0.1 lecture VI

0.2 Angular Momentum as a Generator of Rotation

0.2.1 Unitary Transformations

Consider the position operator $\hat{x}$, and its eigenstate

$$\hat{x}|x>= x|x> \tag{1}$$

Lets define the operator (we set $\hbar = 1$)

$$U \equiv \exp(-i\hat{p}a) \tag{2}$$

where $\hat{p}$ is the momentum operator $[\hat{p}, \hat{x}] = -i$, and a is a constant real number.

The Hermitian conjugate of U is

$$U^\dagger = \exp(i\hat{p}a) \tag{3}$$

since $\hat{p}$ is Hermitian. Now

$$U^\dagger U = UU^\dagger = 1 \tag{4}$$

therefore $U^\dagger = U^{-1}$ by definition. Operators that have this property are called unitary operators, one way to construct a unitary operator is to exponentiate, as above, a Hermitian operator.

We ask, what is the physical significance of the state $U|x>$? First, we notice that the inner product of $U|x>$ is equal to the inner product $<x|x>$, which follows from the fact that U is unitary.

Lets define a new operator

$$\begin{aligned} \hat{x}' \equiv U\hat{x}U^\dagger &= \\ & \hat{x} - ia[\hat{p}, \hat{x}] + \frac{(-ia)^2}{2!}[\hat{p}, [\hat{p}, \hat{x}] + ... \\ &= \hat{x} - a \end{aligned} \tag{5}$$

or

$$U\hat{x}U^\dagger = \hat{x} - a \tag{6}$$

We can rewrite

$$
\begin{aligned}
\hat{x}|x> &= x|x> \\
U\hat{x}U^{\dagger}U|x> &= xU|x> \\
(\hat{x}-a)U|x> &= xU|x>
\end{aligned}
\tag{7}
$$

Thus $U|x>$ is an eigenstate of $(\hat{x}-a)$ with eigenvalue x, or $U|x>=|x+a>$. From this equation we interpret the action of U on the ket $|x>$. It translates the eigenvalue by the amount a, the constant in the exponent. Suppose we have two operators $U_1 = \exp(-ia\hat{p})$, $U_2 = \exp(-ib\hat{p})$ can you show that $U \equiv U_1U_2$ also generates a translation by an amount $a+b$. We say that the linear momentum operator $\hat{p}$ is a generator of the translation symmetry operator U. This symmetry operator is also, by necessity, a unitary transformation.

We can generalize these ideas by defining a new unitary operator

$$
U \equiv \exp(-iJ_2\theta) \tag{8}
$$

where J_2 is a component of the angular momentum. Prove that U is unitary. Suppose we have an eigenstate $|jm>$ such that

$$
\begin{aligned}
\boldsymbol{J}^2|jm> &= j(j+1)|jm> \\
J_3|jm> &= m|jm>
\end{aligned}
\tag{9}
$$

What is the physical meaning of $U|jm>$?

Since

$$
\begin{aligned}
J_3|jm> &= m|jm> \\
UJ_3U^{\dagger}U|jm> &= mU|jm>
\end{aligned}
\tag{10}
$$

$U|jm>$ is an eigenstate of the new operator $UJ_3U^{\dagger}$ with the eigenvalue m associated with operator J_3.

Now

$$
\begin{aligned}
UJ_3U^{\dagger} &= \exp(-iJ_2\theta)J_3\exp(iJ_2\theta) \\
= J_3 - i\theta[J_2, J_3] &+ \frac{(-i\theta)^2}{2!}[J_2,[J_2,J_3]] + ...
\end{aligned}
\tag{11}
$$

Using angular momentum commutation relations we find,

$$
\exp(-iJ_2\theta)J_3\exp(iJ_2\theta) = J_3 cos\theta + J_1 sin\theta \tag{12}
$$

We note that this operator is just $\boldsymbol{J}\cdot\hat{\boldsymbol{n}}$ where $\hat{\boldsymbol{n}} = sin\theta\hat{\boldsymbol{\imath}} + cos\theta\hat{\boldsymbol{k}}$ is a unit vector in the $x-z$ plane making an angle θ with respect to the z axis. In other words $\exp(-iJ_2\theta)J_3\exp(iJ_2\theta) = \boldsymbol{J}\cdot\hat{\boldsymbol{n}}$ is the component of angular momentum along the $\hat{\boldsymbol{n}}$ direction. Thus $U|jm>$ is an eigenstate of the operator $\boldsymbol{J}\cdot\hat{\boldsymbol{n}}$ and we say that U generates a rotation, around the y axis. As an example, consider a spin 1/2 ($j = 1/2$) system. Then $|j; m = 1/2>$ is a "spin-up" eigenstate with respect to the z-axis. By performing the rotation, $U|j; m = 1/2>$ is still a "spin-up vector" (since $m = 1/2$) but along a new direction. If we set $\theta = \pi/2$ it is an "up" state along the x-axis.

In this manner, we can define a general rotation

$$U \equiv \exp(-iJ_3\phi)\exp(-iJ_2\theta)\exp(iJ_3\phi) \tag{13}$$

As above, we know that the eigenstates of this operator are $U|jm>$, but we need to evaluate

$$UJ_3U^\dagger \tag{14}$$

Since J_3 commutes with the outer operators, we get

$$\exp(-iJ_3\phi)\exp(-iJ_2\theta)J_3\exp(iJ_2\theta)\exp(iJ_3\phi) \tag{15}$$

or

$$\begin{aligned}\exp(-iJ_3\phi)\Big(J_3cos\theta + J_1sin\theta\Big)\exp(iJ_3\phi) = \\ J_3cos\theta + sin\theta\exp(-iJ_3\phi)J_1\exp(iJ_3\phi)\end{aligned} \tag{16}$$

Since

$$\exp(-iJ_3\phi)J_1\exp(iJ_3\phi) = J_1cos\phi + J_2sin\phi \tag{17}$$

We find

$$\begin{aligned}UJ_3U^\dagger &= J_3cos\theta + sin\theta(J_1cos\phi + J_2sin\phi) = \boldsymbol{J}\cdot\hat{\boldsymbol{n}} \\ \hat{\boldsymbol{n}} &= \hat{\boldsymbol{k}}cos\theta + \hat{\boldsymbol{\imath}}sin\theta cos\phi + sin\theta sin\phi\hat{\boldsymbol{\jmath}}\end{aligned} \tag{18}$$

and thus, $U|jm>$ is an eigenstate, with eigenvalue m, of the component of angular momentum $\boldsymbol{J}$ along the dircetion of the unit vector $\hat{\boldsymbol{n}}$.

We pointed out in previous discussions that the basis $|jm>$ form a $(2j+1)$ dimensional matrix representation for the operators $\boldsymbol{J}$. We can also use them to form matrix representations of the rotation operator U. The matrix

$$\begin{aligned}< jm'|U|jm >=< jm'|\exp(-iJ_3\phi)\exp(-iJ_2\theta)\exp(iJ_3\phi)|jm >= \\ < jm'|\exp(-iJ_2\theta)|jm > \exp(-(m'-m)\phi)\end{aligned} \qquad (19)$$

is called the Wigner rotation matrix. For the special case $j = \frac{1}{2}$

$$< jm'|\exp(-iJ_2\theta)|jm >= \begin{pmatrix} cos(\frac{\theta}{2}) & -sin(\frac{\theta}{2}) \\ sin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{pmatrix} \qquad (20)$$

and thus

$$< jm'|U|jm >= \begin{pmatrix} cos(\frac{\theta}{2}) & -sin(\frac{\theta}{2})\exp(-\phi) \\ sin(\frac{\theta}{2})\exp(\phi) & cos(\frac{\theta}{2}) \end{pmatrix} \qquad (21)$$

The matrix representation for the rotation operator U allows us to construct eigenstates for the angular momentum in any arbitrary direction. So if we have a "spin-up" state

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (22)$$

the transformed state

$$\begin{pmatrix} cos(\frac{\theta}{2}) & -sin(\frac{\theta}{2})\exp(-\phi) \\ sin(\frac{\theta}{2})\exp(\phi) & cos(\frac{\theta}{2}) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} cos(\frac{\theta}{2}) \\ sin(\frac{\theta}{2})\exp(\phi) \end{pmatrix} \qquad (23)$$

is an eigenstate of $\boldsymbol{J} \cdot \hat{\boldsymbol{n}}$. Using the values $\theta = \pi/2, \phi = 0$ and $\theta = \pi/2, \phi = \pi/2$ we obtain the eigenstates, with eigenvalue $+1/2$, along the $\boldsymbol{x}$ and $\boldsymbol{y}$ axis respectively.