Cirac - Zoller Scheme

Suppose we have cooled the ions situated along the z-axis so that their xy motion is minimized via Laser cooling. However they are still free to move along the z-axis. Image the set of ions all moving in unison so that the center of mass undergoes harmonic motion. In the quantum realm the excitations of this motion are called phonons. For vibrations with lowest energy is called the ground state and no phonons are present, the next highest energy state has one phonon, and so on. We can describe this by a ladder like stucture of energy levels

Each phonon has an energy  $h \nu$

\[ \begin{align*}
|g\rangle_0 & \quad \text{Ground internal state+no phonons} & \text{Energy} = E_g \\
|g\rangle_1 & \quad \text{Ground internal state+1 phonon} & \text{Energy} = E_g + h \nu \\
|g\rangle_2 & \quad \text{Ground internal state+2 phonons} & \text{Energy} = E_g + 2 h \nu \\
\vdots \\
|e\rangle_0 & \quad \text{Excited internal state+no phonons} & \text{Energy} = E_g + h \omega \\
|e\rangle_1 & \quad \text{Excited internal state+1 phonon} & \text{Energy} = E_g + h \omega + h \nu \\
|e\rangle_2 & \quad \text{Excited internal state+2 phonons} & \text{Energy} = E_g + 2 h \omega + 2 h \nu \\
\vdots \\
\end{align*} \]

Note we assume that the internal state energy difference $h \omega >> h \nu$

For a given ion $m$ we let $|g\rangle$ represent the Qbit $|0\rangle$ and $|e\rangle$ the Qbit $|1\rangle$
so if we have a group of four ions and the first two are in the ground internal state
and the the last two are in an internal excited state they would represent the 
computational basis state

\[ |0011\rangle \]

For a string of ions lets consider the m'th ion and label its internal states as 
\[ |g\rangle_m |e\rangle_m (|0\rangle_m |1\rangle_m). \] Likewise for the n'th ion we use teh notation \[ |g\rangle_n |e\rangle_n \].

Imagine the system starting out as

\[ |g\rangle_m |g\rangle_n |0\rangle_v \text{ total energy } 2E_g \]

Let' shine a laser on ion m with a frequency of \( \omega - \nu \). This frequency is not sufficient 
to excite the ion into its excited state \( |e\rangle_m \) because you need at least an energy \( \hbar \omega \) to that 
so the effect of this laser pulse, which we call shorthand \( U_m \) on this state is

\[ U_m |g\rangle_m |g\rangle_n |0\rangle_v \longrightarrow |g\rangle_m |g\rangle_n |0\rangle_v \]

Likewise

\[ U_m |g\rangle_m |e\rangle_n |0\rangle_v \longrightarrow |g\rangle_m |e\rangle_n |0\rangle_v \]

However

\[ U_m |e\rangle_m |g\rangle_n |0\rangle_v \longrightarrow -i |g\rangle_m |g\rangle_n |1\rangle_v \]

That is the if \( U_m \) acts on state \( |e\rangle_m |g\rangle_n |0\rangle_v \) it changes to a new state where the m'th Qubit changes 
its value form 1 to 0 while a phonon is created. The energy of the initial configuration 
\( |e\rangle_m |g\rangle_n |0\rangle_v \) is \( \hbar \omega + 2E_g \) whereas the energy of 
\( |g\rangle_m |g\rangle_n |1\rangle_v \) is \( 2E_g + \hbar \nu \) and so the difference in energy is \( \hbar \omega - \hbar \nu \) which is exactly the energy 
of the laser pulse photon applied on ion m. Thus by energetics alone we see why

\[ U_m |e\rangle_m |g\rangle_n |0\rangle_v \longrightarrow -i |g\rangle_m |g\rangle_n |1\rangle_v \] is possible. But what about the \(-i\) factor, where does that come from?
It turns out we can adjust the polarization properties of the laser in order the \(-i\) phase factor 
appears (Deeper discussion of this requires a better understanding of quantum optics, which is beyond 
the scope of this course).

Finally we also have

\[ U_m |e\rangle_m |e\rangle_n |0\rangle_v \longrightarrow -i |g\rangle_m |e\rangle_n |1\rangle_v \]

In summary, after the first pulse

\[ U_m \begin{pmatrix} |g\rangle_m |g\rangle_n |0\rangle_v \\ |g\rangle_m |e\rangle_n |0\rangle_v \\ |e\rangle_m |g\rangle_n |0\rangle_v \\ |e\rangle_m |e\rangle_n |0\rangle_v \end{pmatrix} \longrightarrow 
U_m \begin{pmatrix} |g\rangle_m |g\rangle_n |0\rangle_v \\ |g\rangle_m |e\rangle_n |0\rangle_v \\ |e\rangle_m |g\rangle_n |0\rangle_v \\ |e\rangle_m |e\rangle_n |0\rangle_v \end{pmatrix} \]

- \[ U_m |g\rangle_m |g\rangle_n |0\rangle_v \longrightarrow |g\rangle_m |g\rangle_n |0\rangle_v \]
- \[ U_m |g\rangle_m |e\rangle_n |0\rangle_v \longrightarrow |g\rangle_m |e\rangle_n |0\rangle_v \]
- \[ U_m |e\rangle_m |g\rangle_n |0\rangle_v \longrightarrow -i |g\rangle_m |g\rangle_n |1\rangle_v \]
- \[ U_m |e\rangle_m |e\rangle_n |0\rangle_v \longrightarrow -i |g\rangle_m |e\rangle_n |1\rangle_v \]
We now consider a similar laser pulse, which we call $V_n$, on ion n (this will have different polarization properties than the aforementioned pulse). We then have

\[
\begin{align*}
V_n |g\rangle_m |g\rangle_n |0\rangle_v &\rightarrow |g\rangle_m |g\rangle_n |0\rangle_v \\
V_n |g\rangle_m |e\rangle_n |0\rangle_v &\rightarrow |g\rangle_m |e\rangle_n |0\rangle_v \\
V_n |g\rangle_m |g\rangle_n |1\rangle_v &\rightarrow -i |g\rangle_m |g\rangle_n |1\rangle_v \\
V_n |g\rangle_m |e\rangle_n |1\rangle_v &\rightarrow i |e\rangle_m |e\rangle_n |0\rangle_v
\end{align*}
\]

For each state in the first column the photon does not have enough energy to change the state of Qbit n.

Now we repeat pulse $U_m$ once again so that

\[
\begin{align*}
U_m |g\rangle_m |g\rangle_n |0\rangle_v &\rightarrow |g\rangle_m |g\rangle_n |0\rangle_v \\
U_m |g\rangle_m |e\rangle_n |0\rangle_v &\rightarrow |g\rangle_m |e\rangle_n |0\rangle_v \\
U_m |g\rangle_m |g\rangle_n |1\rangle_v &\rightarrow -i |e\rangle_m |g\rangle_n |0\rangle_v \\
U_m |g\rangle_m |e\rangle_n |1\rangle_v &\rightarrow i |e\rangle_m |e\rangle_n |0\rangle_v
\end{align*}
\]

The three-pulses in this sequence $W = U_m V_n U_m$ then lead to the following transformations

\[
\begin{align*}
W |g\rangle_m |g\rangle_n |0\rangle_v &\rightarrow |g\rangle_m |g\rangle_n |0\rangle_v \\
W |g\rangle_m |e\rangle_n |0\rangle_v &\rightarrow |g\rangle_m |e\rangle_n |0\rangle_v \\
W |e\rangle_m |g\rangle_n |0\rangle_v &\rightarrow |e\rangle_m |g\rangle_n |0\rangle_v \\
W |e\rangle_m |e\rangle_n |0\rangle_v &\rightarrow - |e\rangle_m |e\rangle_n |0\rangle_v
\end{align*}
\]

Notice that all states return to their original configurations, except the last state which picks up a negative sign.

Consider the superposition

\[
\frac{1}{\sqrt{2}} ( |g\rangle_n \pm |e\rangle_n )
\]

According to the above table

\[
\begin{align*}
W |g\rangle_m \frac{1}{\sqrt{2}} ( |g\rangle_n \pm |e\rangle_n ) | 0\rangle_v &= |g\rangle_m \frac{1}{\sqrt{2}} ( |g\rangle_n \pm |e\rangle_n ) | 0\rangle_v \\
W |e\rangle_m \frac{1}{\sqrt{2}} ( |g\rangle_n \pm |e\rangle_n ) | 0\rangle_v &= |e\rangle_m \frac{1}{\sqrt{2}} ( |g\rangle_n \mp |e\rangle_n ) | 0\rangle_v
\end{align*}
\]

So let's define the gate

$H_n W H_n$ here $H_n$ is the Hadamard gate acting on Qbit n. Therefore
\[ H_n \otimes H_n \ket{g}_m \ket{g}_n \ket{0}_r = H_n \otimes \ket{g}_m \frac{1}{\sqrt{2}} (\ket{g}_n + \ket{e}_n) \ket{0}_r = \]

\[ H_n \ket{g}_m \frac{1}{\sqrt{2}} (\ket{g}_n + \ket{e}_n) \ket{0}_r = \ket{g}_m \ket{g}_n \ket{0}_r \]

\[ H_n \otimes H_n \ket{g}_m \ket{e}_n \ket{0}_r = H_n \otimes \ket{g}_m \frac{1}{\sqrt{2}} (\ket{g}_n - \ket{e}_n) \ket{0}_r = \]

\[ H_n \ket{g}_m \frac{1}{\sqrt{2}} (\ket{g}_n - \ket{e}_n) \ket{0}_r = \ket{g}_m \ket{e}_n \ket{0}_r \]

\[ H_n \otimes H_n \ket{e}_m \ket{g}_n \ket{0}_r = H_n \otimes \ket{e}_m \frac{1}{\sqrt{2}} (\ket{g}_n + \ket{e}_n) \ket{0}_r = \]

\[ H_n \ket{e}_m \frac{1}{\sqrt{2}} (\ket{g}_n + \ket{e}_n) \ket{0}_r = \ket{e}_m \ket{0}_r \]

\[ H_n \otimes H_n \ket{e}_m \ket{e}_n \ket{0}_r = H_n \otimes \ket{e}_m \frac{1}{\sqrt{2}} (\ket{g}_n - \ket{e}_n) \ket{0}_r = \]

\[ H_n \ket{e}_m \frac{1}{\sqrt{2}} (\ket{g}_n - \ket{e}_n) \ket{0}_r = \ket{e}_m \ket{g}_n \ket{0}_r \]

In summary:

\[ H_n \otimes H_n \ket{g}_m \ket{g}_n \ket{0}_r = \ket{g}_m \ket{g}_n \ket{0}_r \]
\[ H_n \otimes H_n \ket{g}_m \ket{e}_n \ket{0}_r = \ket{g}_m \ket{e}_n \ket{0}_r \]
\[ H_n \otimes H_n \ket{e}_m \ket{g}_n \ket{0}_r = \ket{e}_m \ket{g}_n \ket{0}_r \]
\[ H_n \otimes H_n \ket{e}_m \ket{e}_n \ket{0}_r = \ket{e}_m \ket{e}_n \ket{0}_r \]

Thus we notice that the sequence of operations

\[ H_n \otimes H_n \]

is equivalent to a CNOT gate i.e. Qbit n is flipped if Qbit m is in state \( \ket{e} \) but is left alone if Qbit m is in state \( \ket{g} \).
Ion Trap Quantum Computing with Ca\(^{+}\)

and, with a different formulation, Milburn proposed a scheme for “hot” quantum gates, i.e., their procedures for gate operations do not require ground state cooling of an ion string. Although successfully applied to trapped Be\(^{+}\) ions, with the trapping parameters currently available, these gate procedures are not easily applicable to Ca\(^{+}\) ions.

Other gates based on ac Stark shifts have been suggested by Jonathan et al. and holonomic quantum gates (using geometric phases) have been proposed by Duan et al. A different CNOT-gate operation also based on the ac Stark effect which does not require individual addressing and ground state cooling has been realized with trapped Be\(^{+}\) ions.

3. SPECTROSCOPY IN ION TRAPS

Ions are considered to be trapped in a harmonic potential with frequency \(\nu_z\), interacting with the travelling wave of a single mode laser tuned close to a transition that forms an effective two-level system. Internal state detection of a trapped ion is achieved using the electron shelving technique. For this, one of the internal states of the trapped atom is selectively excited to a third short-lived state thereby scattering many photons on that transition if the coupled internal state was occupied. If, on the other hand, the atom’s electron resides in the uncoupled state of the qubit (i.e., the electron is shelved in that state) then no photons are scattered and thus the internal state can be detected with an efficiency of nearly 100%.

Fig. 1. Level scheme of \(^{40}\)Ca\(^{+}\). The qubit is implemented using the narrow quadrupole transition. All states split up into the respective Zeeman sublevels.

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