0.1 Lecture III

0.2 Some Hilbert space examples

The simplest Hilbert space consists of a basis that spans one-dimension, i.e. we only have a single observable that we measure with 100% certainty.

Lets suppose the eigenstate is given by $|\epsilon_0>$ and is an eigenstate of the Hamiltonian operator H, so that $H|\epsilon_0>=\epsilon_0|\epsilon_0>$. Suppose $<\epsilon_0|\epsilon_0>=1$, show that this state is not unique, that $\exp(i\alpha)|\epsilon_0>$ is also a normalized eigenstate (α is an arbitrary real number). Show that a solution of the Schrodinger eqn is given by

$$|\Psi(t)>=\exp(-i\,\epsilon_0 t/\hbar)|\psi_0> \tag{1}$$

where $|\psi_0>$ is an eigenstate of H.

0.2.1 2 D Hilbert space

An important class of quantum systems can simply be expressed by only 2 basis vectors, lets call them $|+>$ and $|->$. According to the Dirac notation we construct operators.

$$\begin{aligned} X_1 &\equiv |+><+| \\ X_2 &\equiv |-><-| \\ X_3 &\equiv |+><-| \\ X_4 &\equiv |-><+| \end{aligned}$$

Can you show that physical operators are linear combinations of the above operators? We can also multiply these operators using Dirac's procedure, thus e.g. $X_1X_3 = X_4$ but $X_3X_1 = 0$. Note that multiplication is not commutative for the defined operators. Show that the above operators also form a vector space. What is the dimension of this vector space? Since multiplication is not commutative it is natural to define a binary operation called the *Lie bracket* that for two members of this space $[X_a, X_b] \equiv X_aX_b - X_bX_a$. A vector space of operators with a Lie Bracket so that it is linear, i.e.

$$[aX_1 + bX_2, X_3] = a[X_1, X_3] + b[X_2, X_3] \tag{2}$$

and satisfies the *Jacobi identity*

$$[[X_1, X_2], X_3] + [[X_3, X_1], X_2] + [[X_2, X_3], X_1] = 0 \tag{3}$$

is called a *Lie Algebra*. Show that the following operators

$$\begin{aligned} S_1 &= X_3 + X_4 \\ S_2 &= -iX_3 + iX_4 \\ S_3 &= X_1 - X_2 \\ I &= X_1 + X_2 \end{aligned} \tag{4}$$

are physical and form a Lie Algebra. A well know Lie Algebra is the set of vectors in R^3 where the Lie bracket is the cross product.

The above system already shows quite interesting behavior. The fact that we have several physical operators raises the question, how do we characterize our physical states (kets)? In our previous lecture we mentioned that we use the eigenvalues of a physical operator to specify the states, but which operator do we choose? For the system described above we have at least four possible physical operators to consider. Certainly the state $|+>$ is an eigenstate of S_3, and I, but is not an eigenstate S_2 and S_1 (can you see why?).

To answer this we define the concept of **compatible observables**.

Compatible observables

The set of operators $\hat{A}, \hat{B}, \hat{C}, ...$ in Hilbert space are said to be compatible if all possible Lie brackets (commutators) vanish among the set, i.e $[\hat{A}, \hat{B}] = [\hat{B}, \hat{C}] = [\hat{A}, \hat{C}] = 0$, etc. In that case an eigenstate of operator $\hat{A}$ is also an eigensta of $\hat{B}...\hat{C}$, etc. *Exercise: proof this.* A ket that is an eigenstate of all mutually compatible operators is called a simultaneous eigenstate of $\hat{A}$, $\hat{B}$. It is denoted by the symbol $|abc... >$ and has the properties. as

$$\begin{aligned} \hat{A}|abc.. &>= a|abc... > \\ \hat{B}|abc.. &>= b|abc... > \\ \hat{C}|abc.. &>= c|abc... > \end{aligned} \tag{5}$$

If $[\hat{A}, \hat{B}] \neq 0$ and eigenstate $|a>$ will not be an eigenstate of B. In the 2 D Hilbert space find 3-possible sets of commuting observables, to which statee do $|+>, |->$ belong to ? Find the eigenstates for the other sets, and express them in terms of the $|+>, |->$

Projection operators

Armed with the bra-ket formalism we can construct any operator in Hilbert space. The projection operator P_a is defined as

$$P_a = |a >< a|, \tag{6}$$

Note that $P_a^2 \equiv P_a P_a = |a >< a||a >< a| = |a >< a| = P_a$ and in general any projection operator P has the property $P^2 = P$. Consider operator X whose eigenstates are given by the set $\{|a_1 > \; |a_2 > ...|a_n >\}$. If we define the projection operators $P_{a_n} = |a_n >< a_n|$, show that operator X can be expressed as a sum of projection operators, i.e.

$$X = \sum_n P_{a_n} a_n \tag{7}$$

Matrix representation of operators

Above we suggested that operators in Hilbert space can be written as a sum of outer products. However, it is much easier to work with operators that are expressed as matrices. Given operator Y and a basis set of kets $\{|a_1 > \; |a_2 > ...|a_n >\}$, we can construct an $n \times n$ table of complex numbers whose $n\,m$th element is given by $< a_n|Y|a_m >$. The set of n^2 complex numbers can most conveniently be written as a square matrix, where m is a column index and n a row index. We use the notation $\underline{\mathrm{Y}}$ to represent this matrix, and the $n\,m$th element of that matrix is $\underline{\mathrm{Y}}_{nm}$. The matrix $\underline{\mathrm{Y}}$ is called the matrix representation of operator Y. Show that we can express Y in the form

$$Y = \sum_{nm} |a_n >< a_m| \underline{\mathrm{Y}}_{nm} \tag{8}$$

Consider the following equation in ket space,

$$Y|\Psi >= |\Psi' > \tag{9}$$

where Y is an operator and $|\Psi >$ an arbitrary ket. We take the inner product of this equation with state $|a_m >$ (Remember, taking an inner product of two kets involves multiplying on the left by a bra), or

$$< a_m|Y|\Psi >=< a_m|\Psi' > . \tag{10}$$

Evaluating this expression we obtain,

$$\sum_n Y_{m\,n} < a_n|\Psi> = < a_m|\Psi'> \tag{11}$$

which can be written as a matrix equation

$$\underline{Y}\ \underline{\Psi} = \underline{\Psi'} \tag{12}$$

where we defined a column matrix $\underline{\Psi}$ whose $n'th$ row has the entry $< a_n|\Psi >$. The utility of a matrix representation for operator Y and ket $|\Psi>$ is now apparent. We can replace any abstract operator equation, such as Eq. (9), with a more familiar matrix equation (12).

We can do the same in bra space. For example, consider the adjoint of Eq. (9) where $\underline{\Psi}^\dagger$ is the adjoint *matrix* of the column matrix $\underline{\Psi}$ (note: you can consider the adjoint of a column matrix to be a row matrix). Note that the nm 'th element of matrix $\underline{A}^\dagger$ is the complex conjugate of the mn'th element of matrix $\underline{A}$, i.e. $\underline{A}^\dagger_{nm} = \underline{A}^*_{mn}$

$$< \Psi| = < \Psi'|Y^\dagger. \tag{13}$$

Show that the matrix representation of this equation is given by

$$\underline{\Psi}^\dagger = \underline{\Psi}^\dagger \underline{Y}^\dagger \tag{14}$$

Matrix representation of 2 D system

Lets reconsider the operators S_1, S_2, S_3 discussed previously. We now seek a representation of these operators with respect to some basis. Which basis do we use?, we have infinite choices but a convenient one is just $|+>$, $|->$. Lets call the matrix representation of the operators S_i, $\underline{\sigma}_i$ defined so that

$$\begin{aligned}
\underline{\sigma}_1 &\equiv \begin{pmatrix} <+|S_1|+> & <+|S_1|-> \\ <-|S_1|+> & <-|S_1|-> \end{pmatrix} \\
\underline{\sigma}_2 &\equiv \begin{pmatrix} <+|S_2|+> & <+|S_2|-> \\ <-|S_2|+> & <-|S_2|-> \end{pmatrix} \\
\underline{\sigma}_3 &\equiv \begin{pmatrix} <+|S_3|+> & <+|S_3|-> \\ <-|S_3|+> & <-|S_3|-> \end{pmatrix}
\end{aligned} \tag{15}$$

Note that $\underline{\sigma}_3$ is diagonal with respect to this basis and so it's eigenvalues are on the diagonal. However, $\underline{\sigma}_1$, $\underline{\sigma}_2$ are not. How would we find the eigenvalues of the other matrices? Is $|+>$ a simultaneous eigenket of another operator? These matrices are very famous and are called the Pauli spin matrices, show that they satisfy the Jacobi identities, as well as the relations

$$[\underline{\sigma}_i, \underline{\sigma}_j] = 2i\underline{\sigma}_k \epsilon_{ijk} \tag{16}$$

Hw Problem

Instead of basis $|+>, |->$ consider the linear combinations

$$\begin{aligned} |\uparrow> &\equiv \frac{1}{\sqrt{2}}(|+> + |->) \\ |\downarrow> &\equiv \frac{1}{\sqrt{2}}(|+> - |->) \end{aligned} \tag{17}$$

a) Do $|\uparrow>, |\downarrow>$ form a valid basis set in Hilbert space?

b) Find the matrix representation of operators S_1, S_2, S_3 in this basis. Do they satisfy the Jacobi identity and the Lie Algebra as the matrices in the $|+>, |->$ representation?

c) Find the eigenstates of S_3 in this representation, and express them as column matrices.

d) Construct the following matrix

$$\underline{U} \equiv \begin{pmatrix} <+|\uparrow> & <+|\downarrow> \\ <-|\uparrow> & <-|\downarrow> \end{pmatrix} \tag{18}$$

e) Evaluate the matrix multiplication

$$\underline{U}\ \underline{u} \tag{19}$$

where $\underline{u}$ are the eigenstates obtained in part c). Comment on the significane of this equation.

f) Construct $\underline{U}^\dagger$, evaluate $\underline{U}\,\underline{U}^\dagger$. Comment.

g) A quantum state is given by the vector $|\Psi> = |+>$.

i) Bob uses measuring device S_3 to make a measurement, what result does he obtain?

ii) After Bob's measurement, Alice uses S_2 to make a measurement. What are her possibel results?, what is the expectation value of those measurements?

iii) After Alice's measurement, Bob repeats his measurements. Re-discuss part i).

iv) Comment on Bob's ability to infer if someone else, with devices S_1, S_2, made a measurement on the system while he was not "looking". Assume he obtained the value $+1$ for his first measurement. Is the same true if someone snooped with an S_3 device?