0.1 Lecture I

Postulate (I) Associated with a physical observable (something that can be measured in the lab) is an operator $\hat{A}$. If a measurement is made, the value obtained must be an eigenvalue of $\hat{A}$. If

$$\hat{A}\phi = \lambda\phi \tag{1}$$

where λ is a constant (eigenvalue), then ϕ is called an eigenstate of operator $\hat{A}$. If $\lambda = a$ then it is common to label the eigenstate by that value [1] e. g.

$$\hat{A}\phi_a = a\,\phi_a. \tag{2}$$

Postulate (II) Suppose a measurement is made, corresponding to operator $\hat{A}$ and the value $\lambda = b$ is obtained. Immediately after that measurement the system is said to be in *quantum state* ϕ_b. Any subsequent measurement (If the system is not disturbed) with $\hat{A}$ will yield the value $\lambda = b$ with 100% certainty.

Class discussion: Quantum Zeno effect, Itano et al. Phys. Rev. A ***41****, 2295 (1990).*

0.1.1 Linear vector spaces

Suppose a physical operator $\hat{A}$ has n possible, countable, outcomes or eigenvalues. For each eigenvalue there is an eigenstate ϕ_m so that,

$$\hat{A}\phi_m = \lambda_m\phi_m \quad m = 1, 2, ...n \tag{3}$$

We claim the eigenstates are members of a linear vector space V that we define below.

1. If ϕ_m is an element of V so is $c\,\phi_m$ where c is an arbitrary complex number.

2. A binary operator of addition is defined. If $\phi_n, \phi_m \in V$ then so is a new vector $\phi_k = \phi_n + \phi_m = \phi_m + \phi_n$.

[1] If several, linear independent ϕ_n correspond to a single eigenvalue, they are called degenerate eigenstates.

3. There exists a null vector 0 so that $\phi_m + 0 = 0 + \phi_m = \phi_m$.
4. The following must hold $\phi_k + (\phi_n + \phi_m) = (\phi_k + \phi_n) + \phi_m$

Class discussion: examples of common vector spaces

Some definitions

1. If the relation
$$c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n = 0 \tag{4}$$
holds only for $c_1 = c_2 = ... c_n = 0$, the set of vectors $\phi_1, \phi_2...\phi_n$ are said to be *linear independent.* Otherwise, they are linear dependent.
2. A linear vector space is n-dimensional if it contains n linear independent vectors but not $n+1$ vectors.
3. If V contains n linear independent vectors for every integer, it is an infinite dimensional space.
4. A set of vectors $\phi_1, \phi_2...\phi_n$ is said to *span* the space if each vector in V is a linear combination of them.

We call the vector space spanned by the eigenstates of physical operator $\hat{A}$ Hilbert space[2]. Since we know, according to postulate II, that ϕ_a is associated with a physical measurement that produced eigenvalue a, what does the new vector
$$\psi \equiv c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n \tag{5}$$
represent ? The answer is given by an additional postulate of the Copenhagen interpretation.

Postulate (III) A physical system is represented by a vector Ψ in Hilbert space. According to postulate I, the measurement $\hat{A}$ will result in one of its eigenvalues b, but with a probability given by the quantity $|c_b|^2$.

It is our intention, for a given physical system, to be able to predict the set of probabilities $|c_1|^2, |c_2|^2...|c_n|^2$. To that end we define the inner product in our vector space.

[2]more precisely, Hilbert space is an infinite dimensional space that is non-denumerable, but we needn't be fuss-budgets with the definition

Definition: An inner product of two vectors ϕ_m and ϕ_n is a mapping into a complex number. We can represent this mapping by the symbol

$$(\phi_m, \phi_k) \tag{6}$$

or, in this course, we will use the bra-ket notation

$$< \phi_m|\phi_k > . \tag{7}$$

This mapping is linear, if $\psi = c_m\phi_m + c_n\phi_n$ then

$$< \phi_k|\psi >= c_m < \phi_k|\phi_m > +c_n < \phi_k|\phi_n > . \tag{8}$$

However

$$< \psi|\phi_k >= c_m^* < \phi_m|\phi_k > +c_n^* < \phi_n|\phi_k > \tag{9}$$

where c^* is the complex conjugate of c. The latter relation is anti-linear. From the definition above we require $< \phi_m|\phi_n >=< \phi_n|\phi_m >^*$, thus $< \phi_m|\phi_m >=< \phi_m|\phi_m >^*$ and therefore $< \phi_m|\phi_m > \ \geq 0$ i.e. it is a real number. The equality is satisfied only for the case where the vector ϕ is the null vector 0.

Postulate (IV) The null vector in Hilbert space does not represent a physical state.

Using the latter relation we can ascribe a length to each physical state in Hilbert space.

$$||\Psi|| \equiv \sqrt{< \Psi|\Psi >} \tag{10}$$

For any state Ψ with length $||\psi||$ we can always find a renormalized state $\Psi' \equiv \Psi/\sqrt{||\Psi||}$ so that $< \Psi'|\Psi' >= 1$.

More definitions

- Any two vectors (states) ϕ_m, ϕ_n in Hilbert space that have the property $< \phi_m|\phi_n >= 0$, are said to be orthogonal.

- If a set of states $\phi_1, \phi_2 ... \phi_n$ are mutually orthogonal, and if we normalize each state to unity, so that

$$< \phi_k|\phi_m >= \delta_{k\,m} \tag{11}$$

  then the set is said to be an orthonormal set.

- A set of vectors that are linear independent, span the Hilbert space, and are orthonormal are said to constitute a *basis* for the Hilbert space.

Finally, we offer, without proof, the Schwarz inequality. For any two quantum states ϕ_m, ϕ_m the following inequality must be true

$$| < \phi_m|\phi_n > | \leq ||\phi_m|| \, ||\phi_n|| \tag{12}$$