,d.

   **NOTE:** This question has **MULTIPLE PAGES** on an exam.

   a) The Friedmann equation and acceleration equation in forms appropriate for solving for the Einstein universe and investigating its stability are

$$H^2 = \left(\frac{\dot{x}}{x}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a_0^2 x^2} + \frac{\Lambda}{3} = \frac{8\pi G\rho_0}{3}\left[\Omega_{\rm M} x^{-3} + \Omega_k x^{-2} + \Omega_\Lambda\right]$$

   and

$$\frac{\ddot{x}}{x} = -\frac{4\pi G\rho}{3} + \frac{\Lambda}{3} = -\frac{4\pi G\rho_0}{3}\left[\Omega_{\rm M} x^{-3} - 2\Omega_\Lambda\right]$$

   (Li-55 *mutatis mutandis*), where $x = a/a_0$, $a_0$ is the Gaussian curvature radius of the Einstein universe (as aforesaid), $\rho_0$ is the density of Einstein universe, $k = 1$ for a positive curvature universe,

$$\Omega_{\rm M} = 1\ , \qquad \Omega_k = -\frac{kc^2}{a_0^2(8\pi G\rho_0/3)}\ , \qquad \text{and} \qquad \Omega_\Lambda = \frac{\Lambda}{8\pi G\rho_0}\ .$$

   Note we cannot use the Hubble parameter $H$ in defining the density parameter $\Omega_i$ quantities since $H = 0$ for the Einstein universe.

   The Einstein universe has $x = 1$, $\dot{x} = 0$, $\ddot{x} = 0$ and $\rho = \rho_0$. Given the Einstein-universe values, determine formula for $\Lambda$ from the first form of the acceleration equation and the numerical value of $\Omega_\Lambda$ from the second form.

   b) Given the Einstein-universe values, determine the formula for $a_0$ as function of $\rho_0$ and then the formula for $a_0$ as a function of $\Lambda$. **Hint:** Start from the second form of the Friedmann equation and recall the given formula for $\Omega_k$.

   c) Given $G = 6.67430(15) \times 10^{-11}$ MKS, vacuum light speed $c = 2.99792458 \times 10^8$ m/s, $\rho_0 = 0.85 \times 10^{-26}$ kg/m$^3$ (which is suggest value of the critical density circa 2021), and 1 Gpc $= (3.085677581\ldots) \times 10^{25}$ m, calculate the Gaussian curvature radius $a_0$ in units of gigaparsecs (Gpc). You can use your phone for the calculations—but only for those.

d) Now write the Friedmann equation in the dimensionless form

$$\frac{dx}{d\tau} = \pm\sqrt{f(x)} \ ,$$

where the dimensionless time $\tau$ is given by

$$\tau = t\sqrt{\frac{8\pi G\rho_0}{3}} \ .$$

Sketch a plot the radicand $f(x)$ for $x \geq 0$ going left from $x = 1$ to $x = 0$ and right from $x = 1$ to $x = \infty$. Using the first two derivatives of $f(x)$ as a function of $x$ (not $\tau$) prove that the Einstein universe (i.e., the $x = 1$ case) is a unique static universe for $x \geq 0$.

e) For the initial condition $x_1$ greater/less than 1 at $\tau_1$ and the positive/negative case for $x' = \pm\sqrt{f(x)}$, describe the evolution of $x$ with $\tau$ increasing and in particular what happens if $x \to 0$. Explain the evolutions and describe the stability of the Einstein universe to perturbations in these cases. **Hint:** It might help to draw a figure of the evolutions.

f) For the initial condition $x_1$ greater/less than 1 at $\tau_1$ and the negative/positive case for $x' = \mp\sqrt{f(x)}$, describe the probable evolution of $x$ with $\tau \to \infty$. Prove these evolutions and describe the stability of the Einstein universe to perturbations in these cases. **Hint:** The proof requires that you show that all orders of derivative of $x$ are zero when $x$ is stationary. You will need to determine the $x''$, $x'''$, and $x^{(4)}$, notice some things about these orders of derivative, and add some explanatory words. Also, it might help to draw a figure of the evolutions.

g) From parts (d) and (e), what is the stability of the Einstein universe to general perturbations of $a$? Note a solution is unstable to general perturbations if it is unstable to any kind of perturbations.

h) Given all the answers to the other parts, discuss how an Einstein universe filled with real gas (including dark matter gas) and/or stars might evolve.

2. First order (ordinary) differential equations that are autonomous (meaning they have no explicit dependence on the independent variable) can only have stationary points at infinity (i.e., plus or minus infinity) and each such stationary point corresponds to a static solution. Hereafter for brevity, we call such differential equations 1st order DEs and the rule they obey the 1st order DE rule. The form of these 1st order DEs is

$$x' = f(x) \ ,$$

where $x$ is the dependent variable and $t$ is the independent variable and we assume $f(x)$ is infinitely differentiable and contains no fractional roots. There are exceptions to the 1st order DE rule. The ones known to yours truly are of the form

$$x' = \pm[g(x)]^P \ ,$$

where $P = (1 - 1/n)$ with $n \in [2, \infty)$ and we assume $g(x)$ is infinitely differentiable with respect to $x$. Note $g(x)$ may go negative as a function of $x$, but we assume it does not negative as function of $t$ at stationary points. The most obvious and most important exception is for $n = 2$ (i.e., $P = 1/2$) which gives

$$x' = \pm[g(x)]^{1/2} \ ,$$

which is exampled by the Friedmann equation. In fact for $n \geq 3$, yours truly know of no interesting cases at all. There may other exceptions to the 1st order DE rule yours truly knows not of. In this problem, we only treat the cases that obey the 1st order DE rule.

**NOTE:** There are parts a,b,c,d.

a) Given $x_i$ (or in the time variable $t_i$) is a stationary point of $x' = f(x)$ (i.e., $x'(x_i) = f(x_i) = f[x(t_i)] = 0$), prove without words that $x''(x_i) = 0$.

b) The part (a) answer gives the base case (or 1st step) for a proof by induction that all orders of derivative of $x$ with respect to $t$ at $x_i$ (or in the time variable $t_i$) are zero. The proof follows by inspection if your math intuition is good enough. However, do a formal proof by induction. **Hint:** For the proof, you do **NOT**, in fact, need the full general Leibniz rule for the derivative of a product (Ar-558)

$$\frac{d^m(fg)}{dx^m} = \sum_{k=0}^{m} \binom{m}{k} \frac{d^k f}{dx^k} \frac{d^{m-k} g}{dx^{m-k}} \ .$$

Using it actually makes the proof a bit more tricky to follow. But you do need to know that the $n$th order derivative of $x$ (i.e., $x^{(n)}$) is obtained by applying the general Leibniz rule for $m = n - 2$ to the result of the part (a) answer and that highest derivative of $x$ on the right-hand side of that application is $x^{(n-1)}$. Note that $f(x)$ is general to the degree specified in the preamble, and so the proof is unchanged if any order of derivative $f(x)$ with respect to $x$ is zero at $x_i$.

c) Given the part (b) result, give an argument for why the stationary point $t_i$ must be all points (i.e., is actually a static solution) or at time equals infinity.

d) A 1st order DE system given a small perturbation from a static solution either asymtotically goes back to it (i.e., is asymptotic to it at positive infinity, and so is called stable) or grows away from it (i.e., is asymptotic to it at negative infinity, and so is called unstable). Assuming the $df/dx$ is nonzero at $x_i$, prove without words that a 1st order DE system given a small perturbation (i.e., a perturbation $\Delta x_0$ which requires only 1st order expansion of $f(x)$ in small $\Delta x = x - x_i$) varies exponentially and determine the condition for stability.

3. Consider the 1st order (ordinary, autonomous) differential equation

$$x' = f(x) \ ,$$

where $x$ is the dependent variable and $t$ is the independent variable and we assume $f(x)$ is infinitely differentiable and contains no fractional roots. The 1st order DE rule (as yours truly calls it) applies to this DE. We have $f(x_i) = 0$ and therefore $x_i$ yields a constant solution and a stationary point at either of $\pm\infty$.

**NOTE:** There are parts a,b.

a) Assuming $(df/dx)(x_i) \neq 0$, solve without words for the 1st order perturbation solution in small $\Delta x = x - x_i$. Let $\Delta x_0$ be the initial perturbation, time zero is 0, and $R_1 = (df/dx)(x_i)$ for compactness. What is the condition for convergence/divergence in the future to the constant solution? What is the condition for convergence/divergence in the past to the constant solution? **Hint:** Recall the antiderivative of $1/y$ is always $\ln(|y|)$.

b) Now assume the lowest order nonzero coefficient in the expansion of $f(x)$ in small $\delta x$ is $(d^k f/dx^k)(x_i)$ where $k \geq 2$. The write the solution only in terms of $|\Delta x|$ and $|\Delta x_0|$ since that seems most clear and start from the differential form

$$\frac{d|\Delta x|}{|\Delta x|^k} = hR_k \, dt \ ,$$

where for $k$ even $h = \pm 1$ with upper case for $\Delta x > 0$ and lower case for $\Delta x < 0$ and for $k$ odd $h = 1$, and $R_k = (d^k f/dx^k)(x_i)$ for compactness. Show why this differential form is correct before you use it.

c) What happens as $hR_k t$ **INCREASES/DECREASES** from 0? At what time $t$ is there an infinity?

4. First order (ordinary) differential equations that are autonomous (meaning they have no explicit dependence on the independent variable) can only have stationary points at infinity (i.e., plus or minus infinity) and each such stationary point corresponds to a static solution. Hereafter for brevity, we call such differential equations 1st order DEs and the rule they obey the 1st order DE rule. The form of these 1st order DEs is

$$x' = f(x) \ ,$$

where $x$ is the dependent variable and $t$ is the independent variable and we assume $f(x)$ is infinitely differentiable. There are exceptions to the 1st order DE rule. The ones known to yours truly are of the form

$$x' = \pm[g(x)]^P \ ,$$

where $P = (1 - 1/n)$ with $n \in [2, \infty)$ and we assume $g(x)$ is infinitely differentiable with respect to $x$. Note $g(x)$ may go negative as a function of $x$, but we assume it does not negative as function of $t$ at stationary points. The most obvious and most important exception is for $n = 2$ (i.e., $P = 1/2$) which gives

$$x' = \pm\sqrt{g(x)} \ ,$$

which is exampled by the Friedmann equation. In fact for $n \geq 3$, yours truly know of no interesting cases at all. There may other exceptions to the 1st order DE rule yours truly knows not of. In this problem, we only treat the cases that obey the 1st order DE rule.

**NOTE:** There are parts a,b,c,d,e.

a) Given $x_i$ (or in the time variable $t_i$) is a stationary point of $x' = \pm\sqrt{g(x)}$ (i.e., $x'(x_i) = \pm\sqrt{g(x_i)} = \pm\sqrt{g[x(t_i)]} = 0$), prove without words that $x''(x_i) \neq 0$ for $g(x_i) \neq 0$.

b) What does the part (a) answer imply about $x_i$? What does the part (a) answer imply about $x_i$ given the sign of $dg/dx(x_i)$?

c) Given $(dg/dx)(x_i) = 0$, prove by induction that for general $n \in [1\infty]$ that $x^{(n)}(x_i) = 0$. **Hint:** Consider $x^{(4)}(x_i) = 0$ as step 1 (i.e., the base case) of the proof. Note that the right-hand side of the expressions in the proof will always have a derivative of $x$ two orders lower than the left-hand side.

d) Given $(dg/dx)(x_i) = 0$, what does the part (c) answer imply about $x_i$?

e) Given $(dg/dx)(x_i) = 0$, and therefore there is a static solution $x = x_i$ for all time $t$, we can consider what the lowest order solution is for a small perturbation from the static solution. The expansion of the differential equation in small $\Delta x = x - x_i$ is

$$\frac{d\Delta x}{dt} = \pm\sqrt{\sum_{k=\ell}^{\infty} \Delta x^k \left[\frac{d^k g}{dx^k}(x_i)\right]} ,$$

where $\ell$ is the lowest power for which there is a nonzero coefficient $(d^\ell g/dx^\ell)(x_i)$. What possible signs can $\Delta x$ when $\ell$ is even and $(d^\ell g/dx^\ell)(x_i) > 0$? What possible signs can $\Delta x$ when $\ell$ is even and $(d^\ell g/dx^\ell)(x_i) < 0$? What possible signs can $\Delta x$ when $\ell$ is odd?

5. The logistic function (called that for a darn good reason) turns up in many contexts looking like:

$$f(x) = \begin{cases} \dfrac{f_{\rm M}}{1+e^{-r(x-x_0)}} = \dfrac{f_{\rm M}}{1+(f_{\rm M}/f_0 - 1)e^{-rx}} & \text{in general form;} \\ \dfrac{1}{1+e^{-x}} = \dfrac{e^x}{e^x+1} = \dfrac{1}{2}\left[\tanh(x/2)+1\right] & \text{in natural or reduced form.} \end{cases}$$

In this question, we only use the natural form for simplicity and elegance.

There are parts a,b,c,d. **NOTE:** This question has **MULTIPLE PAGES** on an exam.

a) Determine $f'$ (which is, in fact, called the logistic distribution), $f''$ (also write it as an explicitly even function which it is), the antiderivative of $f$ (easy if you write $f$ in terms of $e^x$), and the integral of $f'$ from $-x$ to $x$. Use the natural form of the function.

b) Determine stationary points of $f$ and $f'$ and the values of $f$ and $f'$ at those points. Use the natural form of the function.

c) The logistic function can be used as a smooth replacement for the Heaviside step function:

$$H(x) = \begin{cases} 0 & x < 0; \\ 1/2 & x = 0; \\ 1 & x > 0. \end{cases}$$

Show that logistic function becomes the that Heaviside step function with the appropriate limiting procedure. **Hint:** This is really easy.

d) The logistic function is actually the solution of a 1st order nonlinear differential equation. This equation shows up, for example, in population dynamics. Say you have population $N$ that grows at rate (per population) $r$ with unlimited resources. However, the rate with resources limited by carry capacity (or maximum population) $K$ is modeled as $r(1 - N/K)$ which is zero when $N \to K$. The growth differential equation for $N$, sometimes called the Verhulst-Pearl equation, is

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right) N ,$$

Reduce this equation to natural form and find the solution. Then write the solution out in population-dynamics form for general initial population $N_0$ at $t = 0$ and show the small $N/K$ and $t \to \infty$ asymptotic limiting cases explicitly. **Hint:** You'll need a table integral.

6. The Friedmann equation is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$$

(Li-55). Let's consider the matter-positive-curvature universe (i.e., a universe with $\rho \propto 1/a^3$, $k > 0$, $\Lambda = 0$). The geometry of this universe is the surface of hypersphere (specifically a 3-sphere) which is finite, but unbounded. Here, however, we are only interested in the solution for cosmic scale factor $a$, not in the geometry.

There are parts a,b,c,d,e.

a) Rewrite the Friedmann the form $\dot{a} = f(a)$ with $\Lambda = 0$, $\rho = \rho_{\rm M}(a_{\rm M}/a)^3$. We define $a_{\rm M}$ to be the $a$ value for the minimum density $\rho_{\rm M}$ that is allowed by the differential equation. Determine the value for $k$ in terms of the minimum density $\rho_{\rm M}$. What is $a_{\rm M}$ in the solution $a(t)$?

b) Given that the Friedmann equation is of the form $f' = \pm\sqrt{g(f)}$ and that for small $a$ we must have the Einstein-de-Sitter universe behavior ($a \propto t^{2/3}$ assuming $a(t = 0) = 0$), describe what the solution must look like qualitatively.

c) Rewrite the Friedmann equation in natural units: $\sqrt{k}\,t \to t$ and $a/a_{\rm M} \to a$.

d) An approximate simple analytic solution for the Friedmann equation (in natural units) suggested by part (b) is

$$a = \sin^{2/3}\left(\frac{\pi}{2}\frac{t}{t_{\rm M}}\right) ,$$

where $t_{\rm M}$ is the location of the maximum. This approximate solution is an interpolation formula since it gives the right behavior at the endpoints and the maximum. But $t_{\rm M}$ has to be determined. What are natural guesses for $t_{\rm M}$? Now use a 1-step Euler method to obtain a reasonable estimate of a good value for the approximate solution.

e) Actually, an exact analytic solution can be obtained to the differential equation in terms of a new independent variable $\eta$. One needs a trick:

$$\dot{a} = \frac{da}{d\eta}\dot{\eta} = \frac{da}{d\eta}\frac{1}{a} \qquad \text{with requirement} \qquad \dot{\eta} = \frac{1}{a} .$$

The trick gets rid of an $a$ in a denominator, but in the way that clairvoyance says is the Tao. Using the trick solve for $a(\eta)$ using a table integral and with the constant of integration chosen so that $a(\eta = 0) = 0$. Then find $t(\eta)$. What the limits of $\eta$? Why can we write an analytic formula for $a(t)$? but it has no analytic form