Condensed Matter Physics
NAME:

Homework 2: Crystals: Due as announced on the course web page in the tentative schedule. Homework solutions will be posted sometime after the due date in the tentative schedule. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

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001 qfull 01320 1 3 0 easy math: polygons tiling a plane

1. As a brain exercise in thinking about crystals, lets consider tiling a plane with regular polygons without gaps or overlap.

   a) First derive the general formula for the interior vertex angle θ of a regular n-sided polygon. **HINT:** In constructing a regular polygon by laying down line segments, each segment must deviated from the previous one's direction by the same angle θ_{dev}. To complete the polygon, the sum of the deviation angles must be 2π.

   b) Now assume that we can tile the plane by attaching the regular polygons to each other with exactly overlaping sides between touching polygons. What condition must hold on the sum of the interior angles at each vertex? Derive the general formula for the number of polygons m that meet at a vertex as a function of n. What condition on m is necessary (but not obviously sufficient) in order for our tiling assumption to hold? For what values of n does m satisfy the condition? What the polygons with those n values?

   c) Give an argument that the tiling assumption of part (b) is correct for the regular polygons that meet the condition on m we found in part (b). The regular polygons that don't meet the condition on m are already excluded.

   d) Now consider the case of tiling the whole plane (without gaps or overlap) with regular polygons that do not have exactly overlaping sides between touching polygons. The polygons have touch along sides that do not exactly overlap (i.e., are slide along from exact overlap) or one polygon's vertex just touches anothers side. Derive which cases for which complete tiling is possible.

SUGGESTED ANSWER:

a) To complete the n-sided polygon requires n deviations. Thus,

$$\theta_{\text{dev}} = \frac{2\pi}{n} \ .$$

The interior vertex angle is given by

$$\theta = \pi - \theta_{\text{dev}} = \pi - \frac{2\pi}{n} \ .$$

b) The sum of the interior vertex angles at each vertex must be 2π in order to tile without gaps or overlap. Thus,

$$2\pi = m\theta = m\left(\pi - \frac{2\pi}{n}\right)$$

and thus

$$m = \frac{2}{1 - 2/n} = \frac{1}{1/2 - 1/n} \ .$$

The necessary condition on m is that it must be an integer greater than equal to 3 since only an integral number of polygons can meet at a vertex and 3 is the obvious minimum number of polygons that can so meet. It only 2 polygons met a vertex, then their interior angles would have to be π and they couldn't be polygons. We note that m strictly decreases with n. The smallest value of n allowed is 3 which gives a triangle and the maximum $m = 6$. We can solve for the maximum allowed value of n by setting $m = 3$. We find

$$n = \frac{1}{1/2 - 1/m} = 6 \ .$$

Thus only regular polygons possible so far are those with n in the range $[3,6]$. For $n=4$, $m=4$, and so the $n=4$ polygon is as allowed. For $n=5$, $m=10/3$ which is not an integer, and so is not allowed.

So given our assumption about tiling, the only possible regular polygons are triangle, square, and hexagon. The pentagon is notably excluded.

c) Well we only have to consider whether we can, in fact, tile the whole plane with triangles, squares, and hexagons.

For squares, the answer is obviously yes. Just make an infinite row of squares and then add infinitely many rows.

For triangles, construct an infinite row of tight fitting alternating up and down triangles. Now add infinitely many rows.

Hexagons present a slightly trickier case. Take three rows of our triangles from above. One can make an infinite up-and-down row of hexagons out of them with bites excluded. The excluded bites can be incorporated into adjacent up-and-down rows. The up-and-down rows can the be piled to infinity in both up and down directions.

So regular triangles, squares, and hexagons can all tile all space with their line segements exactly overlapping for touching polygons. No other regular polygons can do this. Not pentagons.

d) Consider the slide case first. If the wedge opened when you slide equal $\pi-\theta$. In order to put any other polygon into that wedge we require $\theta \leq \pi-\theta$ or $\theta \leq \pi/2$. Using the formula for θ from part (a), we find

$$\pi - \frac{2\pi}{n} leq \frac{\pi}{2} \qquad \text{or} \qquad n \leq 4 \ ,$$

We conclude that the slide case can possibly give complete tiling only for triangles and squares. In both cases, obviously completely tiling is possible. Hexagons cannot completely tile in this case nor any other polygons.

In the vertex touching case, there are two wedges. Let them have angles θ_1 and θ_2 Note

$$\pi = \theta + \theta_1 + \theta_2$$

In order for polygons to fill the wedges, we require $\theta_1 \geq \theta$ and $\theta_2 \geq \theta$. Thus,

$$\pi = \theta + \theta_1 + \theta_2 \geq 3\theta$$

or

$$\theta \leq \frac{\pi}{3} \ .$$

So the vertex touching case only works for triangles.

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