# Cosmology & Galaxies                                                NAME:

**Homework 3: The Friedmann Equation**

1. "Let's play *Jeopardy*! For $100, the answer is: It was derived from general relativity in 1922 with the assumptions of a homogeneous and isotropic universe and that all mass-energy in the universe could be modeled by a perfect fluid. A Newtonian derivation (which required extra natural hypotheses) was given in 1934.

    What is the _____, Alex?

    a) Einstein equation      b) Milne-McCrea equation      c) Synge equation      d) Bondi equation
    e) Friedmann equation

2. A Newtonian derivation of the Friedmann equation (with extra natural hypotheses) could easily have been done in the 19th century, but it wasn't. There were probably 3 reasons why 19th century astronomers did not think of such a derivation. First, many were still thinking of a universe that was static on average even though dynamic equilibrium seemed hard to arrange, even though the universe was obviously not in thermodynamic equilibrium (and so why should be in dynamic equalibrium), and even though idea existed that the Milky was held up by rotation around the center of mass located somewhere. Second, they did not know that other galaxies existed though some believed this and they had not observed the general redshifts of the objects they thought might be other galaxies. Third, they thought in terms of Newton's absolute space (i.e., a single fundamental inertial frame) and did not think of the alternative idea completely compatible with their data that all _____ unrotating with respect to the observable universe were elementary inertial frames (i.e., frames with respect to which Newtonian physics and all other physics could be referenced to). The elementary inertia frames could be incorporated into more general inertial frames (e.g., center-of-mass inertial frames) and the whole observable universe could organized into the more general inertial frames. There is whole hierarchy of general inertial frames that tops out with the comoving frames of the expanding universe.

    What is the _____, Alex?

    a) star frames      b) planet frames      c) free-fall frames      d) thermodynamics frames
    e) gravity frames

3. "Let's play *Jeopardy*! For $100, the answer is: This theorem (originally proven by Newton by primitive means) allows one to show by means of a **COROLLARY** that spherically symmetric masses should interact gravitationally as though they are point masses as long as they are do not interpenetrate.

    What is the _____, Alex?

    a) Newton theorem      b) shell theorem      c) point-mass theorem      d) sphere theorem
    e) waste book theorem

4. "Let's play *Jeopardy*! For $100, the answer is: The theorem that states that the only attractive central forces that give closed orbits for all bound orbits are the inverse-square law force and the attractive linear force (AKA Hooke's law force or the radial harmonic oscillator force). All attractive central forces give closed **CIRCULAR** orbits, of course."

    What is _____, Alex?

    a) the virial theorem      b) Euler's theogonic proof      c) the brachistochrone problem
    d) Schubert's unfinished symphony      e) Bertrand's theorem

5. The solutions of the Friedmann equation have characteristic cosmological quantities some of which are called Hubble quantities since the Hubble constant is one of their ingredients. The table below displays some the cosmological quantities. Since the currently determined values of the quantities always fluctuate a bit depending on whose analysis is used, we have written the quantities as fiducial values with correction factors that are 1 to within a few percent: $h_{70}$ is the Hubble constant divided by $70 \, (\text{km/s})/\text{Mpc}$ (i.e., $H_0/(70 \, (\text{km/s})/\text{Mpc})$), $\omega_{\text{m},0} = \Omega_{\text{m},0}/0.3$, and $\omega_\Lambda = \Omega_\Lambda/0.7$. The asymptotic Hubble quantities are those that will be the Hubble quantities as cosmic time goes to infinity if the $\Lambda$-CDM model is correct.

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Table: Cosmological Quantities

Cosmic scale factor for the present cosmic time $a_0 = 1$ by convention

Hubble constant $H_0 = 70h_{70}$ (km/s)/Mpc

Hubble time $t_H = 1/H_0 = (13.968\ldots)/h_{70}$ Gyr

Hubble length $\ell_H = c/H_0 = (13.968\ldots)/h_{70}$ Gly $= (4.2827\ldots)/h_{70}$ Gpc

Critical density $\rho_{\text{critical}} = [3H_0^2/(8\pi G)] = (9.2039 \times 10^{-27})h_{70}^2$ kg/m$^3$

$\qquad\qquad\qquad\quad = (1.3599 \times 10^{11})h_{70}^2$ M$_\odot$/Mpc$^3$

AKA Hubble density (i.e., the density implied by the Hubble constant at cosmic present)

Cosmological constant matter density parameter $\Omega_{\text{m},0} = 0.3\omega_{\text{m},0}$

Cosmological constant $\Lambda$ density parameter $\Omega_\Lambda = 0.7\omega_\Lambda$

Asymptotic $\Lambda$ Hubble parameter $H_\Lambda = H_0\sqrt{\Omega_\Lambda} = \sqrt{\Lambda/3} = (58.566\ldots)h_{70}\sqrt{\omega_\Lambda}$ (km/s)/Mpc

Asymptotic $\Lambda$ Hubble time $t_{H_\Lambda} = (16.6955\ldots)/(h_{70}\sqrt{\omega_\Lambda})$ Gyr

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Given that the $\Lambda$-CDM model is correct, to 1st order, the observable universe is already expanding like a cosmological-constant universe with $a = a_0 \exp(\Delta t/t_{H_\Lambda})$ (where $\Delta t = t - t_0$) and this formula becomes more correct as time advances. On what time scale $\Delta t$ will the matter mass-energy density of the observable universe fall to of order 2 % of the total mass-energy? Note you have to solve for $a/a_0$ from

$$\Omega_{\text{m}} = \Omega_{\text{m},0}\left(\frac{a_0}{a}\right)^3 \approx 0.02\Omega_\Lambda$$

and then solve for $\Delta t$.

 a) $t_{H_\Lambda}$.     b) $2t_{H_\Lambda}$     c) $3t_{H_\Lambda}$.     d) $4t_{H_\Lambda}$.     e) $5t_{H_\Lambda}$.

6. In this problem, we will derive the generic Gauss' law in its integral form and then specialize to the gravity and Coulomb force cases.

   **NOTE:** There are parts a,b,c,d. Some of the parts can be done independently, and so do not stop if you cannot do a part.

   a) Consider the generic inverse-square law central force

   $$\vec{f} = \frac{q}{r^2}\hat{r} \ ,$$

   where $q$ is a generic charge for the force located at the origin. Now consider the differential surface area vector $d\vec{A} = dA\,\hat{n}$ for a **CLOSED** surface. The unit vector $\hat{n}$ is normal to the differential surface and points in outward direction. The differential solid angle subtended by the differential surface area is $d\Omega$. Prove

   $$\vec{f} \cdot d\vec{A} = q(\pm\, d\Omega)$$

   where the upper/lower cases are for the solid angle cone going outward/inward through the differential surface area. Note the charge could be inside or outside the closed surface. **HINT:** This is an easy question, but a few words of explanation are needed. But **NO** words are during exams.

   b) Consider a differentally small cone extending from the origin. It intersects the closed surface $n$ times. Note that closed surface is finite, and so the cone must exit the closed surface for good at some point. We form the sum

   $$\sum_{i=1}^{n} \vec{f} \cdot d\vec{A}_i \ ,$$

   where sum is over all intersections. What is the sum equal to in terms of solid angle for all cases?

   c) Say you had multiple charges $q_i$ with total charge $Q$ and total charge $Q_{\text{enclosed}}$ inside a closed surface. Evaluate

   $$\oint \vec{f} \cdot d\vec{A} \ .$$

   The result is the generic Gauss' law in its integral form. Specialize the result for the cases of gravity and the Coulomb force.

   d) What is the necessary condition for a force to obey Gauss' law?

7. Remarkably the linear force obeys analogues to Gauss's law and shell theorem for the inverse-square law force. Let the linear-force field (force per unit charge) for a point charge be

$$\vec{f} = kqr\hat{r} \ ,$$

where $k$ is a constant which could be positive or negative, $q$ is the charge (of some unspecified kind), and $r$ is the distance from the point charge. We assume Newtonian physics, and so to maintain Newton's 3rd law, we require

$$\vec{F}_{1,2} = kq_1q_2r_{1,2}\hat{r}_{1,2} \ ,$$

where $\vec{F}_{1,2}$ is the force of point charge 1 on point charge 2.

There are parts a,b,c,d,f. Some of the parts can be done independently, and so do not stop if you cannot do a part. Omit part (f) during exams.

a) Without words, for a close surface derive the linear-force Gauss' law

$$\oint \vec{f} \cdot d\vec{A} = kQ \ ,$$

where $\vec{f}$ is the field due to the entire charge distribution, the integral is over the whole close surface, and $Q$ is the total charge of the charge distribution wherever it is in space. **HINT:** Recall the divergence theorem (AKA Gauss' theorem)

$$\oint \vec{Y} \cdot d\vec{A} = \int \nabla \cdot \vec{Y} \, dV \ ,$$

where $Y$ is a general vector field and the volume integral is over all volume $V$ inclosed by the closed surface (Wikipedia: Divergence theorem). Recall also the divergence operator for spherically symmetric system in spherical coordinates obeys

$$\nabla \cdot \vec{Z} = \frac{1}{r^2} \frac{\partial(r^2 Z_r)}{\partial r} \ ,$$

where $Z$ is spherically symmetric, but otherwise general, and $Z_r$ is the radial component of $\vec{Z}$ (Arfken-104).

b) For what symmetries can the linear-force field be easily solved for directly from the linear-force Gauss' law?

c) Without words, solve for the linear-force field for a spherically symmetric charge distribution. What simple charge distribution would give an equivalent linear-force field for all radius $r$? What can this result be called? How is this equivalent linear-force field different from the analogue result with the inverse-square-law force?

d) Without words, show for a general charge distribution 1 and a spherical symmetric charge distribution 2 that the force of distribution 1 on distribution 2 is exactly the same as when distribution 2 is replaced point-charge 2. If charge distribution 1 were also spherically symmetric, what be the force between them be equal to and what would it be if their centers coincided exaclty?

e) Say you had a charge distribution that maintained spherically symmetry no matter what, that had its center of mass at its center, and the only external forces that acted on it were external linear forces. How would described its motion? Recall Newton's 2nd law:

$$\vec{F}_{\text{net external}} = m\vec{a}_{\text{cm}} \ ,$$

where $\vec{F}_{\text{net external}}$ is the net external force on a body of mass $m$ and $a_{\text{cm}}$ is the center of mass of the body. Given the result of part (d) Without words, show for two spherically symmetric distribution charges that the force of distribution 1 on distribution 2 is exactly the same **HINT:** Recall the part (d) answer.

f) Is the linear force for spherically symmetric mass distribution with mass as its charge consistent with linear force that occurs in the Newtonian derivation of the Friedmann equation:

$$\vec{F} = \frac{\Lambda}{3}mr\hat{r} \ ,$$

where $m$ is a test particle mass. There is no right answer. This is a discussion question.

8. The Friedmann equation of general relativity (GR) cosmology in its most standard form (e.g., Wikipedia: Friedmann equations: Equations) is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \ ,$$

where $H$ is the Hubble parameter (which at current cosmic time is the Hubble constant $H_0$ and has fiducial value $70\,(\mathrm{km/s})/\mathrm{Mpc}$), $a$ is the cosmic scale factor, $\dot{a}$ is the time derivative of the cosmic scale factor with respect to cosmic time $t$, $G = 6.67430(15) \times 10^{-11}\,\mathrm{J\,m/kg^2}$ is the gravitational constant, $\rho$ is the density of a uniform perfect fluid (in old-fashioned jargon AKA the cosmological substratum: Bo-75–76) which is used to model the universal mass distribution, $k$ is called the curvature (Li-24,28) $k/(c^2a^2)$ is called Gaussian curvature (CL-12,29), $c = 2.99792458 \times 10^8\,\mathrm{m/s}$ is the vacuum light speed as usual. and $\Lambda$ is the cosmological constant which is the simplest form of the dark energy even though is only a from of energy in one interpretation. Note $k$ is often defined with an unabsorbed $c^2$: i.e., the shown $k$ is replaced by $kc^2$.

There are parts a,b,c. Some of the parts can be done independently, and so do not stop if you cannot do a part. During exams do **ONLY** parts a,b,c,d.

a) Without words prove the Friedmann equation starting from the work-energy theorem

$$E_{\mathrm{mechanical}} = \frac{1}{2}mv^2 - \frac{GMm}{r} - \frac{1}{2}\frac{\Lambda}{3}mr^2 \ ,$$

where $m$ is the mass of a test particle.

b) Without words prove the general Hubble law $v = Hr$, where $v$ is recession velocity (i.e., the velocity between comoving frames) and $r$ is proper distance (i.e., the distance measurable in with a ruler at one instant in cosmic time).

c) What is the asymptotic Hubble law (i.e., Hubble law valid in the limit $z \to 0$)?

9. The scaled Friedmann equation for multi-component (power-law) density components is

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \sum_p \Omega_{p,0} x^{-p} \ ,$$

where $0$ indicates the fiducial time which may be cosmic present, $h = H/H_0$ is the scaled Hubble parameter with $H_0$ being the Hubble constant, $x = a/a_0$ is the scaled cosmic scale factor, $x_0 = 1$, $\dot{x} = dx/d\tau$ is the rate of change of the scaled cosmic scale factor, $\tau = H_0 t = t/t_{H_0}$ is the scaled time with $t_{H_0}$ being the Hubble time, the $\Omega_{p,0}$ are the density parameters for the density components at the fiducial time with their sum being 1, and $p$ are the powers of the power-law density components.
**NOTE:** There are parts a,b,c,d,e,f,g. On exams, do **ONLY** parts a,b,c,d.

a) Without words, derive the general asymptotic solution $\tau(x)$ and its inverse $x(\tau)$ for the leading density component as $\tau \to 0$ (i.e., the density component with highest $p$). As a shorthand, this solution can be called the early universe solution. Assume $p > 0$. To avoid pointless generality, assume $x(\tau = 0) = 0$ (i.e., there is a point origin at time zero).

b) Without words, derive early universe formula for $\Omega_p(\tau)$ for $p > 0$.

c) Without words, derive the special case early universe solutions for $p = 1, 2, 3, 4$.

d) Without words, derive the Hubble parameter $h = \dot{x}/x$ and the deceleration parameter $q = -\ddot{x}x/(\dot{x})^2 = -\ddot{x}/(xh^2)$ for the general early universe with $p > 0$. Simplify the latter as much as possible. For what $p$ values is the universe in positive/negative acceleration? For what $p$ value is the universe coasting?

e) We now assume the universe has only one density component with power $p > 0$. Without words, derive the generic age of the universe formula (which we assume to the fiducial time where $x = 1$) for $\tau$ and $t$ and give the fiducial value version for $t$ with the Hubble time $t_{H_0} = (13.968 \ldots \mathrm{Gyr})/h_{70}$, where $h_{70} = H_0/[70\,(\mathrm{km/s})/\mathrm{Mpc}]$. special case solutions for $p = 1, 2, 3, 4$. Note the fi

f) We assume the universe has only one density component with power $p = 0$. Without words, derive $x(\tau)$ and $x(t)$ assuming $x(0) = 1$. Note this universe is the de Sitter universe and the Hubble constant $H_0 = \sqrt{\Lambda/3}$.

g) Students are now welcome to view a table in the answer to this part that presents the single density component solutions plus relevant features for powers $p = 4, 3, 2, 1, 0$. Note that if we assume that the dependence of the density components on the scale factor is due to a perfect fluid pressure obeying the equation of state $p_{\text{pressure}} = w\rho c^2$ where $w$ is a constant parameter (with no special name), then power

$$p = 3(1 + w) .$$

The $w$ values are included in the table.

10. The differential equation (DE) for the perfect fluid of cosmology of the Friedmann equation is

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) ,$$

where $\rho$ is mass-energy in the comoving frames of Friedmann cosmology and $p$ is isotropic pressure in those frames (in some sense) (Liddle 26). The perfect fluid DE can be derived rigorously from general relativity (Carroll 333–334) or, perhaps fudgily, from classical thermodynamics. Remarkably, this equation does not guarantee conservation of energy in the ordinary sense of classical physics: it does embody the general relativity feature that the covariant derivative of the energy-momentum tensor is zero (Carroll 117,120): i.e., the energy-momentum conservation equation. General relativity may or may not in some sense conserve energy for cosmology, but certainly gravitating mass-energy is allowed to appear and disappear by the perfect fluid DE.

Multiple perfect fluids can exist and if they are assumed to act independently (which is the usual cosmological assumption), then they all obey there own perfect fluid DE: i.e., for perfect fluid $i$

$$\dot{\rho}_i = -3\frac{\dot{a}}{a}\left(\rho_i + \frac{p_i}{c^2}\right) .$$

In current standard cosmology (i.e., the $\Lambda$CDM model or simple variations thereof), it is assumed that the perfect fluid equation of state (EOS) is of the form

$$p = w\rho c^2 ,$$

where $w$ is a constant parameter that seems to have no special name. Most standard/interesing values of $w$ are given by

$$w = \begin{cases} 0 & \text{for nonrelativistic (NR) mass-energy (AKA "matter"} \\ & \text{or "dust": Liddle-40);} \\ 1/3 & \text{for extreme relativistic (ER) mass-energy: most obviously photons,} \\ & \text{but also the ER neutrinos of the Big Bang era and early cosmic dark ages;} \\ -1 & \text{for cosmological constant (which name can also be used for constant dark energy;} \\ -1/3 & \text{for zero-acceleration (or constant } \dot{a}) \text{ universes such as} \\ & \text{Fulvio Melia's } R_{\text{h}} = ct \text{ universe or a universe with cosmic scale} \\ & \text{determined only by negative curvature } k. \end{cases}$$

Solve for the formula for $\rho(a)$ for general $w$ and the 4 special cases of $w$ listed above. Assume $a_0$ and $\rho_0$ for cosmic present values.

11. Here we do the quick derivations of the Friedmann equation, the fluid equation, the Friedmann acceleration equation, and some other results.
    **NOTE:** There are parts a,b,c,d,e,f,g. On exams, do **ONLY** parts a,b,c,d,e. Some of the parts can be done independently, and so do not stop if you cannot do a part.

a) Without words, derive the Friedmann equation in standard form (with the cosmological constant force $F_\Lambda = (\Lambda/3)mr$ included) from classical physics with the hypotheses that all free-fall frames are elementary inertial frames (as told to us by general relativity) and that the shell theorem for a spherically symmetric mass distribution can be extended to infinite distance (which is validated by Birkhoff's theorem from general relativity). The derivation makes use of classical conservation of mechanical energy. You should end up with a $-k/a^2$ term among other things. You can a draw

diagram if you like. **HINT:** Start with the conservation of mechanical energy of a test particle of mass $m$:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} - \left(\frac{1}{2}\right)\frac{\Lambda}{3}mr^2 \ .$$

b) Without words and starting from the 1st law of thermodynamics

$$dE = T\,dS - p\,dV + \mu\,dN \ ,$$

derive the cosmological fluid equation in standard form (which means with $dS = 0$ adn $dN = 0$) and in a form with $\dot{\rho}a/\dot{a}$ equal to something for use in part (d). Recall the rest-frame energy is $E = \rho c^2 V$.

c) Specialize the fluid equation to the special case where the equation of state is $p = w\rho c^2$ where $w$ is the constant equation of state (which seems to have no special name). Determine the explicit solution $\rho(a)$ for the special case where $\rho_0 = \rho(a_0)$. **HINT:** You will have to eliminate the time derivative.

d) Without words, derive the acceleration equation (or Friedmann acceleration equation) in standard form using parts (a) and (c). A subtle point is that you have to assume that the graviational potential energy formula continues to be valid (though perhaps with a different meaning) for cases where mass is not conserved. There is an argument why it should, but that is beyond the scope of this question.

e) Without words, derive from the Friedmann equation the de Sitter universe solution which has $\rho = 0$ and $k = 0$, but $\Lambda \neq 0$.

f) Without words, derive the scaled Friedmann equation

$$h^2 = \left(\frac{\dot{x}}{x}\right)^2 = \Omega_{\text{non-}k,\Lambda} + \Omega_k + \Omega_\Lambda$$

with the scalings $x = a/a_0$, $\tau = H_0 t$, $h = H/H_0$, $k_a = k/a_0^2$, and $\rho_c = 3H_0^2/(8\pi G)$. Note the subscript 0 indicates fiducial time $t_0$ which is often cosmic present and is not in general the Hubble time. Implicitly show expressions for $\rho_k$, and $\rho_\Lambda$ and the density parameters in the derivation. What is the curvature equation at the fiducial time: i.e., the formula for $\Omega_{k,0}$. What does it mean if $\Omega_{k,0} = 0$ exactly.

g) Without words, derive the scaled accelertion equation using the same scalings and expressions as in part (f) and $p = w\rho c^2$.

12. Consider the following linear 1st order differential equation (DE):

$$x' = A - kx \ ,$$

where $t$ is the independent variable, $A > 0$ is a constant, and $k > 0$ is the rate constant.
There are parts a,b,c,d. Parts (a) and (b) can be done independently at least.

a) Solve for the constant solution $x_A$. **HINT:** This is easy.

b) We can now write the DE as
$$x' = k\,(x_A - x) \ .$$

Without solving for non-constant solution describe what it must look like as a function of $t$ for arbitrary initial value $x_0 = x(t = 0)$. In particular, where are its stationary points if any? **HINT:** Consider the continuity of all orders of derivative of $x$.

c) Given $x_0 = x(t = 0)$, solve for the solution $x(t)$, $x'(t)$, and the 1st order in small $t$ solution $x_{1\text{st}}(t)$. **HINT:** You can use an integrating factor, but there is a more straightforward way.

d) What is the $e$-folding time $t_e$ of your solution and what does it signify? What is the $x(t_e)$? What is the $x_{1\text{st}}(t_e)$? What is remarkable about $x_{1\text{st}}(t_e)$?

13. First order autonomous ordinary differential equations (FAODEs), linear or nonlinear, only have solutions with stationary points at infinity (SPIs), (except for special cases which are not all that

rare) and constant solutions. Actually, each SPI corresponds to a constant solution which could also be viewed as a continuum of stationary points. Note an autononous differential equation depends only on functions of the dependent variable, and so has no explicit dependence on the independent variable.

To investigate the SPI behavior of FAODEs consider the (somewhat general) FAODE

$$x^{(1)} = [f(x)]^{1/k} \ ,$$

where $t$ (not necessarily time) is the independent variable, the superscript (1) means 1st derivative with respect to $t$, $f(x)$ is an infinitely differentiable function with zeros at set of values $\{x_i\}$, and $k > 0$. We limit $k$ to being greater than zero to avoid uninteresting generality. Since $f(x)$ is infinitely differentiable at (general) $x_i$, we can expand $f(x)$ about $x_i$ with some radius of convergence: i.e.,

$$f(\Delta x) = \sum_{j=\ell}^{\infty} \Delta x^j f_j = \Delta x^{\ell} f_{\ell} + \dots \ ,$$

where $\Delta x = x - x_i$, the $f_j$ are expansion constants, and $\ell > 0$ is the lowest (nonzero) order in the expansion. Note $\ell \neq 0$ since we have assumed $x_i$ is a zero of $f(x)$: i.e., $f(x_i) = 0$.

We will primarily be examining the lowest order solutions in $\Delta x$, and so we will be dealing with $\Delta x^{\ell/k} f_{\ell}^{1/k}$ and related expressions. Mathematically, if $\ell/k$ is not an integer, complex numbers can arise in these expressions. However, we are only interested FAODEs and their solutions corresponding to physical systems involving real numbers. In these systems, the solutions just never evolve into the complex number realm. So we are not going to concern ourselves with question what happens mathematically if some our expressions can give rise to complex numbers. They never give rise to complex numbers physically.

**NOTE:** There are parts a,b,c,d,e,f,g,h,i,j,k. On exams, only do parts i,j.

a) What is the behavior of $x$ as a function of $t$ between the points in the set $\{x_i\}$.

b) In this question we are only interested in the SPI behavior and constant solution behavoir, and so we are only interested in the behavior of $x(t)$ when it is arbitrarily close to $x_i$ where SPI and constant solutions occur. Therefore expand the FAODE about $x_i$ with dependent variable $\Delta x$ to lowest order in the exponent.

c) Determine the formula $p(n)$ for the exponent of $\Delta x$ in the $n$ derivative of $\Delta x$ (for the lowest order of the FAODE) with respect to $t$. **HINT:** Drop all constants that turn up in the differentiations.

d) What is behavior of the $t$ derivatives of $\Delta x$ when $x = x_i$ for $\ell/k \geq 1$? What solutions $x(t)$ are implied by $\ell/k \geq 1$?

e) What is behavior of the $t$ derivatives of $\Delta x$ for $f(x_i)$ for $\ell/k < 1$ assuming the formula $p(n)$ never equals zero? What solution $x(t)$ behavior is implied by $\ell/k < 1$ in this case? Only a short answer is expected to the last question.

f) If $\ell/k < 1$ and the formula $p(n)$ goes to zero for a stopping $n_{\text{st}}$, what is the formula for $\ell/k$ as a function of $n_{\text{st}}$ and what are the values of $\ell/k$ for the set $n_{\text{st}} = 1, 2, 3, \dots, \infty$ and what do the $n_{\text{st}} = 1$ and $n_{\text{st}} = \infty$ cases mean? What is the formula $n_{\text{st}}$ as a function of $\ell/k$? What is this formula good for?

g) What is implied by a stopping $n_{\text{st}} \in [2, \infty)$ (i.e., an actual integer $n_{\text{st}}$ in this range)? Give the solution for small $\Delta x(t)$ with with initial condition $\Delta x(t = 0) = 0$. Describe the function behavior at $\Delta x(t = 0) = 0$: i.e., maximum or minimum stationary point or rising or falling inflection point.

h) What would you expect the two likeliest values for $\ell$ to be for physically relevant FAODEs? What would you expect the two likeliest value for $k \neq 1$ to be for physically relevant FAODEs?

i) Now we intuited for the case of $\ell/k \geq 1$ that the stationary point would be a SPI, but we did not prove this directly. To prove directly, we need to show that the small $\Delta x$ (meaning small in absolute value) solutions of

$$\Delta x^{(1)} = \Delta x^{\ell/k} f_{\ell}^{1/k}$$

that go to zero only do so as $t \to \infty$. Solutions that go to zero are convergent solutions. This means that the constant solutions they correspond to are stable solutions: small perturbations from the constant solutions damp out. Those that do not go to zero are divergent solutions. This means

that the constant solutions they correspond to are unstable solutions: small perturbations from the constant solutions cause non-stopping divergence from the constant solutions. Here consider the $\ell/k = 1$ case and the solutions for $\Delta x(t)$ starting from $t = t_0$ and $\Delta x = \Delta x_0$ as initial conditions. Determine the solutions and under what conditions they are convergent/divergent. Does convergent solution, in fact, have a SPI? **HINT:** Let $y = \pm\Delta x$ where the upper/lower case is for positive/negative $\Delta x_0$.

j) Repeat part (i) for the case of $\ell/k > 1$.

k) An optional continuation of the discussion of the part (h) answer.

14. In this problem, we will get some more insight into first order autonomous ordinary differential equations (FAODEs) with stationary points that are not stationary points at infinity (SPIs) by examining a solution beyond solution to lowest (nonzero) order around the stationary points. Consider the FAODE

$$x^{(1)} = f(x) \ ,$$

where $f(x_i) = 0$ (i.e., $x = x_i$ gives a stationary point of some kind) and the independent variable is $t$ (not necessarily time). However,

$$x^{(2)} = \frac{df}{dx}x^{(1)} = \frac{df}{dx}f(x) \neq 0$$

for $x = x_i$. This means the stationary point is not a SPI.
   **NOTE:** There are parts a,b,c,d. On exams, only do parts a,b,c.

a) Let

$$g(x) = \frac{df}{dx}f(x)$$

and determine a formal solution for $f(x)$.

b) Assume $x(t)$ has maximum and minimum at, respectively, $x_i$ and $-x_i$. Now invent the simplest $f(x)$ you can starting from the part (a) answer, except it has a general constant coefficient so as to give a general scale to the derivative $x^{(1)}$.

c) Now solve for $x(t)$ given the part (b) answer. **HINT:** You could do this by integrating $x(t)$, but differentiating $x(t)$ lead to solution by inspection.

d) Say a FAODE is given by

$$x^{(1)} = [f(x)]^{1/k} \ ,$$

where $t$ is the independent variable (not necessarily time), $k > 0$, $f(x)$ is infinitely differentiable, and $f(x) = \Delta x^\ell f_\ell + \ldots$ is the expansion of $f(x)$ around the stationary point $x_i$ with $\Delta x = x - x_i$ starting with the lowest nonzero order. Then the lowest order FAODE is

$$\Delta x^{(1)} = x^{\ell/k} f_\ell^{1/k} \ ,$$

In order for a solution of the FAODE to have stationary point that is not a SPI, there must be a stopping (derivative order) $n_{\text{st}}$ given the formula

$$n_{\text{st}} = \frac{1}{1 - \ell/k}$$

where an actual stopping $n_{\text{st}}$ must be an integer. If the formula gives a non-integer value, then there is a singularity in the behavior of some order of derivative of $x(t)$ at $x = x_i$ and that behavior takes some analysis to determine. An actual stopping $n_{\text{st}}$ gives the only nonzero derivative order of $x(t)$ at $x = x_i$. What are the $\ell$ and $k$ values for the FAODE used in the part (c) and are they consistent with a nonzero derivative order $n = 2$ which is what we imposed in the preamble?