0.1 lecture VII

0.2 Angular Momentum Coupling

Consider a Hilbert space spanned by a direct product of kets $|j_1 m_1 > |j_2 m_2 >$ $j_1\, m_1$ are the angular momentum quantum numbers for particle 1 whereas $j_2\, m_2$ are the angular momentum quantum numbers associated with particle 2. The dimension of this Hilbert space is $(2j_1+1)(2j_2+1)$. As an example suppose we have two spin-1/2 particles i.e. $j_1 = j_2 = 1/2$. Then the kets that span Hilbert space are

$$\begin{array}{c} |\frac{1}{2}\frac{1}{2} > |\frac{1}{2}\frac{1}{2} > \\ |\frac{1}{2}\frac{1}{2} > |\frac{1}{2} - \frac{1}{2} > \\ |\frac{1}{2} - \frac{1}{2} > |\frac{1}{2}\frac{1}{2} > \\ |\frac{1}{2} - \frac{1}{2} > |\frac{1}{2} - \frac{1}{2} > \end{array} \tag{1}$$

These product state are eigenstates of the set of commuting operators $\boldsymbol{J}_1^2, \boldsymbol{J}_2^2, J_{1z}, J_{2z}$ and we introduce the new notation

$$|j_1 j_2 m_1 m_2 > \equiv |j_1 m_1 > |j_2 m_2 > \tag{2}$$

and we find (in this section, we set $\hbar = 1$)

$$\begin{aligned} \boldsymbol{J}_1^2 |j_1 j_2 m_1 m_2 > &= j_1(j_1+1)|j_1 j_2 m_1 m_2 > \\ \boldsymbol{J}_2^2 |j_1 j_2 m_1 m_2 > &= j_2(j_2+1)|j_1 j_2 m_1 m_2 > \\ J_{1z} |j_1 j_2 m_1 m_2 > &= m_1 |j_1 j_2 m_1 m_2 > \\ J_{2z} |j_1 j_2 m_1 m_2 > &= m_2 |j_1 j_2 m_1 m_2 > \end{aligned} \tag{3}$$

We also note that these kets are complete, i.e.

$$\sum_{m_1 m_2} |j_1 j_2 m_1 m_2 >< m_2 m_1 j_2 j_1| = I \tag{4}$$

here the identity I is a direct product of I_1, the identity operator spanned in the sub space of $|j_1 m_1 >$ and I_2.

We now define new operators

$$\boldsymbol{J} \equiv \boldsymbol{J}_1 + \boldsymbol{J}_2 \tag{5}$$

Since both $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ obey the angular momentum Lie algebra, and $[\boldsymbol{J}_{i1}, \boldsymbol{J}_{j2}] = 0$, it is obvious that

$$[J_i, J_j] = i\epsilon_{ijk} J_k. \tag{6}$$

That is, $\boldsymbol{J}$ is also an angular momentum operator. It is called the total angular momentum for the system. We proved in class that

$$[\boldsymbol{J}^2, \boldsymbol{J}_1^2] = [\boldsymbol{J}^2, \boldsymbol{J}_2^2] = 0 \tag{7}$$

and

$$[J_z, \boldsymbol{J}_1^2] \;\; = \;\; [J_z, \boldsymbol{J}_2^2] = 0 \tag{8}$$

$$[J_z, \boldsymbol{J}^2] \;\; = \;\; 0 \tag{9}$$

or the set of operators $\boldsymbol{J}^2, J_z, \boldsymbol{J}_1^2, \boldsymbol{J}_2^2$ are mutually commuting. Thus they have common eigenstates itemized by the quantum numbers $|jmj_1j_2>$ so that

$$\begin{aligned}
\boldsymbol{J}^2|jmj_1j_2> &= j(j+1)|jmj_1j_2> \\
\boldsymbol{J}_z|jmj_1j_2> &= m|jmj_1j_2> \\
\boldsymbol{J}_1^2|jmj_1j_2> &= j_1(j_1+1)|jmj_1j_2> \\
\boldsymbol{J}_2^2|jmj_1j_2> &= j_2(j_2+1)|jmj_1j_2>
\end{aligned} \tag{10}$$

We now ask the question: What is the relationship between the eigenstates $|jmj_1j_2>$ and $|j_1j_2m_1m_2>$?

Since the states $|j_1j_2m_1m_2>$ are complete we have

$$\begin{aligned}
|jmj_1j_2> &= I|jmj_1j_2> \\
|jmj_1j_2> &= \sum_{m_1m_2} |j_1j_2m_1m_2> (< m_2m_1j_2j_1|jmj_1j_2>)
\end{aligned} \tag{11}$$

where we have used Dirac's associative axiom. The coefficient $< m_2m_1j_2j_1|jmj_1j_2>$ is called a **Clebsch-Gordon coefficient**. In modern applications it is common to use the $3j$ symbol. A $3j$ symbol is related to the Clebsch-Gordon coefficient via

$$< m_2m_1j_2j_1|jmj_1j_2> = (-1)^{j_2-j_1-m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} \tag{12}$$

In most instances, for arbitrary values j, m, j_1, j_2, m_1, m_2, the $3j$ symbols vanish. Selection rules help us determine what values for these parameters result in a non-trivial value for the $3j$ symbol.

Consider the operator J_z acting on $|jmj_1j_2>$,

$$
\begin{aligned}
J_z|jmj_1j_2 >= m|jmj_1j_2 >= \\
m\sum_{m_1m_2}|j_1j_2m_1m_2 >< m_2m_1j_2j_1|jmj_1j_2 >= \\
\sum_{m_1m_2}(m_1+m_2)|j_1j_2m_1m_2 >< m_2m_1j_2j_1|jmj_1j_2 >
\end{aligned}
\tag{13}
$$

which can only be satisfied if $m = m_1 + m_2$ or $m_1 + m_2 - m = 0$. Thus

$$
\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0 \tag{14}
$$

iff $m_3 + m_1 + m_2 = 0$.

Using this we can re-write the above relations

$$
|jmj_1j_2 > \;=\; \sum_{m_1}|j_1j_2m_1m-m_1 > (< m-m_1m_1j_2j_1|jmj_1j_2 >) \tag{15}
$$

or

$$
|jmj_1j_2 > \;=\; \sum_{m_2}|j_1j_2m-m_2m_2 > (< m_2m-m_2j_2j_1|jmj_1j_2 >) \tag{16}
$$

Now

$$
-j_1 \le m_1 \le j_1
$$

$$
-j_2 \le m_2 \le j_2
$$

but

$$
m_1 + m_2 = m
$$

so

$$
-j_1 \le m - m_2 \le j_1
$$

$$
-j_2 \le m - m_1 \le j_2
$$

Lets take m to have its maximum value j, and m_1, m_2 to have their maximum values j_1, j_2. Thus

$$
-j_1 \le j - j_2 \le j_1
$$

$$
-j_2 \le j - j_1 \le j_2
$$

or

$$j_2 - j_1 \leq j \leq j_1 + j_2$$

$$j_1 - j_2 \leq j \leq j_2 + j_1$$

which is equivalent to

$$|j_1 - j_2| \leq j \leq j_1 + j_2$$

This is called the triangle condition and sometimes it is written as $\triangle(jj_1j_2)$. Since the eigenvalues j differ by an integer, the above condition allows possible values

$$j = j_1 + j_2,\ j_1 + j_2 - 1,\ j_1 + j_2 - 2 \ldots |j_1 - j_2| \tag{17}$$

Thus we find that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0 \tag{18}$$

unless $m_3 + m_1 + m_2 = 0$, and $\triangle(jj_1j_2)$.

There are other important relations that the $3j$ symbols satisfy including (orthogonality relations),

$$\sum_{j_3 m_3} (2j_3 + 1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta(m_1, m_1')\delta(m_2, m_2') \tag{19}$$

and

$$\sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{\delta(m_3, m_3')\delta(j_3, j_3')}{(2j_3 + 1)} \tag{20}$$

These relations can be used (can you show this?) to derive the inverse relation

$$|j_1 j_2 m_1 m_2 >= \sum_{jm} (-1)^{j_2 - j_1 - m} \sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} |j_1 j_2 j m > \tag{21}$$

0.2.1 More than 2 angular momenta

In the previous section we showed that we can form a total angular momentum operator composed as the sum of the two constituent angular momenta. We can generalize this procedure for any number of particles. Suppose we have three particles and the three angular momentum operators $\boldsymbol{J}_1$, $\boldsymbol{J}_2$, $\boldsymbol{J}_3$. We can first couple the angular momentum for particle 1 & 2 or

$$\boldsymbol{J}_{12} \equiv \boldsymbol{J}_1 + \boldsymbol{J}_2 \tag{22}$$

and we can think of $\boldsymbol{J}_{12}$ as the total angular momentum operator associated with the sub-system composed of particle 1 & 2. We now add $\boldsymbol{J}_{12}$ with $\boldsymbol{J}_3$ to form the total angular momentum for all three particles,

$$\boldsymbol{J} \equiv \boldsymbol{J}_{12} + \boldsymbol{J}_3 \tag{23}$$

We can show, as in the previous section, that the set of operators $\boldsymbol{J}^2, \boldsymbol{J}_z, \boldsymbol{J}_{12}^2, \boldsymbol{J}_3^2, \boldsymbol{J}_2^2$ mutually commute. Therefore we can define an eigenstate

$$|jm; (j_{12})j_1 j_2 j_3>$$

where the notation is an straightforward generalization from that discussed in the previous section. Alternatively, we can find the mutually commuting set $\boldsymbol{J}^2, \boldsymbol{J}_z, \boldsymbol{J}_{23}^2, \boldsymbol{J}_2^2, \boldsymbol{J}_3^2, \boldsymbol{J}_1^2$ with eigenstates

$$|jm; (j_{23})j_1 j_2 j_3>$$

Because the first set of angular momentum operators does not commute with the second set,

$$|jm; (j_{12})j_1 j_2 j_3> \neq |jm; (j_{23})j_1 j_2 j_3> \tag{24}$$

However, the kets can be written as a linear combination of each other, i.e.

$$\begin{aligned} |j_1 j_2 j_3 (j_{23}) j> &= \sum_{j_{12}} (-1)^{j_1+j_2+j_3+j} \times \\ &\sqrt{(2j_{12}+1)(2j_{23}+1)} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} |j_1 j_2 (j_{12}) j_3 j> \end{aligned} \tag{25}$$

The quantity

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}$$

is called a $6j$ symbol. It is also related to (older literature) the Racah W coefficient. The $6j$ symbol has numerous symmetries and selection rules. For a given symbol

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \tag{26}$$

we require the conditions,

$$\triangle(j_1 j_2 j_3), \quad \triangle(l_1 l_2 j_3), \quad \triangle(l_1 j_2 l_3), \quad \triangle(j_1 l_2 l_3)$$

for it to be non-vanishing.