Updated 2019jul18

3001

updated 2021(sep

3 Newtonian Gravity & ~~Physics~~ & The Friedmann Equations

3.0a Historical Intro

⇒ followed by ~~other~~ intros—review—curious quirk

The Friedmann equations for a(t) scale factor

(i.e., the ~~Friedmann equation~~ plus the Friedmann acceleration equation)

governs the overall dynamics of the Observable universe

——provided General Relativity (GR) is a correct theory

well a correct emergent theory since we believe it is ~~the~~ macroscopic limit of quantum gravity—for which there is no established theory.

Diff eq of motion of universal expansion 1st order ODE nonlinear (>can't add solution

Notes drawn from

Liddle (2015)

Wik

Bondi (1960) for historical tidbits

Carroll (2004)

Coles & Lucchin (2002)

Penrose (2004)

Weinberg (1972)

Barden (2002)

lots of variations but all are complex none have required useful for field theorists to develop their power

Though fairly simple

Because nonlinear—in general solutions of Friedmann eqn will not add to make a solution

al

$r = r_0 a$

$a_0 = 1$ convention

$H_0 = \frac{\dot{a}}{a}\Big|_0$ relative rate of expansion of universe inverse time in $\frac{km/s}{Mpc}$

In fact, the Friedmann equations can be derived from Newtonian physics with plausible/natural ad hoc assumptions

{ If only ordinary matter is considered
{ -Radiation & ~~Cosmological constant~~ ~~constant~~ Dark energy

However, one needs the GR perspective to understand that they include the

All kinds of mass-energy with enough natural assumptions

At universal level Λ and constant energy same ~~Beat~~ But differ in theory microscopically

constant or other ~~sense as~~ Λ-effectively

002

effects of the curvature of space → and radiation + Λ constant + dark energy)

(but he derived them in Russian)

Historically, the Friedmann equations ~~was~~ as first derived by Alexander Friedmann (1888–1925) in 1922 in Russian (Wiki Friedmann eqs)

Georges Lemaître derived ~~them~~* independently a bit later in the 1920s (Einstein himself inimical to work of Friedmann

The Newtonian derivation came later remarkably by Milne & McCrea in 1934. (Bondi-75

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It's Remarkable that a lot of progress in cosmology could have been made before GR (1915), but that didn't happen.

~~Two~~ Three main hold-ups

1) People seemed to belive the universe was static on average

— Newton thought this

— odd since the universe is not in thermodynamical equilibrium

↳ as manifested by Olbers' Paradox

— a dark night sky is inconsistent with an eternal unchanging static universe — homogeneous & isotropic too

⇒ if true in every direction you should see a star (homework problem ___)

bmy
people
that it's hard
to know, no
opinion polls
from the past
before ~1950

2) People didn't yet know there were other galaxies until 192~~3~~4 when Hubble established that the Andromeda nebula was the Andromeda galaxy (Wiki Hubble) — so then all spiral nebulae were spiral galaxies and all elliptical nebulae too since they came clustered with spirals

people had suspected this for a long time — since the 18th century (Wiki galaxy)

Even the 17th Chr. Wren in ~1660 suggested before leaving astr. + physics for "rubbish"!

In any case, it was hard to ~~get this~~ get an expanding universe idea without knowing galaxies and observing their redshifts which came along with Slipher ~~after~~ starting in 1912 (but very slowly)

3) Need GR idea of free-fall frame being the 'true' inertial frame

Of course a ~~vigorous~~ vigorous derivation of the Friedman equations must be from GR — but we leave that to another course — one on GR

A vigorous demonstration as to why ~~the~~ GR Friedman eqn ~~GR~~ and Newtonian analogs ~~are the same~~ should be the same is given by Wells (2014)

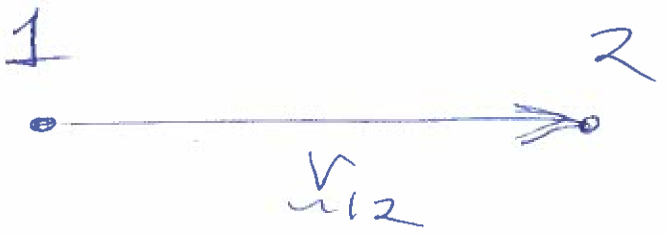
To become an expert in cosmology, you should study GR
(not me, just a specialist in teaching intro cosmology)

3004

~~But~~ Before we jump to the derivation, we do a [illegible] in a lot useful results — I felt I had to do these to know what I meant.

3.06 Newtonian Gravity, Gauss' Law, & The Shell Theorem

Newton's law of universal gravitation



$$\hat{r}_{12} = \frac{\underset{\sim}{r}_{12}}{r_{12}}$$

$$\underset{\sim}{F}_{12} = -\frac{G m_1 m_2}{r_{12}} \hat{r}_{12}$$

an ideal limit law

— formally it's for point masses

Newton never wrote it down like this. He used an obscure formalism in the Principia (1687) and a few year later Pierre Varignon translated Newton's results into Leibniz calculus formalism, but vector notation didn't come until the later 19th century (Wiki: Euclidean Vector)

were recognizable to us

We can derive Gauss' law from it now — nonrigorously

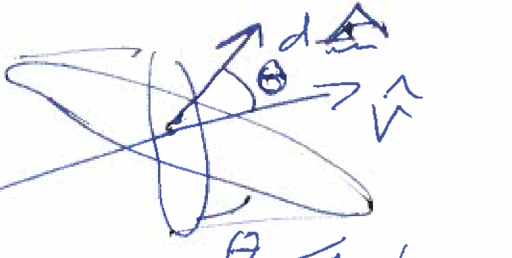
Gauss' Law

3005

For a force field (force per unit generic charge) from a point source

$$\underset{\sim}{f} = \frac{Q}{r^2}\hat{r}$$

Only field of the charge. There may be other fields source around!!



θ < Angle from radial direction

Very special results due to inverse square-law nature

$d\underset{\sim}{A}$ is differential vector area

Consider

Now

$$dA_R = dA \cos\theta$$

$$\underset{\sim}{f} \cdot d\underset{\sim}{A} = \frac{Q\hat{r}\cdot d\underset{\sim}{A}}{r^2}$$

$$= Q\frac{dA\cos\theta}{r^2}$$

$$= Q(\pm d\Omega)$$

We are actually assuming a lot about 3-d Euclidean geometry here but geometry is another course

differential solid angle

+ for ~~outward~~ $(\theta<\frac{\pi}{2})$ radial

− for anti-radial $(\theta>\frac{\pi}{2})$

since $d\Omega$ is conventionally always +ve

$\oint \underset{\sim}{f}_i\cdot d\underset{\sim}{A} = Q_i 4\pi$

one charge

$\oint \underset{\sim}{f}\cdot d\underset{\sim}{A} = 4\pi Q$

only 4π for $\oint\Omega$ because of cancelation

See p. 3006

Consider now a closed surface. It has a definite inside and outside

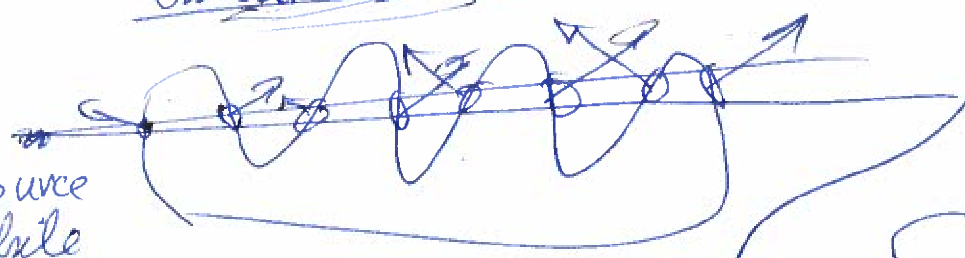
A very special result dependent on the inverse-square law

3006

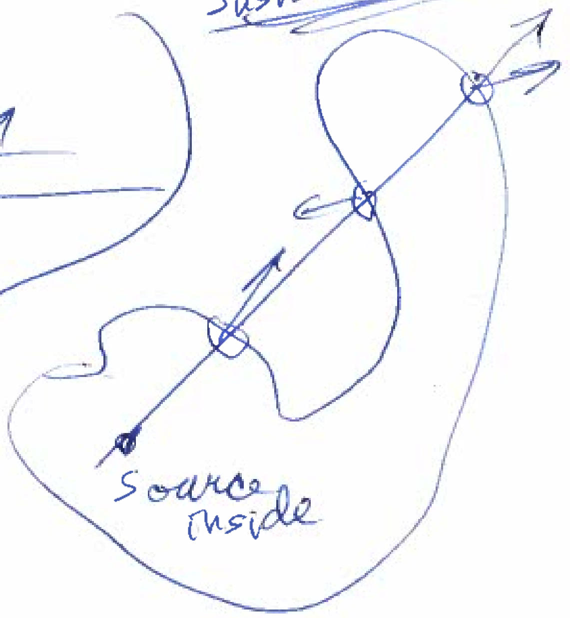
Question
Is topology among other things, a way to get yes/No answers
Just musings

Any radial cone from an inside source or outside source too. that goes thru the surface must pass from inside to outside or vice versa and must ultimately pass to the outside

outside Source



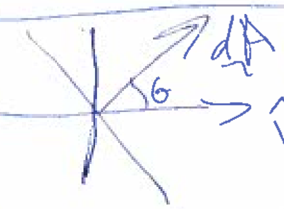
Inside Source



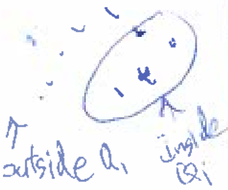
"in" always -ve ?
"out" always +ve

and so must ~~cancel~~? all differentiated solid angle bits cancel? yes

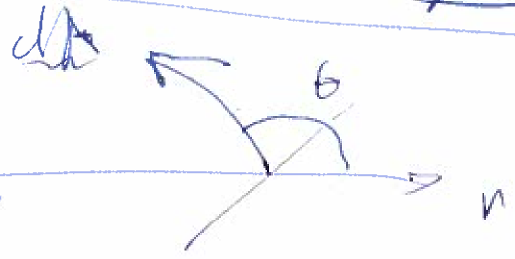
Always one more "out" than "in" and so one solid angle bit uncanceled? yes


$d\underset{\sim}{A}$ out arrow
θ → $\hat{r}$ radial direction

$\oint f_i \cdot dA$
$= 4\pi \sum_i Q_i$ inside



$\theta \in [0, 90°]$ always an "out" and always positive
$\theta \in [90°, 180°)$ always an "in" and always negative



Gauss' generic form law $\oint \underset{\sim}{f} \cdot d\underset{\sim}{A} = 4\pi Q_{\text{encl}}$ Sup. 300s 300

Gauss' law (Wik

Only field of G and any other field involving Q [illegible]

~~integral~~ form

Gravity

$$\underset{\sim}{g}_i = -\frac{GM_i}{r_i^2}\hat{r}_i$$

gravitational field

$Q \leftrightarrows -GM$

Gravity always a -ve charge

So make minus explicit

M_i

point source i

$\underset{\sim}{\hat{r}}_i$

$\oint \underset{\sim}{g} \cdot d\underset{\sim}{A}$

No Einstein summation

$$\sum_i \oint \underset{\sim}{g}_i \cdot d\underset{\sim}{A} = -G\sum_i M_i 4\pi$$

inside

And mass is always -ve unlike electric charge

in total; Can have -ve contributions

$$= -4\pi G M_{\text{inside}}$$

$\boxed{\text{only -ve}}$

minus because gravity always attractive

Coulomb force

k = coulomb constant $\doteq 8.99 \times 10$ $\doteq 10^{10}$ [illegible] (Wik

$$\underset{\sim}{E} = \frac{kq_i}{r_i^2}\hat{r}_i$$

Electric field

q_i

$\underset{\sim}{\hat{r}}_i$

$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A}$

No Einstein summation

$$\sum_i \oint \underset{\sim}{E}_i \cdot d\underset{\sim}{A} = k\sum_i q_i 4\pi$$

$Q \leftrightarrows kq$

$$= 4\pi k q_{\text{inside}}$$

$$= \frac{q_{\text{inside}}}{\varepsilon_0}$$

+ve

Vacuum Permitivity

$f = r^2 d\Omega$

Very special results due to special nature of inverse-square law forces

For high symmetry cases, One can use the integral Gauss' law to obtain solutions

1) spherical sym,
2) cylindrical sym
3) planar sym

and that's all I think

We'll look at this case in a moment

convergent evolution of physics

that Both grav & EM have this was one reason Einstein saw ~~it~~ his unified field theory approach was true path But no. One needs the quantum mechanics + ~~field theory~~ QFT idea

3008

Gauss' theorem

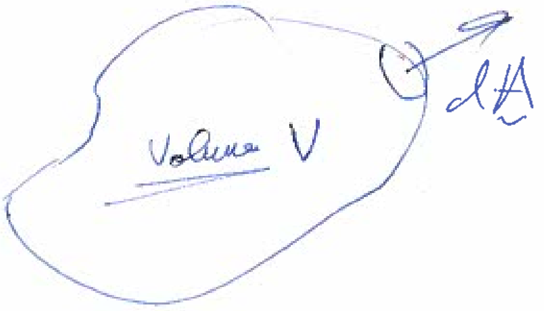
(AKA divergence theorem)

vector field

For differential form of Gauss Law

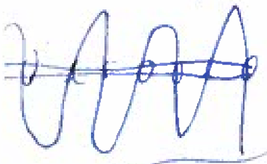
3010

$$\int \nabla\cdot\underset{\sim}{F}\, dV = \oint \underset{\sim}{F}\cdot d\underset{\sim}{A}$$

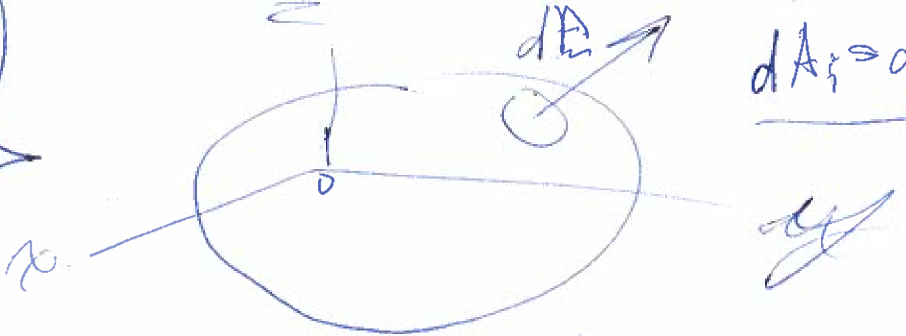


Proof Consider a sufficiently small volume to allow 1st order expansion to be accurate

unny shape



lways u-out pair



$dA_i = dA\cdot\hat{x}_i$

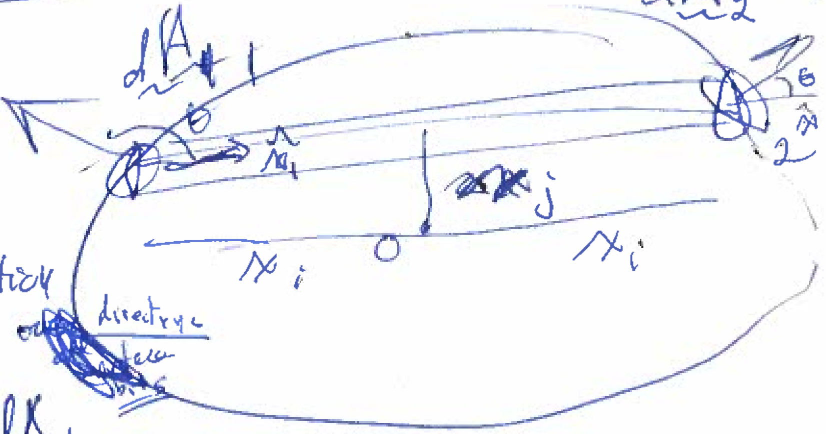
$$\oint \underset{\sim}{F}\cdot d\underset{\sim}{A}$$

$$= \oint F_i\, dA_i$$ with Einstein summation on i

Taylor expand

$$= \oint \underbrace{\left(F_{i0} + x_j \left.\frac{\partial F_i}{\partial x_j}\right|_0\right)}_{} dA_i$$

This term cancels pairwise — along and x_i line $dA_{i_1} = -dA_{i_2}$ $F_{i0}\oint dA_i = 0$

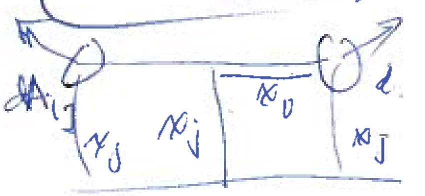


This are constants over all small volume

$$= \oint x_j \left.\frac{\partial F_i}{\partial x_j}\right|_0 dA_i$$

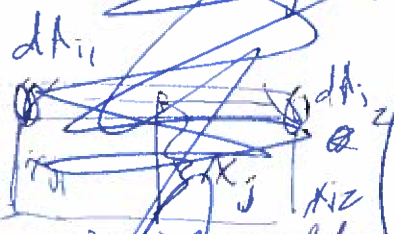
$$= \frac{\partial F_{i0}}{\partial x_j} \oint x_j \, dA_i$$

3009



~~$x_j dA_{i1} + x_j dA_{i2}$~~ ~~but if $j \neq i$ $x_{j1} = x_{j2}$~~

~~$\delta_{ij} dV$~~



pair wise cancelation

cancels pairwise also if $i \neq j$

$$\sum_{k=1,2} (x_j \, dA_j)_k = \delta_{ij} dV$$

~~No sum~~ ~~i~~

$dA_{i1} x_{i1} + x_{i2} dA_{i2}$
$= |dA_{i2}| (x_{i2} - x_{i1})$
But $dA_{i1} x_j + dA_{i2} x_j$
$= x_j (dA_{i2} - |dA_{i1}|)$

No Ein S

a volume element tube-like $= C$

No Einstein sum

$dA_{i1} x_{i1}$
$+ dA_{i2} x_{i2}$
$= dA_{i2} (x_{i2} - x_{i1})$
dV

$$= \oint \left.\frac{\partial F_i}{\partial x_i}\right|_0 d\cancel{V}$$

$$= \oint \nabla \cdot F \, dV$$

~~$\nabla \cdot F \, dV$~~

$$= \nabla \cdot F \; V$$

for a sufficiently small volume

d/ Einstein summation clouds the issue

$$\sum_{ij} \oint x_j \frac{\partial F_i}{\partial x_j} dA_j$$

$$= \sum_i \oint \frac{\partial F_i}{\partial x_i} dV_i$$

$$= \left(\sum_i \frac{\partial F_i}{\partial x_i} \right) V$$

$$= \nabla \cdot F \; U$$

~~$dA_{j1} x_{j1} + dA_{j2} x_{j2}$~~

~~if $i \neq j$ then $x_{j1} = x_{j2}$ and $x_j (dA_{i2} + dA_{i1}$~~

~~if $i = j$ then $dA_{i2} (x_{i2} - x_{i1}) = dV$~~

3010

Finite Volume

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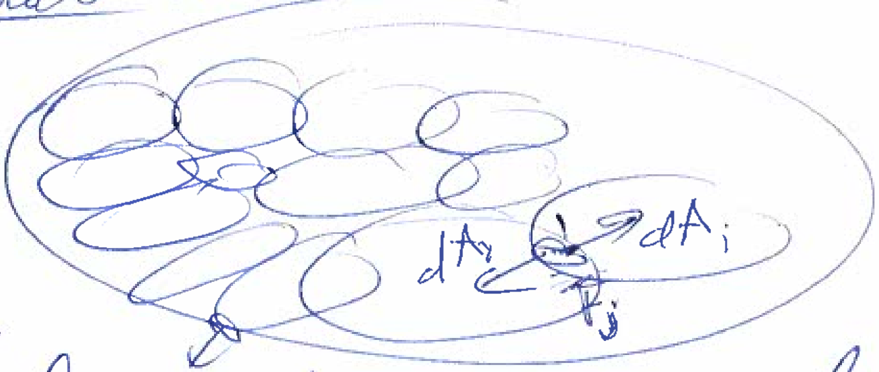

Now just add but small volume.

the internal surfaces cancel pairwise

and only the outer surfaces contribute and you get Gauss' theorem QED.

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airwise

$$\oint \underset{\sim}{F}\cdot d\underset{\sim}{A} = \int \nabla\cdot\underset{\sim}{F}\, dV$$

Gauss' Law Differential form

$$\oint \underset{\sim}{f}\cdot dA = 4\pi \sum_{i\ \text{inside}} Q_i$$

see proof p.3005 _3007

total field per unit charge

sum of inside ~~internal~~ generic charges

see p.3007

Say ~~these~~ from a continuum

$$\sum_i Q_i = 4\pi\int \rho\, dV$$

ρ is a density

Point source

$\odot$

$\int \nabla\cdot\underset{\sim}{f}\,dV = 4\pi Q$

$4\pi Q\,\delta(\underset{\sim}{r})$

for consistency Dirac Delta function

$$\therefore \oint \underset{\sim}{f} \cdot d\underset{\sim}{A} = 4\pi \int \rho \, dV$$

$$\oint \nabla \underset{\sim}{f} \, dV = 4\pi \int \rho \, dV$$

using divergence Thm p.3008 ~~[illegible]~~ or Gauss Thm (p.3008)

but since the shape is general

$$\nabla \cdot \underset{\sim}{f} = 4\pi \rho$$

{General Gauss' law differential form

Gravity ⟷ ~~Coulomb's force~~

see p.3007

$$\nabla \cdot \underset{\sim}{g} = -4\pi G \rho_{mass} \qquad \nabla \cdot E = 4\pi k \rho_{charge}$$

$$= \frac{\rho_{charge}}{\varepsilon_0}$$

(Wik (Gauss law))

For point charge

$$\nabla \cdot \underset{\sim}{g} = \begin{cases} 0 \\ \text{except for Delta function at the origin} \end{cases}$$

{See p.3016 Bottom

$\int \nabla \cdot \underset{\sim}{g} \, dV = -4\pi G m$

$\int \underset{\sim}{g} \cdot d\underset{\sim}{A}$

recover Point gravity law at p.3007

$= -4\pi G \delta(\underset{\sim}{r}-\underset{\sim}{r}_1) M$

~~the Mass distribution~~ ~~[illegible]~~ ~~[illegible]~~

~~Shell Theorem~~

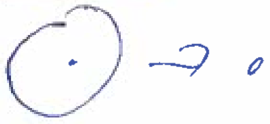
See p.3012 for general statement c

a & b are corollaries

Shell Theorem: Gravity Case

a/b/c are 3 perspectives

a) A spherically symmetric body acts as if all mass were concentrated at a point; i.e., it acts like a point mass at the center of symmetry



3012

b) A spherically symmetric shell exerts no force on a body inside no matter where inside

(b) is really a corollary of (a).

c) General statement

Given a spherical symmetric distribution

at r only that interior if all very concentrated at centre

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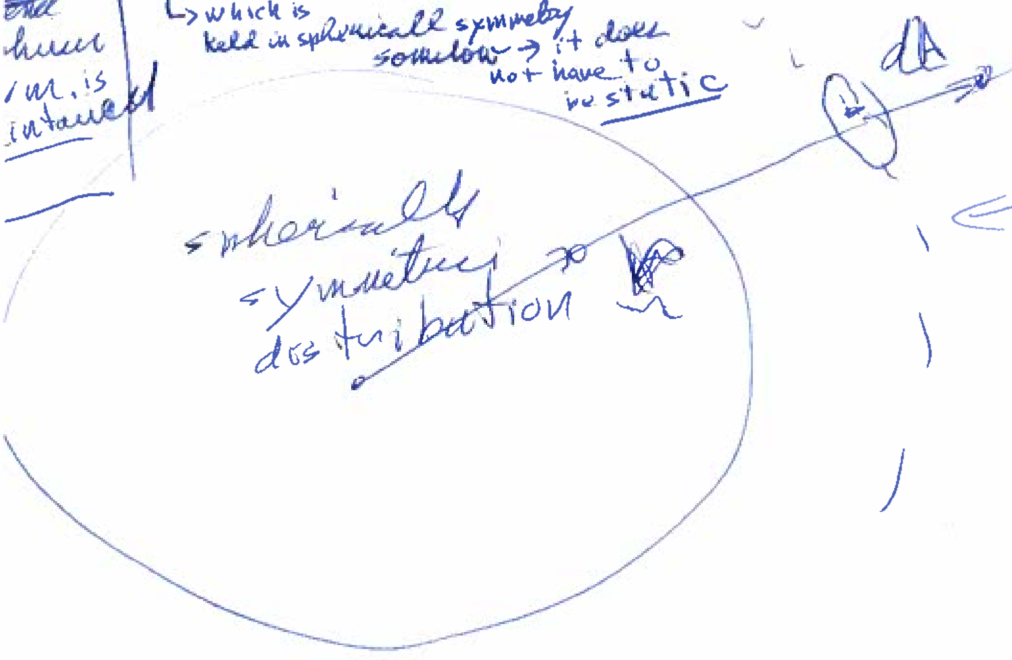
Proof from Gauss' law for gravity integral form

Note $\underset{\sim}{g}$ is not the total gravitational field, just the field of the mass considered

$\underset{\sim}{g}$ + $\underset{\sim}{dA}$ point opposite

$$\oint \underset{\sim}{g} \cdot \underset{\sim}{dA} = -4\pi G M_{enclosed}$$

→ which is held in spherical symmetry somehow → it does not have to be static



could extend to here

Gaussian surface.

By symmetry $\underset{\sim}{g}$ must be radial and have equal magnitude at $r = |\underset{\sim}{r}|$

∴ ~~$\oint \underset{\sim}{g} \cdot \underset{\sim}{dA} = g(r)\, 4\pi r^2 = -4\pi G M_{encl}$~~

~~$|g(r)| =$~~

That minus sign

means $\underset{\sim}{g}$ and $d\underset{\sim}{A}$ point in opposite directions: $\underset{\sim}{g}\cdot d\underset{\sim}{A} < 0$

minus signs cancel and

$$\therefore \quad -g(r)\,4\pi r^2 = -4\pi G M_{\text{encl}}$$

$\Leftarrow g = \frac{GM_{\text{encl}}}{r^2}$

$$\therefore \quad \underset{\sim}{g}(\underset{\sim}{r}) = -\frac{G M_{\text{intern}}}{r^2}\hat{r}$$

so nothing which is ~~positive~~ g(r), M both the they hold be

Which proves part (c) QED

As a & b follow

Note since if $M_{\text{enclosed}} = 0$, $\underset{\sim}{g}(\underset{\sim}{r}) = 0$ as in cavity

~~$g(r' < r) = 0$~~ ext

$\therefore$ ~~$\underset{\sim}{g}(r' < r) = 0$~~ if there is no mass for $r' < r$

~~Which proves part (c)~~

QED. Shell theorem.

See p. 3012 (c), a b follow from (c) (3010–3011)

Note Mincl, Mexter could be moving – pulsing – anything as long as they maintain spherical symmetry the shell Thm ho

~~Do two spherically symmetric masses interact like point masses iff they do NOT touch?~~

~~Yes but I think it's~~ not quite obvious and needs a proof

– Birkhoff's theorem is almost enough to show why Newtonian derivation of FE works (subtle point) Wells 2014

– The analogous GR Theorem Birkhoff's theory is the same – Motion that doesn't break spherical symmetry changes nothing

See p. 3051

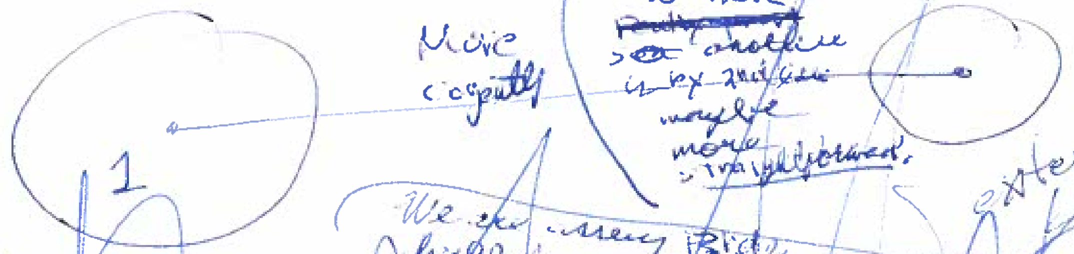
3014a) Proof

Note the internal forces on 2 cancel out by 3rd law and so only from 1 affects CM motion

and $F_{net\,2} = m a_{cm}$
 ~~stu~~

But F_{12} contributes to $F_{net\,2}$ and ~~[illegible]~~ ~~so~~ another is by 2nd law maybe more straightforward.

Move cogently



extended

We are using Bickham [illegible] here for purposes replacement → may be worse

maybe implicitly assuming 1st law here

$$\underset{\sim}{F}_{12} = \underset{\sim}{F}_{1\text{point}\,2} = -\underset{\sim}{F}_{2\,1\text{point}}$$

by shell theorem

by 3rd law (which holds explicitly for Newton's gravity law)

$$= -\underset{\sim}{F}_{2\text{point}\,1\text{point}}$$

by shell thm again

$$= \underset{\sim}{F}_{1\text{point}\,2\text{point}}$$

By 3rd law again

of a gait.

Q.E.D.

Newton had to work really hard to prove ~~this~~ and ~~the~~ shell thm using his klutzy formulation

{general replacement of spherically-symmetric bodies by point mass

But he needed ~~these~~ proofs in order to solve solar system motions tractably (i.e., celestial mechanics

→ Exact solution for the 2-body system

→ his early perturbation theory for general solar system motions

About spherically symmetric bodies

a) The shell theorem is only the gravity field of the spherically symmetric body.
There can be other grav. fields around, of course

b) Ideally the spherically symmetric body is that by magic.

c) Real spherically symmetric astro-bodies are only approximately that and are held that way by a combination of self-gravity and pressure force, Centrigul force, solid body force, tidal force of other gravitating bodies etc. cause perturbations.

d) Do spherically-symmetric bodies interact as point masses?
Yes, but need a proof.

Proof

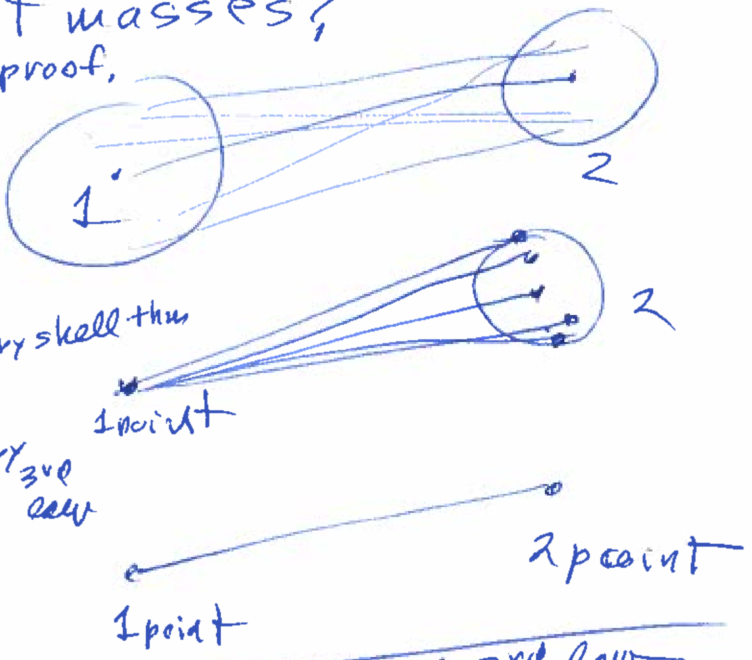
$$\underset{\sim}{F}_{1,2} = F_{1\text{point},\,2} \quad \text{by shell thm}$$

$$= -F_{2,\,1\text{point}} \quad \text{by 3rd law}$$

$$= -F_{2\text{point},\,1\text{point}} \quad \text{by shell thm}$$

$$= F_{1\text{point},\,2\text{point}} \quad \text{by 3rd law}$$

QED.



So the force of 1 on 2 is exactly the force of 1point on 2point.

Note the 3rd law holds explicitly for gravity.

3014c

But does this fact mean that body 2 responds to body 1 as if it were a point mass?

~~I~~ Body 2 center of mass does since

$$F_{\substack{net\\ external\\ on\ 2}} = m_2 \, a_{cm_2}$$

is Newton's 2nd law for finite bodies

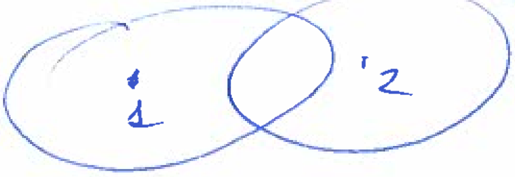
and the Center-of mass of a spherically symmetric body is at the center.

So that completes the proof.

~~Note we assume the bodies don't interact except through gravity [illegible] so no non-point like affects~~

~~Th~~

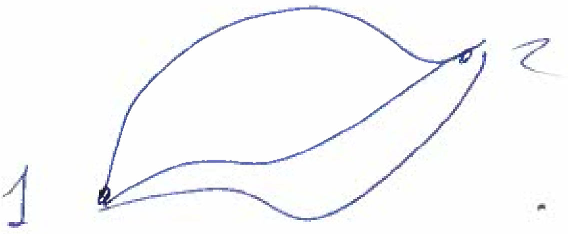
We assume the bodies don't penetrate each other ← They can't be point-like for gravity in this case.

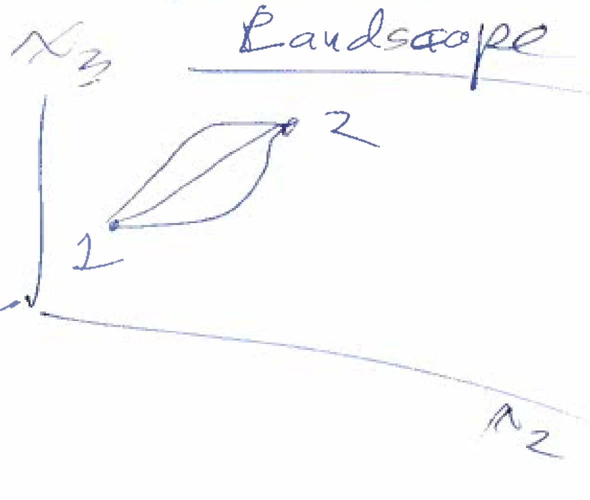


Potential Theory

L3015

Landscape





Say $W_{12} = \int_1^2 \underset{\sim}{F} \cdot d\underset{\sim}{s}$ is path independent, for ~~the~~ space or some subspace

Then we are free to define a potential energy.

$$PE = U$$

by $U_{12} = -W_{12} = -\int_1^2 \underset{\sim}{F} \cdot d\underset{\sim}{s}$

The minus sign needed to make mechanical energy conserved.

$$\therefore \quad dU = -\underset{\sim}{F} \cdot d\underset{\sim}{s} = -F_i \, dx_i$$

using Einstein summation

$$\frac{\partial U}{\partial x_i} dx_i = -F_i \, dx_i$$

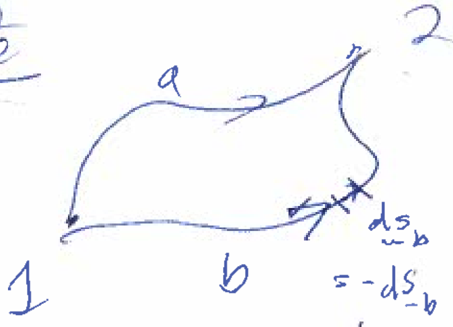
$$\therefore \quad F_i = -\frac{\partial U}{\partial x_i}$$

$$\underset{\sim}{F} = -\nabla U$$

actually I like these element forms best and think ~~when~~ we should use them all the time and teach them first to students

3016

Note



Our PE require

U_{12} path a $=$ U_{12} path b — by hypothesis of PE theory

$= -\int_1^2 \underset{\sim}{F} \cdot d\underset{\sim}{s}_b$

$= +\int_1^2 \underset{\sim}{F} \cdot (-d\underset{\sim}{s}_b)$

and add minus

$= \int_2^1 \underset{\sim}{F} \cdot d\underset{\sim}{s}_{(-b)}$

~~$= \int_2^1 \underset{\sim}{F} \cdot ds_{-b}$~~

$= -U_{21}$ path $(-b)$

path $(-b)$ flipping $-ds_b \rightarrow ds_{(-b)}$ flips the limits too

$U_{12a} = -U_{21b}$

$U_{12a} + U_{21b} = 0$

$$\boxed{U_{12a} + U_{21b} = U_{12a} - U_{12a} = 0}$$

so $U_{12} + U_{21} = 0$

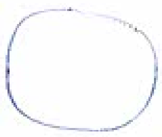
$\therefore \Delta U_{\text{closed path}} = 0$ QED

The reverse proof (converse) is easily proven too

For gravity with a spherically symmetric mass distribution (outside of it)

See p. 3011 $\nabla \cdot \underset{\sim}{g} = 4\pi\rho$

$$\underset{\sim}{g} = -\frac{GM_{\text{encl}}}{r^2}\hat{r}$$



By clairvoyance

$$V = -\frac{GM_{\text{encl}}}{r}$$

with $V(\infty) = 0$ by usual convention

~~$\nabla V = -\left(\frac{GM}{r^2}\right)$~~

$$-\nabla V = -\left(\frac{GM}{r^2}\right)\hat{r} = \underset{\sim}{g}$$

Potential is not potential energy, in QM we call potential energy? potential by convention

$\nabla \cdot \underset{\sim}{g} = \frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}(r^2 g_r) + r\frac{\partial}{\partial\theta}(\sin\theta\, g_\theta) + r\frac{\partial g_\phi}{\partial\phi}\right]$ (Arf-104)

$= \frac{\partial g_r}{\partial r} + 2g_r/r = 0$ $\quad = 0 \quad = 0$

except at $r = 0$ delta function. (see p. 3011

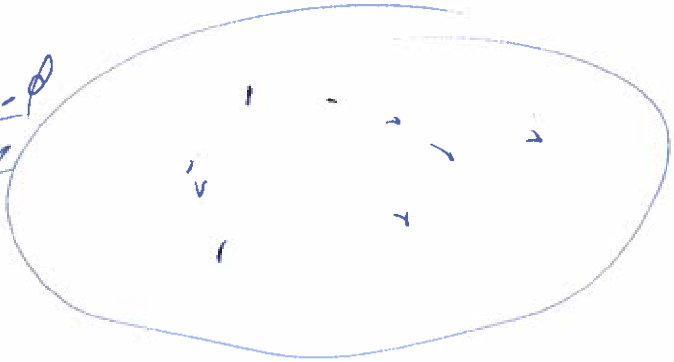
Zero-Point PE

3017

In PE theory alone, you never need to define a zero-point PE. However for a localized set of sources (i.e., ~~they~~ can be put in a finite closed surface)

Not loc
canno
∞ f
be
W
=

Localized sources



$U(\infty) = 0$ for sources at infinity is conventional

What is PE anyway?

For abstract PE theory, PE is just itself. However for real forces (Except gravity!!) I believe the answer is field energy (except gravity).

What else is there that is interacting? Gravity is a special case, it turns out

What else can it be? I wonder? B → Gravity shows
Some
e

Let's consider Electromagnetic first

Not contributions to E+B but total
otherwise because squaring
$E_1^2 + E_2^2 = E^2$
$E_1 + E_2 = E$

$$\mathcal{E}_{density} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$$

energy density

(Wik: energy density)

see p. 302U

We can sort of prove this result. E Radiation proves it!! (?)

~~tech~~
~~energy~~
Local
ener
—so
much
at ea
poin

3018)

Someone has derived this — with the ambiguity

what about point sources?

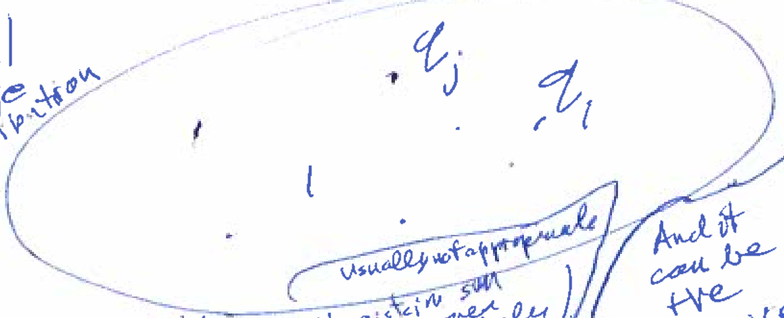
- point charges ⇒ electron, quarks

- point magnetic dipoles → electron ele.

⇒ Somebody in field theory has dealt with this. (maybe by trickoux renormalization? whatever it really is)

Let's ~~consider~~ Derive the E^2 part. electrostatics with charge smeared out into a continuum ⇒ as we usually do for macroscopic treatments. (Wiki: Electrostatics)

ocal harge istribution



$$U = \sum_{\substack{i \\ j<i}} \frac{k q_i q_j}{r_{ij}}$$

$$= \frac{1}{2} \sum_{ij} \frac{k q_i q_j}{r_{ij}}$$

to correct for double counting

Note the U is nowhere here Just an abstract accout so far

$U = \frac{1}{2} \sum_i q_i \sum_j \frac{k q_j}{r_{ij}}$

Usually not appropriate

No einstein sum ~ sum over particles

$$U = \frac{1}{2} \sum_{i} q_i \phi_i$$

And it can be +ve or -ve like +ve ulike -ve

electric potential due to all other charges.

continuous limit (

- gets rid of [illegible] ...ities

$$U = \frac{1}{2} \int \rho(\underset{\sim}{r}) \phi(\underset{\sim}{r}) dV$$

Integral over all space for our localized distribution

work

But also -ve Potential energy → we are counting all PE in assembly of the charge distribution

Now recall p. 3011 and Gauss' law differential form.

$$\nabla\cdot\underset{\sim}{E} = \rho/\varepsilon_0$$

N~~ote the~~ ~~$\nabla\cdot E$~~ total not JUST a ~~contribution~~

$\nabla\cdot\underset{\sim}{g} = -4\pi G\rho_{mass}$ (p. 3011)

and note (with Einstein summation)

$$\frac{\partial}{\partial x_i}(E_i\phi) = \frac{\partial E_i}{\partial x_i}\phi + E_i\frac{\partial\phi}{\partial x_i}$$

$$= (\nabla\cdot\underset{\sim}{E})\phi + \underset{\sim}{E}\cdot\nabla\phi$$

QED

So $$(\nabla\cdot\underset{\sim}{E})\phi = \nabla\cdot(\underset{\sim}{E}\phi) - \underset{\sim}{E}\cdot\nabla\phi$$

[illegible]

$$\therefore\ U = \frac{1}{2}\int(\rho\phi)\,dV = \frac{\varepsilon_0}{2}\int(\nabla\cdot\underset{\sim}{E})\phi\,dV$$

$$= \frac{\varepsilon_0}{2}\int\left[\nabla\cdot(\underset{\sim}{E}\phi) - \underset{\sim}{E}\cdot\nabla\phi\right]dV$$

$$= \frac{\varepsilon_0}{2}\oint\underset{\sim}{E}\phi\cdot d\underset{\sim}{A} + \frac{\varepsilon_0}{2}\int\underset{\sim}{E}\cdot\underset{\sim}{E}\,dV = \frac{\varepsilon_0}{2}\int E^2$$

by Gauss' theorem p. 3008

this term must be +ve

using $\underset{\sim}{E} = -\nabla\phi$

From potential energy theor

(as E^2

Always positive due to our smearing out of charge — Which we assume is a valid macroscopic limit of ~~dealing~~ dealing with point charges — which all 2 of classical E&M verifies / QED may do it by renormalization — by exorcise at my mother

We assume a localized set of charge and just set the enclosing surface to ∞.

$E\phi\sim\frac{1}{r^3}$

$A\sim\frac{1}{r^2}$

$\underset{\sim}{E}\phi \Rightarrow 0$ at infinity $\Rightarrow$ a reasonable assumption

3020

So

$$U = 0 + \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 \, dV$$

$$U = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 \, dV$$

Seems absolutely ~~reasonable~~ to define ~~[illegible]~~ ε and

$$\varepsilon_{\text{density}} = \frac{\varepsilon_0}{2} E^2$$

if is right in classical E&M.

For E and B fields it is right

But can we do this for gravity.

$U_g = \int \quad dV$

$= -\frac{1}{4\pi G}\frac{1}{2}\int g^2 \, dV$

Always -ve

Just would integral say ~~[illegible]~~

The answer is No.

since like attracts like in gravity

Is this correct

Yes + No:

It sort of works in class. calc.

But U_g does [illegible]

So this seems concrete

PE is field energy in this case

Oddity this formula says $U \geq 0$ always.

but we do have $\Delta U < 0$

Delta U

Consider two point charges

q $\quad E > 0 \quad r \quad$ $-q$

$E < 0$

V_{close}

close to ~~their~~ each other $|E| \to \infty$ ~~infinitely~~ close regions

But ~~let's~~ just omit the [illegible] together

Then as we bring

$U(r = \infty) > U(r \text{ finite} > V_{close})$

So $\Delta U < 0$

But have to deal with infinities

{ So U total always +ve avoiding infinities

But ΔU's can be -ve.

~~two~~ distributions

U_2

$U_1 + U_2$

U total

since not counting the [illegible] between interaction

E^2

total

[illegible]

$E_{1,2}$

$+ E_{total}$

of 3018

~~[illegible]~~ continuous

sort of charge

Just there relative U

Not total of their self fields

[illegible]

[illegible]

[illegible] +ve energy

$E = mc^2$ so true energy so local: [illegible] [illegible]

Somewhat local

How you define r_{close} to cut out ~~the~~ infinities might be tricky. 302

mean out pos cha

However, I think just smeari charge out to be continuous at the macro scale as we often do works.

Tot ene alw +

In any case, we seldom calculate ΔU from fields ~~best from~~ integrated over all space, but by some ~~the~~ line integral

$$U_{a4} = -\int_{a}^{4} q\underset{\sim}{E}\cdot d\underset{\sim}{s} = -W_{done}$$

which is easier and avoids infinities and also adding the energy of assembli the continuo chau which in E+M prob we may not want but we do with capacitors $E = \frac{1}{2}CV^2 = \frac{1}{2}VQ$

~~What About ~~Gravity?~~ Gravitational PE?~~

~~Let's consider ~~the~~ classical limit first where Gravity is Newtonian and is an inverse-square law like the ~~Coulomb force~~~~

3022

~~and where we define gravitational field $g(\underset{\sim}{r})$:~~

~~$F_{grav} = m\, g(\underset{\sim}{r})$ grav force on mass m~~

$U_B = \frac{1}{2\mu_0}\int B^2 dV$ see p. 3026

I don't know of similar ~~formula~~ ~~formula~~ for B-fields, but derivation probably exists.

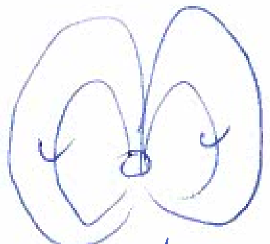
Certainly potential energy for B-fields can be defined in some cases: e.g.,

a) $U = -\underset{\sim}{\mu} \cdot \underset{\sim}{B}$ ~~the~~ PE of a magnetic dipole.

(Wik: PE: Magnetic PE)

current loop

(Wik magnetic dipole)



intrinsic like an electron

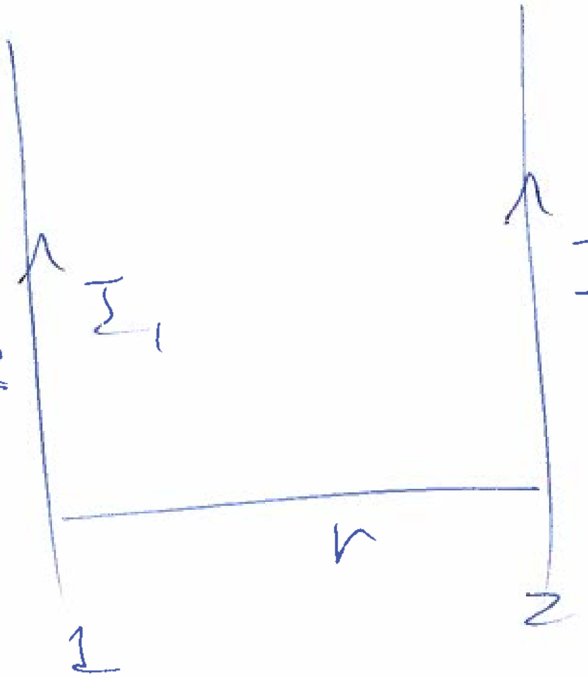
Omit

b) Ampère's force law

3023

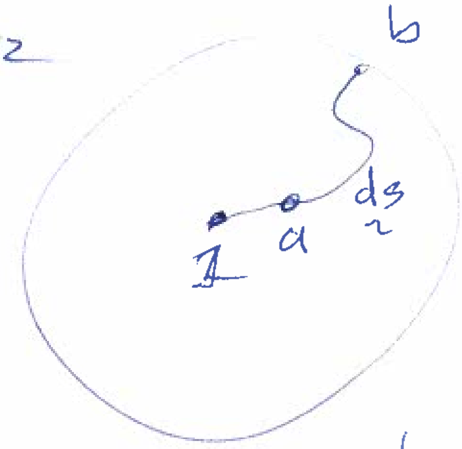
Omit (?)

Parallel wires



$$\frac{F_{12}}{L} = \frac{\mu_0 I_1 I_2}{2\pi r} * \hat{r}$$

attractive for $I_1 I_2 > 0$
repulsive if $I_1 I_2 < 0$



cross section

$$\Delta U = \int_a^b \frac{\mu_0 I_1 I_2}{r} \hat{r} \cdot d\underset{\sim}{s}$$

$$= \int_a^b \frac{\mu_0 I_1 I_2}{r} dr$$

$$= \mu_0 I_1 I_2 \ln(r_b/r_a)$$

~~Gravitational PE~~

$E = mc^2$

both E+M and probably nuclear fields

PE ~~must~~ obey this law

We'd say anyway

So having PE means having mass ⇒ both it's inertial and gravitational effect.

Easiest to think of for Electromagnetic radiation where there is no charge

spread out $\mathcal{E} = \frac{\varepsilon_0}{2}E^2 + \frac{B^2}{2\mu_0} = mc^2$

— and if there is a field energy density, then there is a mass energy density too. ⇒ a mass energy density.

For $E = mc^2$ see Lawden p. 9 ~~B~~ note (i) for "derivation" → with physics micropostulates along the way.

Omit = done on p.3029 ⇒ {so redundant and/or off path

Gravitational Potential Energy

For gravity in the classical limit we do define a gravitational field $\underset{\sim}{g}$

and $\underset{\sim}{F}_{grav} = m\underset{\sim}{g}$ is the gravitational force on mass m.

We can now repeat the derivation for $U_E = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2\,dV$ (see p. 3018–3020) for localized charges for the gravitational case mutatis mutandis

$$E \to g \ , \ \varepsilon_0 \to -\frac{1}{4\pi G} \quad \text{see p. 3011}$$

$$U_G = -\frac{1}{2}\frac{1}{4\pi G}\int_{\text{all space}} g^2\,dV \qquad \rho_{grav} = -\frac{1}{2}\frac{g^2}{4\pi G}$$

This formula in fact true in a sense — derivation fine but can't [illegible] field since $E=mc^2$ since their energy density doesn't gravitate.

3025

Where does $E = mc^2$ come from?

Recall Special Relativity is derived from 2 axiom

(a) Relativity principalle
— all physical laws should have the same formulae in all inertial frames

(b) Vacuum light speed is the maximum physical speed
↳ ~~and~~ is invariant for all inertial frames

exa
inerti
fra
an
free
fa
frame
Ftro

Principle of equi
of GR

The derivation is physicsy with all kinds of micro axioms along the way: e.g., it is natural in

Einstein was guided (1905) by ~~the fact that the~~ classical limit low relative velocity limit must be Newtonian physics

In trying to maintain conservation of mass and conservation of energy
↳ including PE + heat energy

he found it almost unavoidable or naturally

3026) to say $E = mc^2$

which means that there must be rest mass-energy

$$E_0 = m_0 c^2$$

→ the energy just for existing at rest

~~Now as was well~~

at first this is just established for inertial mass for SR

but experimentally

$$m_{inertial} = m_{grav}$$

So it was "natural" to assume all Energy had a gravitational effect.

~~Of course when~~ Einstein ~~wrote~~ famously decided $m_{in} = m_{grav}$ should be a principle

→ principle of equivalence (POE)

one of the axioms of general relativity (GR)

xioms
of GR

a) POE was one motivation for

Einstein to ~~[illegible]~~ pursue GR

3027

b) Another was the Newtonian gravitational field responds instantly everywhere to motion of mass

⇒ He felt that had to be wrong since that violated the light-speed axiom

c) Actually another aspect must have turned up in his thinking: ⇒ A Paradox

The Electromagnetic field has energy ⇒ mass ⇒ gravitational mass

This OK — causes no problem.

Except for those point masses — 'Leave to Quantum field theory

But if the gravitational field has energy (i.e, gravitational potential energy) then the field itself ~~other~~ has mass-energy and its own

3028) gravitational field

But this leads to a tricky situation?

in Newtonian Gravity the mass creates the field.

— Maybe the field is the mass But it's negative situation

Squirrels chasing their tails situation

see ~~p30~~ p.3029 & 3020

From a SR-mixed-with-Newtonian point of view there may be no consistent approach,

But GR does give the consistent approach but with mysteries still

(Carroll-120)

(Penrose 464-469)

Not gravitational mass [illegible] this [illegible]

R, gravitational field energy; ... gravitating mass energy, ... LHS ... of RHS of Einstein field eqn

But before going on to the GR fix let's see What gravity PE probably is ~~in any sense~~: in GR ...

Gravitational Potential Energy (Yes) + Gravitational Field Energy (only so it exist)

In classical limit $\underset{\sim}{g}$ is the gravitational field and

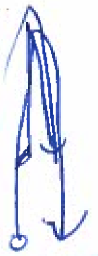
$\underset{\sim}{F} = m\underset{\sim}{g}$ ($\sim$ under means vector.

And of course, we do use Grav. PE an for near E and

$U = mgy$

throw ball

(derived from F=m



and energy is conserved in these classical limit con

$\Delta KE = W$

work-kinetic energy Th

But where is that energy.

$W = W_{non} + W_{con}$

$= W_{non} - \Delta U$

$\Delta KE + \Delta U = W_{non}$

work-energy Theorem

Well for E-field the PE is in the field as we derived p.30~~1~~8-3020

$U_{\underset{\sim}{E}} = \frac{\varepsilon_0}{2}\int_{\substack{\text{all}\\ \text{space for}\\ \text{localized}\\ \text{energy}}} \underset{\sim}{E}^2 dV \rightarrow \rho_E = \frac{\varepsilon_0}{2} E^2$

→ the field energy density

and this is true.

Similarily $\rho_B = \frac{1}{2\mu_B} B^2$

and I assume true for the Nuclear forces (strong & weak too)

mean E=m the fi has iner and gra effect Tru

But can be do this for gravity

mutatis mutandis

$\varepsilon_0 \rightarrow -\frac{1}{4\pi G}$ (p. 3011) and $\underset{\sim}{E} \rightarrow \underset{\sim}{g}$

and same derivation $U_g = -\frac{1}{2}\frac{1}{4\pi G}\int_{\text{all space}} g^2 dV$ [illegible]

(smearing out mass into a contin distribut again implied,

30) $U_g = -\frac{1}{2}\frac{1}{4\pi G}\oint g^2 dV$ < Perfectly correct account of grav. PE

But ~~can the~~ we can localize $\rho_G = -\frac{1}{2}\frac{1}{4\pi G} g^2$ { True? No

Can points in space have -ve mass?

as an energy density which then implies it has a negative inertia & negative ~~mass~~ grav. mass?

pockets of negative mass -energy No!

(Not all contributions are negative → so no way to make sum positive

The answer is no said Einstein

[illegible] all

In ~~sotting~~ setting up GR Einstein effectively had to dispense with ~~the~~ idea

Grav PE being anywhere exactly.

[c]annot say so much here and so much there,

[c]an say maybe this [a]spect of ~~energy~~ PE here,

[th]is aspect there — but

[th]at is a tricky job in GR

[i]t may be useless = unmeaningful

it exists, but nonlocal

→ no grav. field energy density can be defined fundamentally but you may do it as an approximation somehow

in our classical way.

$\Delta U < 0$ but calculated same we do

To ~~E~~lucidate

Let's consider a binary pulsar case → very strong gravity field close to'

[l]ike [e]lement

[me]tric

$= g_{\mu\nu} dx^{\mu} dx^{\nu}$

[R]ic tensor

Carroll-71

[m]ass-energy-momentum tells space-time how to curve and that tell mass energy - how to move under gravity not other forces

gnav field far off

far enough to sort of localize

$g_2 = -G\frac{(m_1 + m_2 + \Delta U)}{r^2}$

~~this~~ true in ~~Newtonian~~ classical limit

* But close too you need GR in strong gravity and the only mass-energy in GR is m_1 and m_2

Einstein Field Equation (to anticipate - ~~4~~ 4 x 4 Tensor ~~PE~~ → true at every point (above QM level)

geometry

$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ < mass energy momentum

[R]ik Einstein tensor Geometry al

Cosmolog constant metric tensor

stress-energy tensor energy-momentum tensor Wik

Not going to really do GR Talk around it

3030

There is no grav. potential energy in it
the mass in the $T_{\mu\nu}$ tensor

Grav. PE
is not in GR
It can be
superimpo
as our
classical
way of
understandi
GR.

classica
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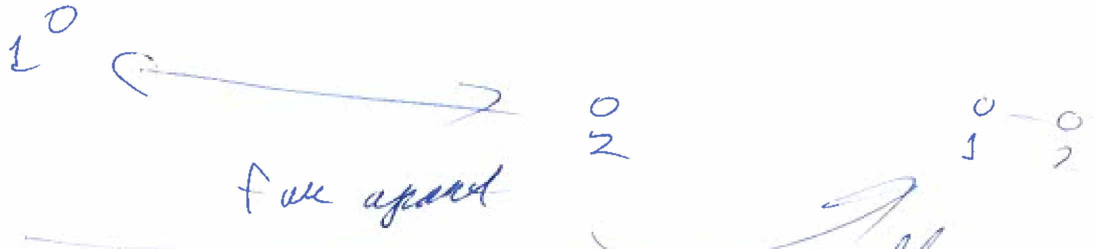


close together
— but rigidly held rigid
and at rest.

— Their contribution to
the Net $T_{\mu\nu}$ are
unchanged (Penrose 464
-465)

All the
difference
is in the LHS
the geometry
of space-time

What happens as the pulsars
in spiral under GR effects
(an unperturbed Newtonian
2-body system is
perpetual, eternal,
no inspiral

Well $\underset{\sim\text{far field}}{g} = -\frac{G(m_1+m_2+\Delta U)}{r^2}$ gets smaller,

In ~~classical~~ / SR terms mass energy
is lost
due to increase
$|\Delta U|$

equivalent
calculation
in a
sense
but
GR justifies
the first

— But the GR
calculation gives the
same - field g
but without any changes formal mass

So there is ΔU, but in GR terms
it can't be said there is so much
here or there except when far enough

30306

away in far field
that
you can
say it's
localized relative to you.

[illegible]

Of course, where does the lost
non-localized energy
go as
ΔU decrease
KU increase

We say carried off by Grav. waves

True but GWaves
travel across the universe
and make no contribution
to the local $T_{\mu\nu}$ as they
travel → unless they
deposit some by shaking
and not squeezing

→ in true empty space
$T_{\mu\nu} = 0$
where the waves
travel. (Penrose – 466 caption
– 467 last paragraph)

Do the Gwaves
eventually deposit
exactly the energy they
carry off?

→ There is a GR proof that they do
in a special case Bondi-Sachs
mass/energy conservation law
(Penrose 467–468)
(Bondi generalized by Sachs)

But this not a general proof.

303L

In fact, GR does not guarantee ordinary conservation of energy as we'll discuss in a bit

— See p. 304 2

And, in fact, in cosmology it ~~[illegible]~~ fails

But all is not lost,

Carroll-120
energy-momentum
conservation
equation, holds
at every point

as far as we can tell
unless you
say it is conserve
in some ~~way~~
~~to satisf~~
unspecified
way to
satisfy your
physical
intuition

Ex PE of spherically symmetric mass-distribution → localiz

$U = \frac{1}{2}\sum_{ij}\frac{Gm_im_j}{r_{ij}}$

Similar to p. 3018 for electric charge

So $U \simeq \frac{1}{2}\frac{GM^2}{R}$

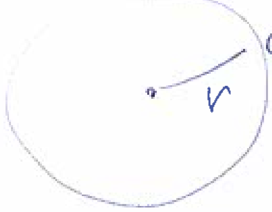
characteristic

R characteristic size

crude estimate

one can drop the $\frac{1}{2}$ as insignificant

Exact calculation for a uniform sphere (a) by assembly, (b) from field energy (leave an exercise for student)



a)

$$dU = -\frac{GM(r)}{r}\,dm = -\frac{Gm}{R}\frac{dm}{x}$$

uniform dens

$$m(r) = \int_0^r \rho\, 4\pi r^2 dr$$

$$m(r) = M\left(\frac{r}{R}\right)^3 = Mx^3$$

$$dm = \frac{3M}{R^3}r^2 dr$$

$$= 3Mx^2dx$$

$$U = -\int_0^1 \frac{3GM^2}{R}x^4dx$$

$$= -\frac{3}{5}\frac{GM^2}{R}$$

- so the crude estimate wasn't so bad.

b) Now from the g-field (Exercise for student)

using field density (p. 3029) $g = -\frac{GM}{r^2}$ outside

~~$g = \frac{GM(\frac{r}{R})^3}{r^2}$ insid~~

$$U = -\int_R^\infty \frac{1}{2}\frac{1}{4\pi G}\frac{G^2M^2}{r^4}4\pi r^2 dr$$

$$= -\frac{1}{2}GM^2\int_R^\infty \frac{dr}{r^2} = -\frac{1}{2}GM^2\left(-\frac{1}{r}\right)\Big|_R^\infty = -\frac{1}{2}\frac{GM^2}{R}$$

3632

Inside from Shell theorem

$$g = -\frac{Gm(r)}{r^2}$$

$$= -\frac{GM\left(\frac{r}{R}\right)^3}{r^2}$$

$$= -\frac{GM}{R^2}x^2$$

use p. 3029 Pdensit

But does it really mean so much PE is inside and so much outside. I don't think so. ce again le classical eatment somehow generate ll the R extent here doesn't make up.

$$U_{inside} = -\frac{1}{2}\frac{1}{4\pi G}\int_0^1 \frac{G^2M^2}{R^4}x^2\,4\pi R^3x^2dx$$

$$= -\frac{1}{2}\frac{GM^2}{R}\int_0^1 x^4dx$$

$$= -\frac{1}{2}\frac{GM^2}{R}\frac{1}{5}$$

This expression can only be true near classical limit I think. So $R >> R_{sch}$. But how much far field mass loss occurs in collapse to Black hole? ~~What~~ Can't find answer NOW

$$U = U_{outside} + U_{inside}$$

$$= -\frac{GM^2}{R}\left(\frac{1}{2} + \frac{1}{10}\right) = -\frac{3}{5}\frac{GM^2}{R}$$

So we get the same answer.

What is the equivalent mass?

$$M_{equivalent} = \frac{U}{c^2} = -\frac{3}{5}\frac{GM^2}{Rc^2} = -\frac{3}{5}\cdot\frac{1}{2}\cdot\frac{R_{sch}}{R}M$$

$$= -\frac{3}{10}\frac{R_{sh}}{R}M$$

$M_{eq} \to \infty$ if $R \to 0$

Nah.

Recall Schwarzschild radius

$0 = \frac{1}{2}mv^2 - \frac{GMm}{R}$ Classical derivation

$R_{sch} = \frac{2GM}{c^2}$

Is this right? If you collapse Sun to BH magically does it mass decrease by $-\frac{3}{10}M_\odot$? ~~Seems so.~~ Nah

3033

Not surprisingly, we find the mass equivalent M_{eq} only becomes comparable to the mass M if the sphere has radius of order the Schwarzschild radius.

Keplerian Sun by solar mass BH and collapse Sun to BH wouldn't be a solar mass BH

But does this formula mean anything? I don't think so at $R \to 0$, $M_{eq} \to -\infty$.

Line Element
=
Metric
$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
Einstien summation
Carroll -71

$g_{\mu\nu}$ is metric tensor

Gravitational Field Energy in General relativity?

Let us Not let gravitate field gravi

Omit to p.3042 (too incoherent)

It's sort of a tricky subject.

Einstein Field Equations → Which gives the curvature of spacetime and dynamics of mass-ene

(recapitulation of p.3030) Universal true → Like all physics DEs true at every point in space but have to solve

for BGs → Recalcin

4x4 Tensor of Differential equations

Features

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

a) Einstien tensor
Wik GR: Einstein field equation
— embodies curvature effects

Wik: GR: Cosmology
Λ is the cosmological constant

Space-time-metric tensor

3034) It's local → ie., it applies at each point in spacetime → above the quantum gravity level → where were that takes over at small scale,

b) The left-hand side

It's a DE → (~~is~~ determines ~~[illegible]~~) ~~curvature~~ of space-time geometry

and the right-hand side is mass-energy and momentum. ~~effect~~ effect.

$T_{\mu\nu}$ is the energy-momentum tensor.

In brief $T_{\mu\nu}$ tells space-time how to ~~curvature~~ curve

and then the curvature tells mass-energy how to move due to gravity

) The tensors correspond to 4 × 4 matrices but

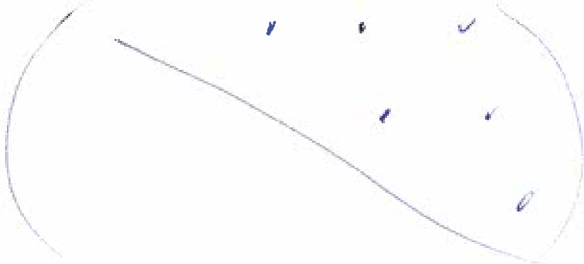
3035

are symmetric

and

4+3+2+1

= 10



independent partial differential eqn.

physical law → what holds point-by-point eternally.

(Wik) Einstein field equations)

4 ~~is~~ (for 3 space and 1 - time dimension of spacetime.

d) Conservation of Mass-energy and Perfect fluids

Follows from the demand ~~that~~

$$\nabla^{\mu} T_{\mu\nu} = 0$$

with Einstein summation

Energy -Momentum Conservation Eq (Carroll-120 Carroll-156) in the form shown here

Covariant differentiation (Carroll 97) (Weinberg-46 -105-1 -1

Relativistic kind of differentiation

~~This~~ This conservation law is a much glorified form of of mass & momentum conservations

in the NR case ~~the~~ the continuity equation for mass

is $\frac{\partial\rho}{\partial t} + \nabla\cdot(\underset{\sim}{v}\rho) = 0$ (Wik)

Omit to bottom of p. 3037

But $T_{\mu\nu}$ includes fields energy too

~~So in~~ except the gravity field

~~So~~ it has forces included except the gravity force.

(see Weinberg 45–46 & 360ff)

For a perfect fluid in the SR limit

Weinberg -48

$$T^{\mu\nu} = p\eta^{\mu\nu} + (p+\rho)U^{\mu}U^{\nu}$$

Right?

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

the SR metric tensor

Weinberg 26

Carroll-34-35 but hard to follow

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Weinberg
p.77–79
in that limit
grav. field energy

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sure
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Jot sure.

If it was
There's be
negative
mass
see p. 3630

Perfect fluid is used in several ways

a) a general way is a ~~fel~~ fluid that is isotropic if you are moving with a fluid element.

b) In cosmology: No turbulence, no turbulence

Weinberg - 47

~~In cosmology we use for more restricted definition~~

Wik characterized only by rest frame mass everY density $\rho = \rho_m c^2$ and pressure $p = p(\rho$

~~$u^\mu u^\nu$~~

$u^\mu u^\nu = \delta_{\mu\nu}$

⇒ EOS, no Temp on T(P) v

$$T^{\mu\nu} = \begin{bmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{bmatrix} = diag(\rho, p, p, p)$$

~~Wik~~

Wik: perfect fluid Liddle - 123

This is for $u^\mu =$

~~and~~

comovi

~~with~~

fru

In cosmology one often (always) uses parameterization $p = w \rho_c c^2$ < can make c^2 disappear $\rho c^2 \to \rho$

where w seems to have no ~~special~~ none EOS parameter?

$p = p(\rho)$ is general equation of state (EOS)

Temperature T heat energy just contribute to P for This perfect fluid.

w = 0 for "dust" = pressureless matter (Li - 40)

Essentially in Cos, Stars, Galaxies, cosmic dust, gas → even if it sort of has pressure but not cosmically

$w = 1/3$ for radiation

⇒ ~~ex~~ extreme relativistic stuff

Li - 40

$w = -1$ for cosmological constant Li - 105, 57

equivalent Dark Energy

$w =$ a function of a scalar field viewed as a perfect fluid (Wiki: EOS cosmological)

Quintessence (Wiki: Quintessence) is a particular theory of dark energy to cause acceleration of the universe

$w = -1/3$

or $\ell = ct$ universe

$\ddot{a} = 0$

(Silvio Melia

and w can just be used as a free parameter in cosmology

ΛCDM has $w = -1$

wCDM has w as a constant free parameter

e) Conservation of Mass-Energy

energy-momentum conservation eqn (Carroll-120, 156) in this form

$$\nabla^{\mu} T_{\mu\nu} = 0$$

(Carroll - 97

Weinberg - 96, 105-107, 153, 156)

$\boxed{303}$

is what GR gives us for conservation of energy ——— that's it.

There is no "integral" of the motion corresponding to energy (Carrol-120) in general

Perhaps conservatio of energy applies only when all energy results can be speci bu wha of 1 mo

Recall the gravitation field energy is excluded from ~~T~~ $T_{\mu\nu}$

Where is it?

Well in a sense it Grav. field energy and the gravitational force is encoded in the Left-hand side of Einstein field equations

encoded then in curvature of spacetime

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

(Wiki: Einstein field equations)

In developing GR Einstein was guided by the idea that since

$$\nabla^\mu T_{\mu\nu} = 0$$

to give en-mo-conservation in a relativistic sense.

then $\nabla^\mu(G_{\mu\nu} + \Lambda g_{\mu\nu}) = 0$

3040) So he had to find ~~the~~ right tensor to determine the geometry of spacetime $G_{\mu\nu}$ ~~the~~ Einstein ~~tensor~~ (Viz Einstein tensor)

The $\Lambda g_{\mu\nu}$ was added on as an after thought for cosmology for reasons we'll discuss later

For damn good reasons, only terms that are linear in 2nd derivative of $g_{\mu\nu}$ or quadratic in 1st derivatives of $g_{\mu\nu}$ (?) can appear in $G_{\mu\nu}$.

Replaced by p. 3030-3030c

f) ~~Gravitational Field energy is non-local~~

Penrose 464-469 esp. 467-469

~~As above, it is encoded in the Left hand side of the Einstein field equations, in the curvature of spacetime. So you can't write down a density for it: e.g., like electromagnetic field~~

...is NOT ...local

In general But can in Newtonian limit

...e ...3029

$\rho = -\frac{1}{2}\frac{1}{4\pi G} g^2$

See p. 3017 $\mathcal{E} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2$

or $\nabla^{\mu}T_{\mu\nu} = 0$ itself.

~~There is a theorem~~

Is energy-momentum conserved by GR. → if not moment by moment at least in start to end of process?

It can't be proven generally (Penrose-467)

but it can in special cases.

e.g. asymptotically flat systems

– remote from the gravitating mass energy one has flat space-time.

Then the Bondi-Sachs mass-energy conservation law can be proven from GR

e.g. δ

Binary Pulsar case

The energy-momentum carried by grav. waves equals the energy lost by some accounting of local Grav PE.

It gets damped somewhere

3042

~~Note the grav waves travel through~~ ~~matter~~ ~~empty space where $T_{\mu\nu} = 0$ or can in principle.~~

grav waves ~~make~~ No contribution to ~~$T_{\mu\nu}$~~ as they travel thru space

Every term in $T_{\mu\nu}$ is zero. But RHS ≠ 0 but Not term by term

[illegible] ~~stalls~~ [illegible] the RHS = 0 [illegible] Just [illegible] –HS ≠ 0 > gives geometry

9) Does GR conserve energy generally?

— as said above it can't be proven generally

— some folks hope this will be restored somehow.

But others think we may have to live without it (Carroll – 120) energy-momentum-conservation eqn.

A good reason for this is cosmological models don't conserve energy. ($\nabla_\mu T^{\mu\nu} = 0$ a differential equation true at all points (above QM scale))

But holds at every point

$\nabla^\mu T_{\mu\nu} = ($ 156

— to explicate ρ_{matter} or dust which only moves w/ expansion of universe

$\rho_{\text{matter}} \propto \frac{1}{a^3}$ where a is the cosmological scale factor

$T_{\mu\nu}$ has No grav PE in it other kind energies

Yes.

Space expands as a(t) 30 43

— Bound systems do not expand but the space between them grows

Now $\rho_{matter} \propto \frac{1}{a^3}$ can be argued to conserve mass-energy (Li-4?)

all ER particles photons neutrinos

But $\rho_{radiation} \propto \frac{1}{a^4}$ (Li-42

where does the "radiation" energy go?

→ I used to say it goes into the expansion of space, but ~~and~~ what does that mean?

It is just gone from the description. Λ cosmological constant or constant dark energy — equivalent to a(t) if not

~~If was no gravita~~ It's gone as gravitating mass

And then there is the c~~osmologica~~l constant to account for the dark energy. to account for the acceleration of the universe

$\rho_\Lambda = \rho_{DE}$ = Constant in the simplest theory

in other respect

emit → absorb photon does photon loss due to redshift just become KE in quasi classical relation sense?

3044

But this means as space expands it grows.

Maybe there is some ~~any~~ way to save simple conservation of energy

but maybe not.

There's a theorem → Noether's thm which shows energy should conserved under time invariance

↪ but an expanding universe doesn't have time (Carroll-120) invariance ~~and so~~ ~~Noether~~ in any obvious way.

← Carroll-120 176

So upshot GR gives us $\nabla^\mu T_{\mu\nu} = 0$ as our energy-momentum conservation law

rdinary Energy onservation ly in classical or SR imit

and i~~ffinsofar~~ insofar as GR is true we may have to live with that.

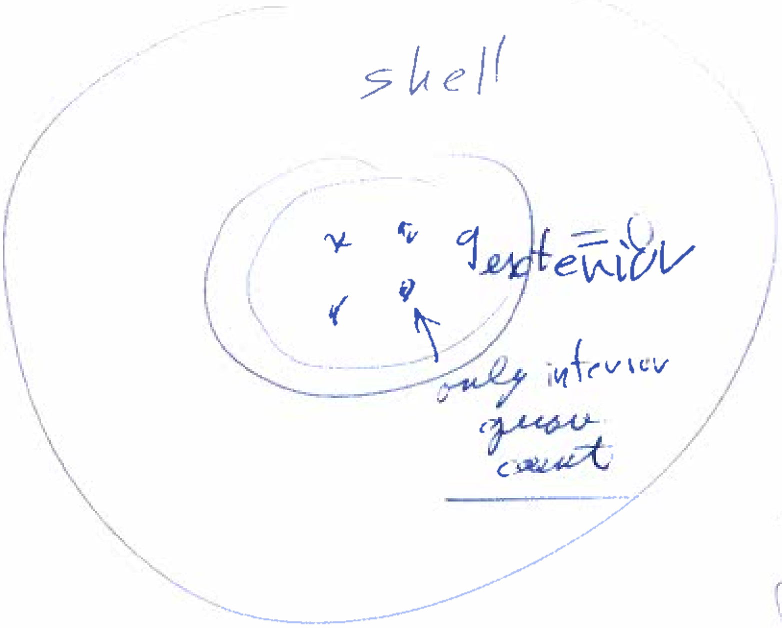
Of course, we expect GR or whatever is the true macroscopic theory of gravity to emerge as the macroscopic limit of quantum gravity whatever that is. → Maybe that will restore conservation of energy

Shell Theorem ~~& Birkhoff Theorem~~ see p. 3051 ~~for Birkhoff's Theorem~~ With ~~infinite~~

- boundless space
- infinite & flat

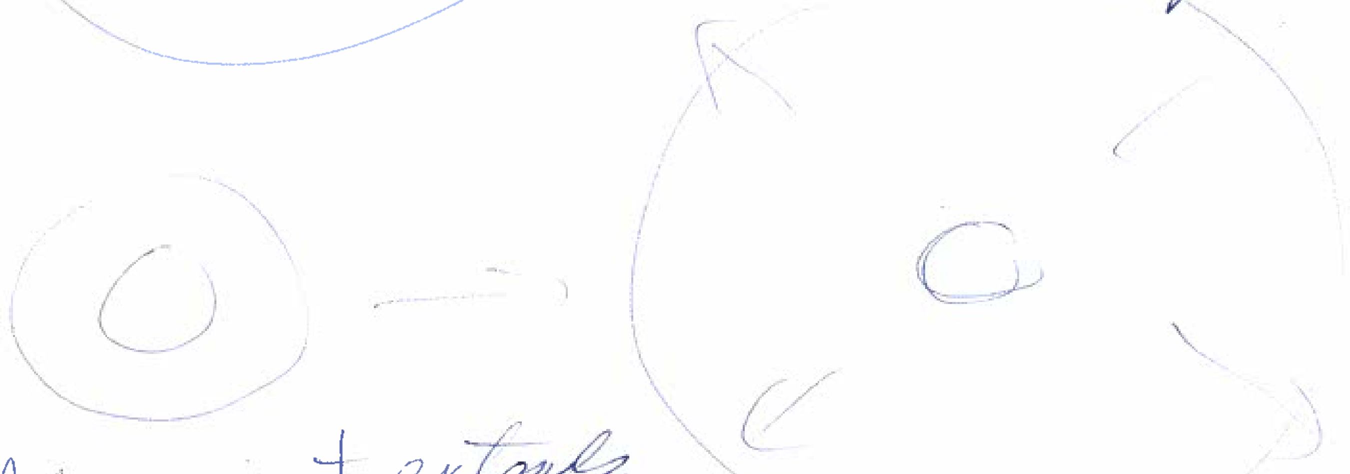
↳ recall p. 3011–3014

One aspect is that a spherical shell has no grav. effects on the cavity.



But what if the shell is infinite?

↳ A consideration for Newtonian cosmology.



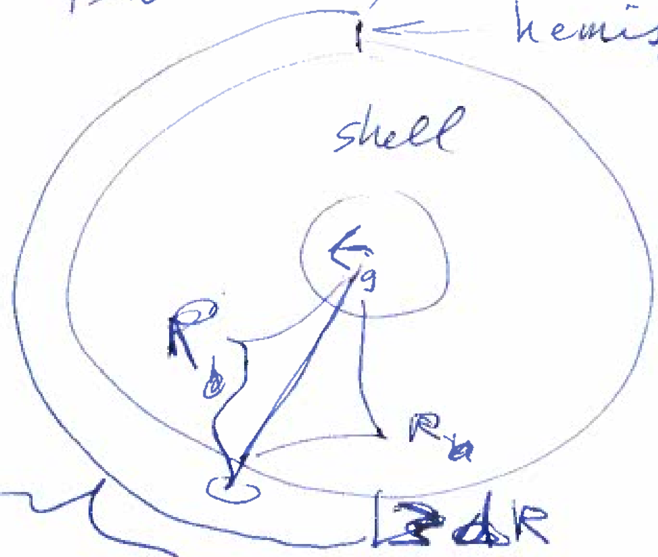
If one just extends the shell to infinity, it seems the shell theorem should still hold since it does

3046

for any step along the way.

But say we had

← hemisphere ~~then~~



- the hemisphere will create a gravity field in the cavity.

- Now what if one increases shell and and the ~~[illegible]~~ hemisphere ~~holding ΔR fixed~~.

$M_h = 4\pi R^2 \sigma$ (holding σ fixed

surface density

~~each layer of the shell grows as R_h^2 in area which cancels the $1/R_h^2$ decreases of Newtonian gravity. $\oint_{hemis} = \int_R^{R_H} GM$~~ $2\pi r^2 dr$

So the

$$g_h^{(\text{at center})} = f \frac{GM_h}{R^2} = f \frac{G(4\pi R^2\sigma)}{R^2}$$

geometric factor $\sim \frac{1}{2}$

$$= f\, G(4\pi\sigma)$$

$$= \text{constant}.$$

In this case ~~there~~ ~~[illegible]~~ is always a net force in the cavity.

and that would be the limit as $R_{ex} \to \infty$.

So, the limit of extending a ~~mass~~ shell around the spherical cavity to infinity depends on the ~~mass~~ ~~shell~~ shell's mass distribution even if an ~~after~~ going to infinity part, if it is spherically symmetric.

Upshot there is no Newtonian physics solution to an infinite universe full of infinite mass without extra ad hoc hypotheses — ad hoc relative to Newtonian physics alone.

→ Which is what Milne & McCrea did in 1934 (p. 3002 & Bondi-7

Relative to GR their hypotheses are valid.

However the most natural ad hoc hypothesis is ~~that~~ ~~the~~ Shell Theorem

extension to infinity is lost.

However Newton himself in unpublished work gave up on finding a static infinite uniform density on average universe (No - 376)

It doesn't sound like he tried very hard, but one must always remember his math tools were ~~[illegible]~~ primitive

He was also aware of Olbers paradox — A static eternal ~~full univer should~~ universe full of stars should have a sky ~~star~~ surface brightness.

Thomas Digges & Kepler even earlier (wik)

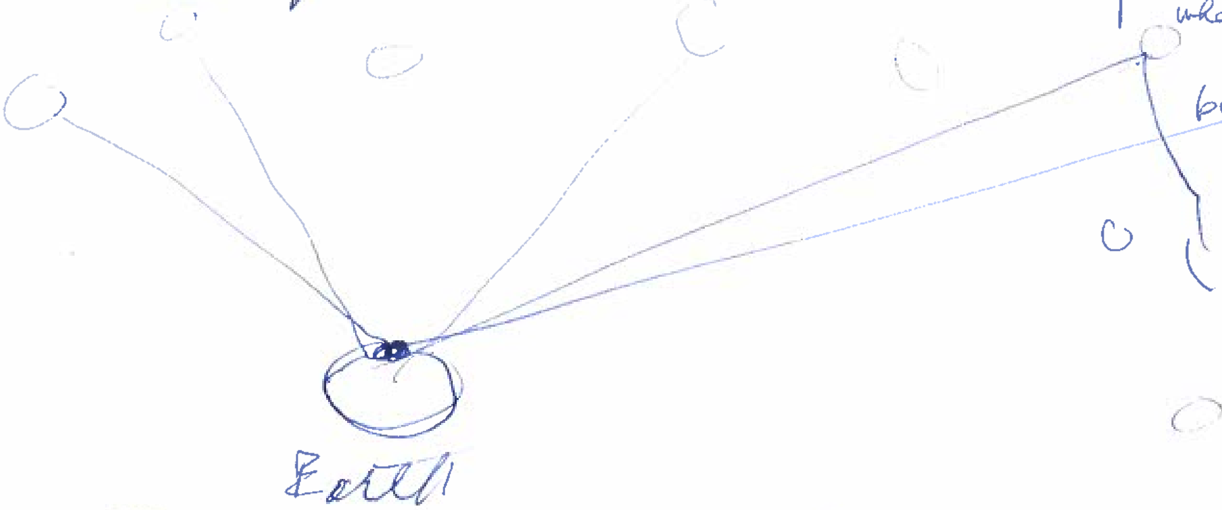
Edmund Halley in printing term discussed it before the Royal Society — in 1721 with 79 year old Newton presiding as president (No-377)

But he attributed it to anonymous David Gregory No-377

304?

From a casual perspective, it is easy to understand

Note pre-thermodynamics one it doesn't seem it worried anyone much till 19th when Edgar A. Poe 1848 but Lord Kelvin 1901 first really found resolution anticipated by Poe



– In every direction, you observe a pencil beam of light from a star surface ⟹ all the photons coming your direction — so you are fried.

In modern radiative transfer terms

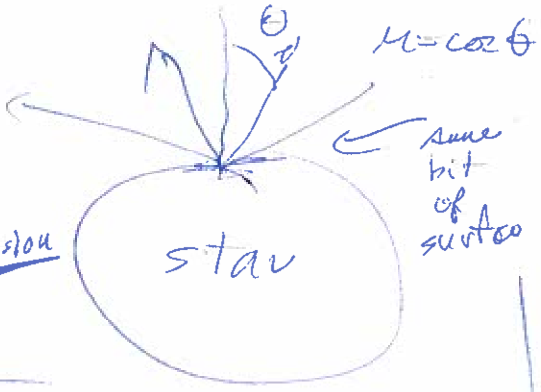
$$I = \frac{E}{\text{tim} * \text{Area} * \text{freq} * \text{Solid Angle}}$$

No spreading out in this picture — geometrical op

3050

~~stari~~

assume I is isotropic on emission



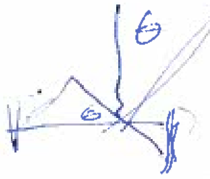
$$\text{Flux} = \int_{2\pi} I \mu \, d\Omega$$

from a surface of a star

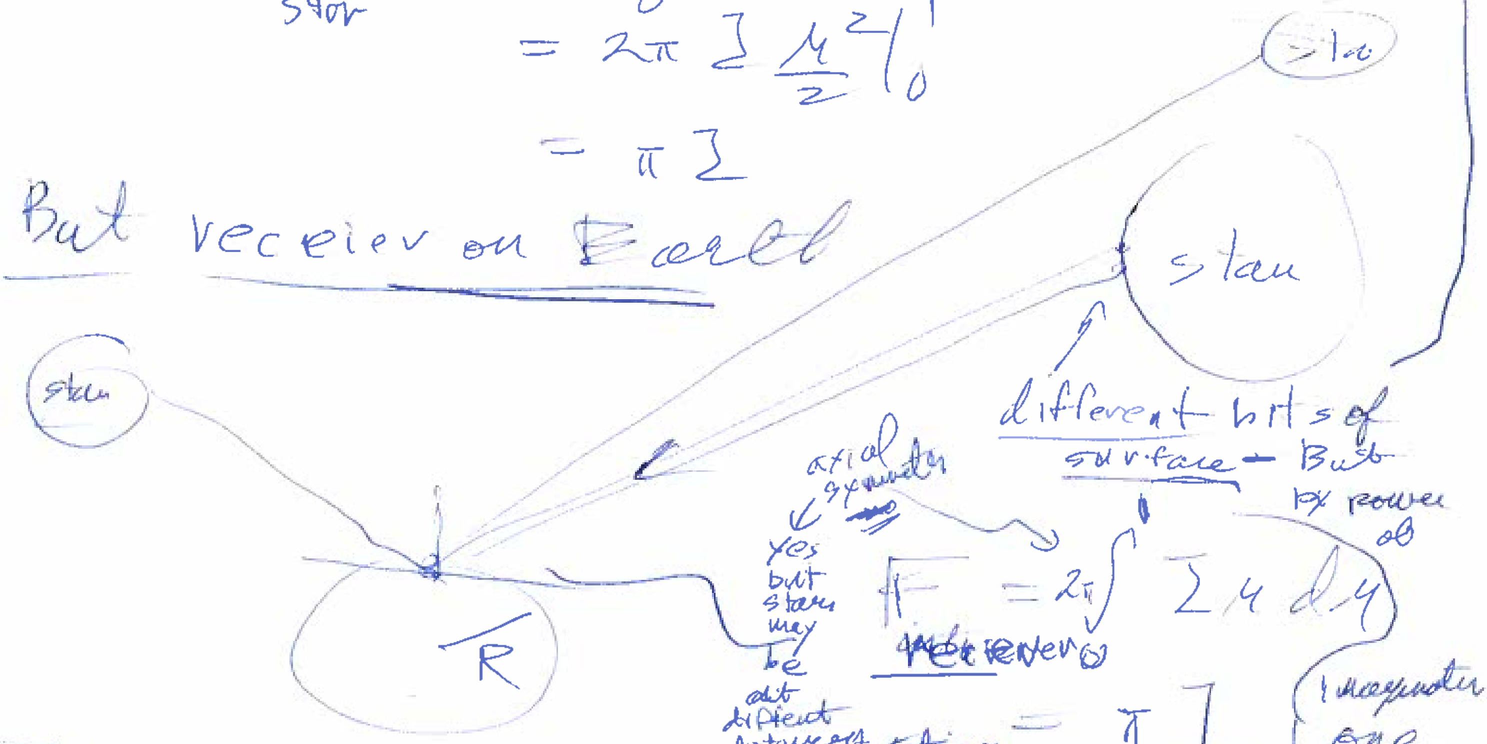
$$= 2\pi \int_0^1 I \mu \, d\mu$$

$$= 2\pi I \frac{\mu^2}{2}\Big|_0^1$$

$$= \pi I$$



But receiver on Earth



different bits of surface — But by power of

axial symmetry

yes but stars may be ~~dit~~ different distances and orientations

$$F = 2\pi \int_0^1 I \mu \, d\mu$$

~~rec~~ receiver θ

$$= \pi I$$

I measure one sees, it makes no differenc

Total input $L_{\text{Earth}} = 4\pi R^2 F_{\text{int}}$

$$= 4\pi R^2 \frac{L_{\text{star}}}{4\pi R_{\text{star}}^2}$$

$$= \left(\frac{R}{R_{\text{star}}}\right)^2 L_{\text{star}}$$

Birkhoff's Theorem

3051

So Newtonian Physics is ambiguous for an infinite universe
→ homogeneous & static or not.

GR rescues us with Birkhoff's Thm which among other things says given

→ spherically symmetric surrounding
— static or not
— finite or infinite

A key point for applying to an expanding universe

leaves ~~the~~ Cavity flat Minkowski space Weinberg -338 -474

$R_c << R_{\text{Gauss curvature radius}}$ CL-12-13 which in the Λ-CDM model is infinite — but we don't know if that model can be applied to infini

So frames ~~that don't rotate~~ ~~or accelerate~~ relative to the surroundings and are in free fall are inertial frames (without inertial forces) Weinberg-474

~~free~~ Fall frames

With inertial forces you can make any local frame an inertial frame

So as long R_c is sufficiently small and the ~~motion~~ contents are in the classical limit and we can use Newtonian Physics (mostly not absolute space,

so they do not perturb the surround

which does span velocity in inertial frame ⇒ c bound ⇒ between no-c

3052

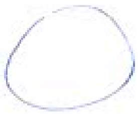
and remarkably we can derive the Friedmann equations → ~~But really~~ ~~Not~~ ~~Even then~~ if you have radiation and $\Lambda \neq 0$ ↓ then yet another special hypothesis is needed arguably natural

— Is this just a lucky accident?

— Wells (2014) shows that it should be so → but it takes a lot of work — so we won't do that.

→ Maybe even without Wells clean it is Wells treats subtleties

By the way Birkhoff's theorem is analogous to the shell theorem in another respect



exterior core

outside of a spherical symmetric mass-energy distribution of any kind static or moving one gets the Schwarzschild metric

Weinberg-337

— so the mass could be replaced by a point mass or black hole.

If you are far enough away, you are in the classical limit and Newtonian ~~gravity~~ & dynamic physics applies.

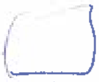
Why does it not matter for cavity and the ~~other~~ ~~treated~~ exterior case that the spherically symmetric mass-energy can be moving.

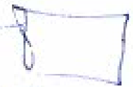
→ I can't give a full answer.

↪ But no gravitational waves to infinity can escape ~~a pot~~ from inside a pulsing spherically-symmetric mass-distribution and so they can't affect the spacetime geometry of the cavity or exterior case.

Quit Discuss p. 3063ff

— in pure Newtonian physics ~~velocity~~ ~~between~~ in. only one true mythical absolute space inertial frame





– in GR ~~only~~ true inertial frames are free fall frames

→ So it emerges that Newtonian physics does spare velocities in ~~inertial~~ a local inertial frame and between them

In an local inertial ~~frame~~ v < c between them ~~the~~ v > c can do happen

3 ~~0~~ 0 4

Inverse square law + linear force duality → Paper on it but forgot to make a note of it.

other
etown

Linear Force

$$\underset{\sim}{F} = q\hat{r}$$

Force Field

q is the point "charge"
+ve repulsive
-ve and you have the SHO force

Hooke's law force.

This sort of turns up in cosmology because you can (~~above~~ insert) in the Newtonian derivation the cosmological constant effect.

But it may just be an ad hoc fudge. I don't know but it's worth a bit of look at.

Radial harmonic oscillator
RHO

Weird aspect
r ↑ $|\underset{\sim}{F}|$ ↑
which mean shouldn't let r → ∞ but we will anyway.

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GR.

a) First curious point (Wik)

Bertrand's ~~theorem~~ (19th century) show that only two central force law give all bound orbits as closed.

these force laws allow some closed orbit

~~& q < 0~~
~~= -k case~~

a) $\underset{\sim}{F} = -\frac{k}{r^2}\hat{r}$ the well known inverse-square law like gravity & coulomb's law

b) $\underset{\sim}{F} = -kr\hat{r}$

$-k = q < 0$

the RHO force.

Of course any central force law give bound circular orbits

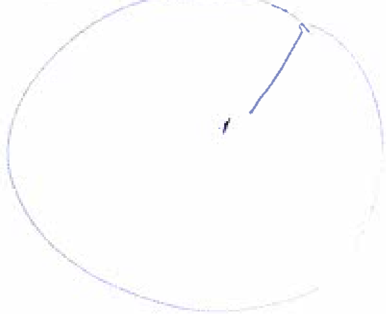
[3065]

Really ma for F=ma specialized for uniform circular motion

$F_{\text{centripetal}} - \frac{mv^2}{r}$

∴ $F_{\text{central}}(r) = -\frac{mv^2}{r}$

anything.



<u>But</u> (a) & (b) cases

(Wik Bertrand's Thm: Radi
Harm
osc
RH

- elliptical orbits ~~with~~ ~~force~~ source at one focus

- also elliptical orbits but the force center at the geometrical center

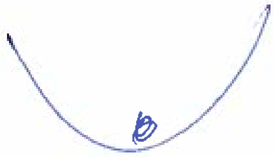
I <u>recall</u> that <u>Newton</u> when investigating central forces laws in Book I of the principia (1687) showed this – in his old-fashioned klutzy formalism

And showed that other central force laws ~~that~~ he looked at didn't

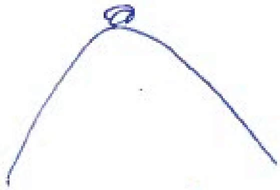
<u>But</u> it was beyond his techniques to prove there were only (a) & (b)

3056

Both kinds of orbit have neutral stability



stable



unstable

metastable

? really all stability is metastable since with enough perturbation anything disrupts

Neutral stable

— the particle will be moved by perturbation displacement but won't go off to infinity

In case of (a) & (b) orbits, a perturbation changes them but only "proportional" to the perturbation.

There was a paper I looked at once that discussed the deep meaning of odd symmetry between the inverse-square force & the RHO force, but I forgot to make a note of it

NASA/ADS draws a blank too