Condensed Matter Physics
NAME:

Homework 1: Atoms and Molecules: Due as announced on the course web page in the tentative schedule. Homework solutions will be posted sometime after the due date in the tentative schedule. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

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001 qfull 00200 1 3 0 easy math: Levi-Civita symbol

1. Let ijk be general integers from the range 1, 2, 3. The Levi-Civita symbol is defined

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ are in cyclic order;} \\ -1 & \text{if } ijk \text{ are in anticyclic order;} \\ 0 & \text{if any two of } ijk \text{ are equal.} \end{cases}$$

The Levi-Civita symbol has many uses in mathematics and physics. There are two identities involving the Levi-Civita symbol that are useful to know. The first because it is useful in proving the second and the second because it turns up in many proofs.

The first identity is

$$\epsilon_{ijk}\epsilon_{\ell mn} = \delta_{i\ell}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{k\ell} + \delta_{in}\delta_{j\ell}\delta_{km} - \delta_{im}\delta_{j\ell}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{k\ell} - \delta_{i\ell}\delta_{jn}\delta_{km} \ .$$

The second is

$$\epsilon_{ijk}\epsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell} \ ,$$

where there is an Einstein summation on the repeated index i.

a) As a first step in proving the first identity, describe the behavior of its left-hand side (LHS). The following parts to part (f) complete the proof of this identity

b) Now show that

$$\text{RHS}(ijk, \ell mn) = \text{RHS}(\ell mn, ijk) \ .$$

This equality shows that the RHS has the same exchange symmetry as the LHS side which is necessary to proving the identity.

What the equality shows is that properties proven for the set of integers in the 2nd argument position (i.e., the 2nd slot) also hold for the set of integers in the 1st argument position (i.e., the 1st slot). For example say that you proved a functional property for ℓmn in RHS$(ijk, \ell mn)$. That functional porperty must also hold for ijk in RHS$(\ell mn, ijk)$ since ijk and ℓmn are general sets of integers. But since RHS$(ijk, \ell mn)$ = RHS$(\ell mn, ijk)$, the property must also hold for ijk in RHS$(ijk, \ell mn)$. Thus, any functional property we prove for the 2nd slot also holds for the 1st slot, and we don't have to repeat the proof nor make a point of not having to repeat the proof.

c) Now show

$$\text{RHS}[ijk, P_\pm(\ell mn)] = \pm\text{RHS}(ijk, \ell mn) \ ,$$

where $P_\pm$ stands for permutation with the upper case being cyclic and the lower case being anticyclic.

d) Now show that the RHS is zero if any two of ℓmn are all equal.

e) Given that all of the ijk have distinct values and all of ℓmn have distinct values, show that the RHS is 1 if the values of ijk and ℓmn have the same cyclicity (i.e., differ in order from each other by an even number of permutations) and -1 if they have different cyclicity (i.e., differ in order from each other by an odd number of permutations).

f) Now complete the proof of the first identity. **HINT:** There is little left to do.

g) Why can't the LHS of the first identity be factored into two identical formulae for the Levi-Civita symbol? **HINT:** The is little left to do.

h) Now prove the second identity from the first identity. into two identical formulae for the Levi-Civita symbol? **HINT:** This is easy.

SUGGESTED ANSWER:

a) The LHS is zero if any two of ijk are equal or any two of ℓmn are equal and otherwise it is not zero. If both ijk and ℓmn are made of distinct values, the LHS is not zero. When not zero, the LHS equal 1 if ijk and ℓmn have the same cyclicity and -1 if they have different cyclicity. Since ϵ_{ijk} and $\epsilon_{\ell mn}$ are just scalars, they do commute. Thus $\epsilon_{ijk}\epsilon_{\ell mn} = \epsilon_{\ell mn}\epsilon_{ijk}$.

b) For the RHS, note that the order of ijk are the same in every term and the order of ℓmn runs through all $3! = 6$ permutations going through all the terms. If you do the same set of permutation operations to put the ℓmn integers all into the order ℓmn order to the ijk integers simultaneously, the sets of ijk integers now run all $3! = 6$ permutations going through all the terms. You then can just commute the indices of the Kronecker delta functions and you get

$$\mathrm{RHS}(ijk, \ell mn) = \mathrm{RHS}(\ell mn, ijk) \ ,$$

QED. It is probably easiest to do this proof by inspection.

But if paranoia afflicts, then explicitly

$$\begin{aligned}
\mathrm{RHS}(ijk, \ell mn) &= \delta_{i\ell}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{k\ell} + \delta_{in}\delta_{j\ell}\delta_{km} - \delta_{im}\delta_{j\ell}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{k\ell} - \delta_{i\ell}\delta_{jn}\delta_{km} \\
&= \delta_{i\ell}\delta_{jm}\delta_{kn} + \delta_{k\ell}\delta_{im}\delta_{jn} + \delta_{j\ell}\delta_{km}\delta_{in} - \delta_{j\ell}\delta_{im}\delta_{kn} - \delta_{k\ell}\delta_{jm}\delta_{in} - \delta_{i\ell}\delta_{km}\delta_{in} \\
&= \delta_{\ell i}\delta_{mj}\delta_{nk} + \delta_{\ell k}\delta_{mi}\delta_{nj} + \delta_{\ell j}\delta_{mk}\delta_{ni} - \delta_{\ell j}\delta_{mi}\delta_{nk} - \delta_{\ell k}\delta_{mj}\delta_{ni} - \delta_{\ell i}\delta_{mk}\delta_{ni} \\
&= \mathrm{RHS}(\ell mn, ijk) \ ,
\end{aligned}$$

where we have used the fact that $\delta_{ij} = \delta_{ji}$. Thus, we have

$$\mathrm{RHS}(ijk, \ell mn) = \mathrm{RHS}(\ell mn, ijk) \ ,$$

QED all over again.

c) The proof is by inspection. But one can elaborate a bit. A cyclic permutation keeps the sum of the first/second three terms the same. An anticycle permutation changes the first/second into the second/first three terms.

d) Now we know that

$$\mathrm{RHS}[ijk, P_-(\ell mn)] = -\mathrm{RHS}(ijk, \ell mn) \ .$$

A particular anticycle permutation case is

$$\mathrm{RHS}(ijk, \ell nm) = -\mathrm{RHS}(ijk, \ell mn) \ .$$

Now if $m = n$, we also have

$$\mathrm{RHS}(ijk, \ell nm) = \mathrm{RHS}(ijk, \ell mn) \ .$$

The conclusion is that

$$\mathrm{RHS}(ijk, \ell mm) = 0 \ .$$

Obviously, the same is true for any anticyclic permutation of any two of ℓmn. Thus, we have proven that if any two of ℓmn are equal, $\mathrm{RHS}(ijk, \ell nm) = 0$.

e) Obviously only 1 term in of the RHS can be non-zero in this case and it must be either 1 or -1. If the values of jkm are cyclic/anticyclic and the values of ℓmn are cyclic/anticyclic, non-zero term is one of the first three in the RHS, RHS = 1, and the orders differ by an even number of permutations (which could be zero permutations). If the values of jkm are cyclic/anticyclic and the values of ℓmn are anticyclic/cyclic, RHS = -1, and non-zero term is one of the second three in the RHS, RHS = -1, and the orders differ by an odd number of permutations.

f) Parts (b) through (e) show that the RHS behaves like the LHS in call cases. Valid is

$$\epsilon_{ijk}\epsilon_{\ell mn} = \delta_{i\ell}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{k\ell} + \delta_{in}\delta_{j\ell}\delta_{km} - \delta_{im}\delta_{j\ell}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{k\ell} - \delta_{i\ell}\delta_{jn}\delta_{km} \ .$$

g) To be factorable, the number of LHS terms would have to be the square of an integer and the number of factors Kronecker delta functions in a term would have to be even. Neither of these things is true. So the LHS can't be factored into identical formulae for the Levi-Civita symbol.

Actually, the Levi-Civita symbol cannot be written as a simple formula containing Kronecker delta functions of the form δ_{jk} and perhaps other straightforward elements. The proof of this is that no one ever presents or hints that such a formula is possible.

One can, however, write a somewhat tricky formula for the Levi-Civita symbol using Kronecker delta functions:

$$\epsilon_{ijk} = \delta_{i+1,j}\delta_{j+1,k} - \delta_{i-1,j}\delta_{j-1,k} \ ,$$

where there is **NO** Einstein summation and where the values obey a sort of cyclic math. The three integers 1, 2, and 3 obey ordinary addition, except 1 is next "above" 3 and 3 is next "below" 1. Thus, for any integer i (out of 1, 2, 3), $i \pm 3 = i$. It is now clear for distinct ijk that the RHS is 1 if ijk are cyclic in their values and -1 if ijk are anticyclic in their values. You could verify this tediously if you afflicted by paranoia. If any two of ijk have the same values, then the RHS is zero. If $i = j$ or $j = k$, then the RHS is clearly is zero. Now i and k have to differ by 2 in order for the two terms both be non-zero. But if $i = k$, the i and k differ by 0 or 3. This $i = k$, the RHS sice is zero. Thus we have proven cyclic-math identity.

h) Begorra:

$$\begin{aligned}
\epsilon_{ijk}\epsilon_{imn} &= \delta_{ii}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{ki} + \delta_{in}\delta_{ji}\delta_{km} - \delta_{im}\delta_{ji}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{ki} - \delta_{ii}\delta_{jn}\delta_{km} \\
&= 3\delta_{jm}\delta_{kn} + \delta_{km}\delta_{jn} + \delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn} - \delta_{kn}\delta_{jm} - 3\delta_{jn}\delta_{km} \\
&= \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \ .
\end{aligned}$$

and thus valid is

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \ .$$

The second identity is very tedious to prove without having the first identity in hand.

The cyclic math formula for the Levi-Civita formula,

$$\epsilon_{ijk} = \delta_{i+1,j}\delta_{j+1,k} - \delta_{i-1,j}\delta_{j-1,k} \ ,$$

can also be used to prove the second identity. Since the Einstein summation rule is turned off for the cyclic math formula, we have to use explicit summation signs. Behold:

$$\begin{aligned}
\sum_i \epsilon_{ijk}\epsilon_{imn} &= \sum_i (\delta_{i+1,j}\delta_{j+1,k} - \delta_{i-1,j}\delta_{j-1,k})(\delta_{i+1,m}\delta_{m+1,n} - \delta_{i-1,m}\delta_{m-1,n}) \\
&= \sum_i [\delta_{i+1,j}\delta_{j+1,k}\delta_{i+1,m}\delta_{m+1,n} - \delta_{i+1,j}\delta_{j+1,k}\delta_{i-1,m}\delta_{m-1,n} \\
&\qquad - \delta_{i-1,j}\delta_{j-1,k}\delta_{i+1,m}\delta_{m+1,n} + \delta_{i-1,j}\delta_{j-1,k}\delta_{i-1,m}\delta_{m-1,n}] \\
&= \sum_i [\delta_{i+1,j}\delta_{j+1,k}\delta_{i+1,m}\delta_{m+1,n} - \delta_i - 1, j - 2\delta_{j+1,k}\delta_{i-1,m}\delta_{m-1,n} \\
&\qquad - \delta_{i+1,j+2}\delta_{j-1,k}\delta_{i+1,m}\delta_{m+1,n} + \delta_{i-1,j}\delta_{j-1,k}\delta_{i-1,m}\delta_{m-1,n}] \\
&= \delta_{j,m}\delta_{j+1,k}\delta_{m+1,n} - \delta_{j-2,m}\delta_{j+1,k}\delta_{m-1,n} - \delta_{j+2,m}\delta_{j-1,k}\delta_{m+1,n} + \delta_{j,m}\delta_{j-1,k}\delta_{m-1,n} \\
&= \delta_{j,m}\delta_{j+1,k}\delta_{m+1,n} - \delta_{j-2,m}\delta_{j+1,k}\delta_{j-3,n} - \delta_{j+2,m}\delta_{j-1,k}\delta_{j+3,n} + \delta_{j,m}\delta_{j-1,k}\delta_{m-1,n} \\
&= \delta_{j,m}(\delta_{j+1,k}\delta_{m+1,n} + \delta_{j-1,k}\delta_{m-1,n}) - \delta_{j,n}(\delta_{j-2,m}\delta_{j+1,k} + \delta_{j+2,m}\delta_{j-1,k}) \\
&= \delta_{j,m}(\delta_{j+1,k}\delta_{j+1,n} + \delta_{j-1,k}\delta_{j-1,n}) - \delta_{j,n}(\delta_{j+1,m}\delta_{j+1,k} + \delta_{j-1,m}\delta_{j-1,k}) \\
&= \delta_{j,m}(\delta_{j+1,k}\delta_{j+1,n} + \delta_{j-1,k}\delta_{j-1,n} + \delta_{j,k}\delta_{j,n}) - \delta_{j,n}(\delta_{j+1,m}\delta_{j+1,k} + \delta_{j-1,m}\delta_{j-1,k} + \delta_{j,k}\delta_{j,m}) \\
&= \delta_{j,m}\delta_{kn} - \delta_{j,n}\delta_{km} \ .
\end{aligned}$$

The proof using the cyclic math formula is not as elegant as I'd hoped.

Redaction: Jeffery, 2014jan01